Constructing coset geometry with MAGMA

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VUB Algebra Research Group Seminar

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Introduction

Francis Buekenhout

Buekenhout's research group was interested in providing a unified geometric interpretation of all finite simple groups, in the spirit of Jacques Tits' theory of buildings.

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- Data were collected in atlases and these atlases led to conjectures that were proven later on theoretically.
- A big emphasis was put on understanding the subgroup structure of groups, something MAGMA can (finally) do very well.

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- 2016 : (string) C-groups and C^+-groups

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- every maximal clique contains n elements.

$\mid I \mid$ is the **rank** of the incidence geometry.

Coset geometries

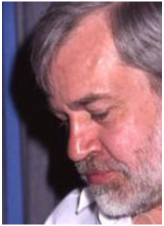
Theorem

(Tits, 1957) Let n be a positive integer. Let $I := \{1, \ldots, n\}$ be a finite set and let G be a group together with a family of subgroups $(G_i)_{i \in I}$. Let X be the set consists of all cosets $G_i g$, $g \in G$, $i \in I$. Let $t : X \to I$ be defined by $t(G_i g) = i$. Define an incidence relation $*$ on $X \times X$ by :

$$G_i g_1 * G_j g_2 \text{ iff } G_i g_1 \cap G_j g_2 \text{ is non-empty in } G.$$

Then the 4-tuple $\Gamma := (X, *, t, I)$ is an incidence structure having a chamber. Moreover, the group G acts by right multiplication as an automorphism group on Γ. Finally, the group G is transitive on the flags of rank less than 3.

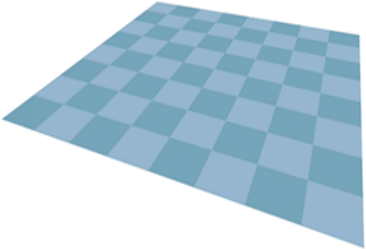
Abstract regular polytopes








Abstract regular polytopes



"Created by Lucas Vieira"

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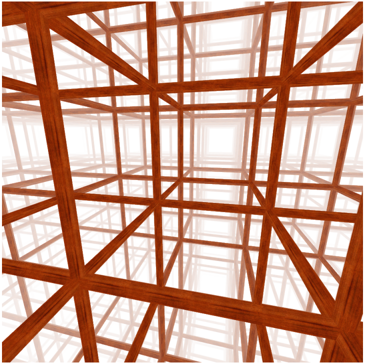

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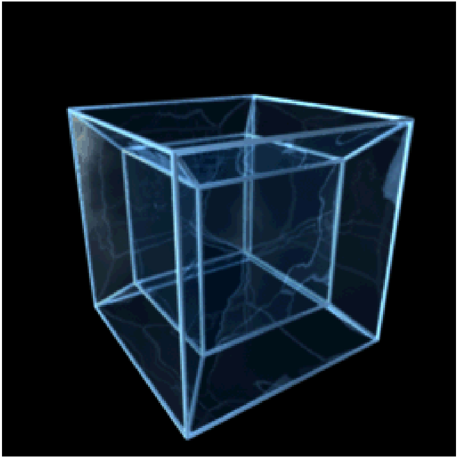


"Created by Jason Hise using Maya and Macromedia Fireworks."

Abstract regular polytopes

An **abstract polytope** $(\mathcal{P}, \leq)$ is a poset satisfying four extra conditions:

- the poset has a least face and a greater face;
- each maximal chain of the poset has same length $r + 2$ (r will be called the **rank**);
- a diamond condition;
- a strong connectedness condition.

An abstract polytope is **regular** if its group of automorphisms is transitive on the set of maximal chains (also called **flags**).

Abstract regular polytopes

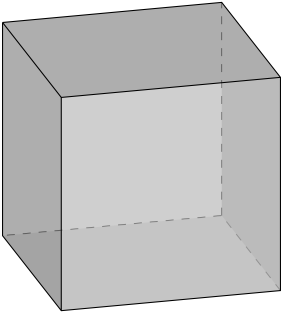


Figure: A Cube

Abstract regular polytopes

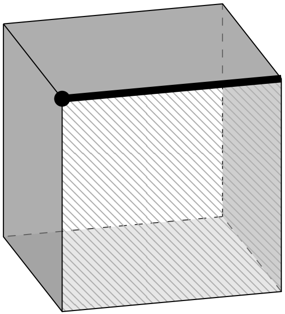
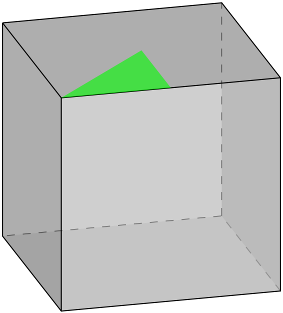


Figure: A chain on the Cube consisting of a vertex, an edge containing that vertex and a face containing the edge

Abstract regular polytopes and String C-groups

There is a natural one-to-one correspondence between abstract regular polytopes and string C-groups.

String C-groups



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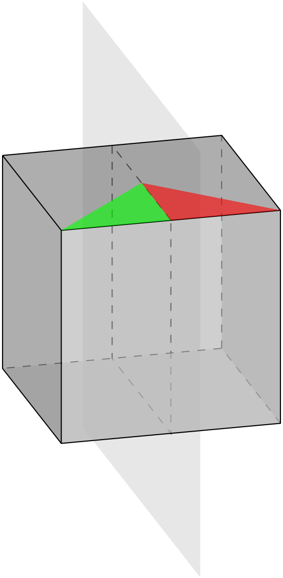

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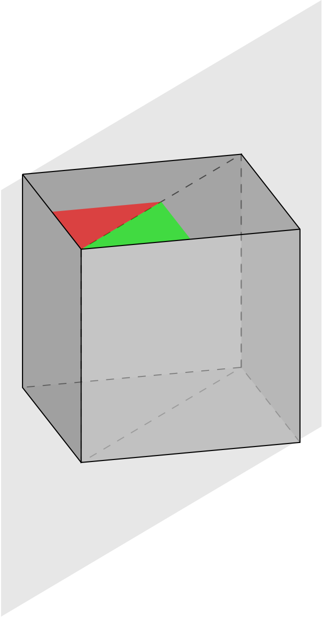

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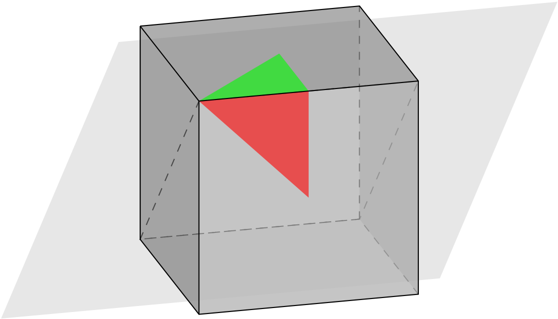


String C-groups

And the other way around ... use Jacques Tits algorithm to construct a coset geometry and order the types.

Abstract regular polytopes - problems

Given a family of groups (e.g. S_n), can we determine

1. what is the highest rank of an abstract regular polytope that has one of the groups of the family as full automorphism group?
2. what are the possible ranks?
3. how many pairwise nonisomorphic polytopes are there?
4. ...

What framework to take?

To solve the problems mentioned in the previous slide, we can choose to work with

- Posets with a set of extra axioms;
- Coset geometries;
- String C-groups.

String C-groups

Definition

A **C-group of rank** r is a pair (G, S) such that G is a group and $S := \{\rho_0, \ldots, \rho_{r-1}\}$ is a generating set of involutions of G that satisfy the following property.

$$\forall I, J \subseteq \{0, \ldots, r-1\}, \langle \rho_i \mid i \in I \rangle \cap \langle \rho_j \mid j \in J \rangle = \langle \rho_k \mid k \in I \cap J \rangle$$

This property is called the **intersection property** and denoted by (IP). We call any subgroup of G generated by a subset of S a *parabolic subgroup* of the C-group (G, S).

String C-groups

Definition

A **C-group** (G, S) **of rank** r is a **string C-group** if its set of generating involutions S can be ordered in such a way that $S := \{\rho_0, \ldots, \rho_{r-1}\}$ satisfies

$$\forall i, j \in \{0, \ldots, r-1\}, o(\rho_i \rho_j) = 2 \text{ if } |i - j| > 1$$

This property is called the **string property** and denoted by (SP).

Definition

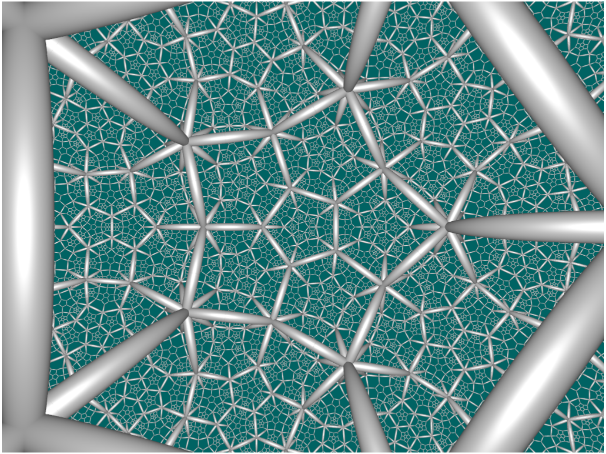
For a given group G, we will call (G, S) a **string C-group representation** of G provided it satisfies (SP) and (IP).

How I decided to switch to these objects ?

- 2002 : Michael Hartley contacts me. He has found an abstract regular polytope of type [5,3,5] whose automorphism group is J_1.
- I knew of the existence of this polytope thanks to a computer search of 1997.
- We decided to study that polytope and found something quite special.

Coxeter groups

Coxeter group of type [5,3,5]



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The Coxeter group of type [5,3,5] is the finitely presented group

$$W =< a, b, c, d | a^2, b^2, c^2, d^2,$$
$$(a * b)^5, (a * c)^2, (a * d)^2, (b * c)^3, (b * d)^2, (c * d)^5 >.$$

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$+ (abc)^5 = (bcd)^5 = 1_W$ gives $L_2(19)$ and Coxeter's 57-cell.
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$+ (bcd)^5 = 1_W$ gives $L_2(19) \times J_1$.
(Hartley, L., Math. Z. 2004)

What groups to look at?

"Small" groups
Soluble groups
Nilpotent groups
2-groups
Simple groups
Sporadic groups
Almost simple groups
Non-solvable groups
etc.

What groups to look at?

- Hartley decided to focus on small groups
- I decided to focus on small almost simple groups

Small groups

Groups of even order ≤ 2000

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Ratio 0.000009% (for soluble) VS 85% (for non-solvable)

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Most groups are 2-groups.
49,910,526,325 - 412,607,930 = 49,497,918,395 2-groups
(99,17% of 2-groups among the groups of even order less than 2001)

String C-group representations of 2-groups

2-groups are also important for abstract regular polytopes as they give the smallest examples of a given rank $n > 8$.

String C-group representations of 2-groups

Theorem (Conder, 2013)

Let F_n be the number of flags in a regular polytope of rank n. Then a lower bound for F_n is given by $F_n \geq 2 \cdot 4^{n-1}$ for all $n \geq 9$, and this bound is attained by a family of tight polytopes of type $\{4| \ldots |4\}$, one for each n. For rank $n \leq 8$, the fewest flags occur for regular n-polytopes as follows:

n	$min(F_n)$	*Type(s) attaining the lower bound*
2	6	$\{3\}$
3	24	$\{3\|3\}$, $\{3\|4\}$ *(and dual* $\{4\|3\}$*)*
4	96	$\{4\|3\|4\}$
5	432	$\{3\|6\|3\|4\}$ *(and dual* $\{4\|3\|6\|3\}$*)*
6	1 728	$\{4\|3\|6\|3\|4\}$
7	7 776	$\{3\|6\|3\|6\|3\|4\}$ *(and dual* $\{4\|3\|6\|3\|6\|3\}$*)*
8	31 104	$\{4\|3\|6\|3\|6\|3\|4\}$

String C-group representations of 2-groups

The **Frattini subgroup** $\Phi(G)$, of a finite group G is the intersection of all maximal subgroups of G.
Let G be a finite p-group for a prime p, and set $\mho_1(G) = \{g^p \mid g \in G\}$.

Theorem (Burnside Basis Theorem)

Let G be a p-group and $|G : \Phi(G)| = p^d$.

(1) $G/\Phi(G) \cong \mathbb{Z}_p^d$. *Moreover, if $N \lhd G$ and G/N is elementary abelian, then $\Phi(G) \leq N$.*

(2) **Every minimal generating set of G contains exactly d elements**[a].

(3) $\Phi(G) = G'\mho_1(G)$. *In particular, if $p = 2$, then $\Phi(G) = \mho_1(G)$.*

[a] d is called the *rank* of G and denoted by $d(G)$.

String C-group representations of 2-groups

Corollary (Hou, Feng, L., 2019)

A given 2-group has only string C-group representations with a fixed rank, that is, the rank of the 2-group.

String C-group representations of $PSL(2, q)$ groups

Linear groups and their automorphism groups

G	Aut(G)	Order of G	Number of involutions	Number of Polytopes
Alt(5) = PSL(2,4) = PSL(2,5)	Sym(5)	60	15	2
Sym(5)	Sym(5)	120	25	5 = 4+1
PSL(3,2) = PSL(2,7)	PΓL(2,7)	168	21	0
PGL(2,7) = PΓL(2,7)	PΓL(2,7)	336	49	16
Alt(6) = PSL(2,9)	PΓL(2,9)	360	45	0
PGL(2,9)	PΓL(2,9)	720	81	14
PΣL(2,9)	PΓL(2,9)	720	75	7 = 2+4+1
M_{10}	PΓL(2,9)	720	45	0
PΓL(2,9)	PΓL(2,9)	1440	111	12
PSL(2,8) = PGL(2,8)	PΓL(2,8)	504	63	7
PΓL(2,8) = PΣL(2,8)	PΓL(2,8)	1512	63	0
PSL(2,11) = PΣL(2,11)	PΓL(2,11)	660	55	4 = 3+1
PGL(2,11) = PΓL(2,11)	PΓL(2,11)	1320	121	42
PSL(2,13) = PΣL(2,13)	PΓL(2,13)	1092	91	11
PGL(2,13) = PΓL(2,13)	PΓL(2,13)	2184	169	59
PSL(2,17) = PΣL(2,17)	PΓL(2,17)	2448	153	16
PGL(2,17) = PΓL(2,17)	PΓL(2,17)	4896	289	110
PSL(2,19) = PΣL(2,19)	PΓL(2,19)	3420	171	18 = 17+1
PGL(2,19) = PΓL(2,19)	PΓL(2,19)	6840	361	140
PSL(2,16) = PGL(2,16)	PΓL(2,16)	4080	255	27
PSL(2,16):2	PΓL(2,16)	8160	323	26 = 21+5
PΓL(2,16) = PΣL(2,16)	PΓL(2,16)	16320	323	0
PSL(3,3) = PGL(3,3) = PΣL(3,3) = PΓL(3,3)	PSL(3,3):2	5616	117	0
PSL(3,3):2	PSL(3,3):2	11232	351	68 = 67+1
PSL(2,23) = PΣL(2,23)	PΓL(2,23)	6072	253	28
PGL(2,23) = PΓL(2,23)	PΓL(2,23)	12144	529	212
PSL(2,25)	PΓL(2,25)	7800	325	17
PGL(2,25)	PΓL(2,25)	15600	625	127
PΣL(2,25)	PΓL(2,25)	15600	455	51 = 34+17

String C-group representations of $PSL(2, q)$ groups

Theorem (Schulte - L., 2007)

The maximum rank of a string C-group representation of $PSL(2, q)$ is 4 and only happens when $q = 11$ or 19.

The corresponding polytopes are

- the 11-cell of Grunbaüm of type [3,5,3], and
- the 57-cell of Coxeter of type [5,3,5].

String C-group representations of sporadic groups

G	Order of G	Rank 3	Rank 4	Rank 5
M_{11}	7,920	0	0	0
M_{12}	95,040	23	14	0
M_{22}	443,510	0	0	0
M_{23}	10,200,960	0	0	0
M_{24}	244,823,040	490	155	2
J_1	175,560	148	2	0
J_2	604,800	137	17	0
J_3	50,232,960	303	2	0
HS	44,352,000	252	57	2
McL	898,128,000	0	0	0
He	4,030,387,200	1188	76	0
Ru	145,926,144,000	21594	227	0
Suz	448,345,497,600	7119	257	13
$O'N$	460,815,505,920	6536	31	0
Co_3	495,766,656,000	21118	1746	44

String C-group representations of symmetric groups

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E. H. Moore (1896) : $(n-1)$-simplex.



String C-group representations of symmetric groups

E. H. Moore (1896) : $(n-1)$-simplex.



Theorem (Moore, 1896)

For every $n \geq 3$, there is a string C-group representation of S_n in its natural permutation representation, of rank $n-1$ whose generating involutions are the transpositions $(i, i+1)$ with $i = 1, \ldots, n-1$.

String C-group representations of symmetric groups

Proposition (Whiston, 2000)

The size of an independent set in S_n is at most $n-1$, with equality only if the set generates the whole group S_n.

String C-group representations of symmetric groups

Sjerve and Cherkassoff (1993) (see also Conder 1980): S_n is a group generated by three involutions, two of which commute, provided that $n \geq 4$.

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Theorem ("Moore, Sjerve, Cherkassoff, Conder")

Every group S_n with $n \geq 4$ has a string C-group representation of rank three and one of rank $n-1$.

String C-group representations of symmetric groups

G	Rank 3	Rank 4	Rank 5	Rank 6	Rank 7	Rank 8
S_5	4	1	0	0	0	0
S_6	2	4	1	0	0	0
S_7	35	7	1	1	0	0
S_8	68	36	11	1	1	0
S_9	129	37	7	7	1	1

Source: http://leemans.dimitri.web.ulb.be/~dleemans/polytopes

String C-group representations of symmetric groups

Theorem (Fernandes, Leemans, 2011)

For $n \geq 5$ or $n = 3$, Moore's generators give, up to isomorphism, the unique string C-group representation of rank $n - 1$ for S_n. For $n = 4$, there are, up to isomorphism and duality, two representations, namely the ones corresponding to the hemicube and the tetrahedron.

String C-group representations of symmetric groups

Theorem (Fernandes, Leemans, 2011)

For $n \geq 7$, there exists, up to isomorphism and duality, a unique string C-group representation of rank $(n-2)$ for S_n.

String C-group representations of symmetric groups

Theorem (Fernandes, Leemans, 2011)

Let $n \geq 4$. For every $r \in \{3, \ldots, n-1\}$, there exists at least one string C-group representation of rank r for S_n.

String C-group representations of symmetric groups

Let $\{\rho_0, \ldots, \rho_{r-1}\}$ be a set of involutions of a permutation group G of degree n. We define the **permutation representation graph** $\mathcal{G}$ as the r-edge-labeled multigraph with n vertices and with a single i-edge $\{a, b\}$ whenever $a\rho_i = b$ with $a \neq b$.

String C-group representations of symmetric groups

Generators	Permutation representation	Schläfli type
(1,2),(2,3),(3,4),(4,5),(5,6),(6,7)	○—1—○—2—○—3—○—4—○—5—○—6—○	$\{3,3,3,3,3\}$
(1,2),(2,3),(3,4),(4,5)(6,7),(5,6)	○—1—○—2—○—3—○—4—○—5—○—4—○	$\{3,3,6,4\}$
(1,2),(2,3),(3,4)(5,6),(4,5)(6,7)	○—1—○—2—○—3—○—4—○—3—○—4—○	$\{3,6,5\}$
(1,2),(2,3)(4,5)(6,7),(3,4)(5,6)	○—1—○—2—○—3—○—2—○—3—○—2—○	$\{6,6\}$

Table: The induction process used on S_7

String C-group representations of symmetric groups

Number of representations, up to duality, for S_n $(5 \leq n \leq 14)$

G\r	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
S_5	4	1	0	0	0	0	0	0	0	0	0
S_6	2	4	1	0	0	0	0	0	0	0	0
S_7	35	7	1	1	0	0	0	0	0	0	0
S_8	68	36	11	1	1	0	0	0	0	0	0
S_9	129	37	7	7	1	1	0	0	0	0	0
S_{10}	413	203	52	13	7	1	1	0	0	0	0
S_{11}	1221	189	43	25	9	7	1	1	0	0	0
S_{12}	3346	940	183	75	40	9	7	1	1	0	0
S_{13}	7163	863	171	123	41	35	9	7	1	1	0
S_{14}	23126	3945	978	303	163	54	35	9	7	1	1

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S_5	4	**1**	0	0	0	0	0	0	0	0	0
S_6	2	4	**1**	0	0	0	0	0	0	0	0
S_7	35	7	**1**	**1**	0	0	0	0	0	0	0
S_8	68	36	11	**1**	**1**	0	0	0	0	0	0
S_9	129	37	7	7	**1**	**1**	0	0	0	0	0
S_{10}	413	203	52	13	7	**1**	**1**	0	0	0	0
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S_8	68	36	11	**1**	**1**	0	0	0	0	0	0
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S_{10}	413	203	52	13	**7**	**1**	**1**	0	0	0	0
S_{11}	1221	189	43	25	**9**	**7**	**1**	**1**	0	0	0
S_{12}	3346	940	183	75	40	**9**	**7**	**1**	**1**	0	0
S_{13}	7163	863	171	123	41	35	**9**	**7**	**1**	**1**	0
S_{14}	23126	3945	978	303	163	54	35	**9**	**7**	**1**	**1**

String C-group representations of symmetric groups

Theorem (Fernandes-Leemans-Mixer, 2018)

For $n \geq 9$, there exists, up to isomorphism and duality, seven string C-group representations of rank $(n-3)$ for S_n.
For $n \geq 11$, there exists, up to isomorphism and duality, nine string C-group representations of rank $(n-4)$ for S_n.

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Conjecture

Let r be a positive integer and $n \geq 2r+3$. The number of pairwise nonisomorphic string C-group representations of rank $n-r$ is independent on n.

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Conjecture

Let r be a positive integer and $n \geq 2r+3$. The number of pairwise nonisomorphic string C-group representations of rank $n-r$ is independent on n.

The sequence, depending on r, looks like 1, 1, 7, 9, 35, 48, 135, ...

String C-group representations of alternating groups

What about alternating groups ?

String C-group representations of alternating groups

G	Rank 3	Rank 4	Rank 5	Rank 6	Rank 7	Rank 8
A_5	2	0	0	0	0	0
A_6	0	0	0	0	0	0
A_7	0	0	0	0	0	0
A_8	0	0	0	0	0	0
A_9	41	6	0	0	0	0

Source: http://leemans.dimitri.web.ulb.be/∼dleemans/polytopes

String C-group representations of alternating groups

G	Rank 3	Rank 4	Rank 5	Rank 6	Rank 7	Rank 8
A_5	2	0	0	0	0	0
A_6	0	0	0	0	0	0
A_7	0	0	0	0	0	0
A_8	0	0	0	0	0	0
A_9	41	6	0	0	0	0
A_{10}	94	2	4	0	0	0
A_{11}	64	0	0	3	0	0
A_{12}	194	90	22	0	0	0
A_{13}	1558	102	25	10	0	0
A_{14}	4347	128	45	9	0	0
A_{15}	5820	158	20	42	6	0

Source: http://leemans.dimitri.web.ulb.be/~dleemans/polytopes

String C-group representations of alternating groups

Theorem (Fernandes, Leemans, Mixer, 2012)

For each $n \notin \{3, 4, 5, 6, 7, 8, 11\}$, there is a rank $\lfloor \frac{n-1}{2} \rfloor$ string C-group representation of the alternating group A_n.

We found a striking example! A_{11} has string C-group representations of rank 3 and 6, but not 4 nor 5!

String C-group representations of alternating groups

A conjecture arose thanks to the collected data and the struggle to construct the above mentioned examples.

Conjecture

The highest rank of a string C-group representation of A_n is $\lfloor \frac{n-1}{2} \rfloor$ when $n \geq 12$.

String C-group representations of alternating groups

Strategy of the proof:

String C-group representations of alternating groups

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Take A_n the group of even permutations of n points.

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First show that a string C-group representations of A_n of rank $r > \lfloor \frac{n-1}{2} \rfloor$, if it exists, must have all its maximal parabolic subgroups (that is the subgroups generated by all but one generator) intransitive.

String C-group representations of alternating groups

Strategy of the proof:
Take A_n the group of even permutations of n points.
First show that a string C-group representations of A_n of rank $r > \lfloor \frac{n-1}{2} \rfloor$, if it exists, must have all its maximal parabolic subgroups (that is the subgroups generated by all but one generator) intransitive.
Then use this fact and permutation representation graphs to show that it is impossible.

String C-group representations of alternating groups

Theorem (Cameron, Fernandes, Leemans, Mixer, 2016)

Let Γ be a string C-group of rank r which is isomorphic to a transitive subgroup of S_n other than S_n or A_n. Then one of the following holds:

1. $r \leq n/2$;
2. $n \equiv 2 \mod 4$, $r = n/2+1$ *and* Γ *is* $C_2 \wr S_{n/2}$. *The generators are*

$$\rho_0 = (1, n/2+1)(2, n/2+2)\dots(n/2, n);$$
$$\rho_1 = (2, n/2+2)\dots(n/2, n);$$
$$\rho_i = (i-1, i)(n/2+i-1, n/2+i) \text{ for } 2 \leq i \leq n/2.$$

   Moreover the Schläfli type is $[2, 3, \dots, 3, 4]$.

3. Γ *is transitive imprimitive and is one of the examples appearing in the next Table.*
4. Γ *is primitive. In this case,* Γ *is obtained from the permutation representation of degree 6 of* $S_5 \cong PGL_2(5)$ *and it is the 4-simplex of Schäfli type* $[3, 3, 3]$.

String C-group representations of alternating groups

Degree	*Number*	Structure	Order	Schäfli type
6	9	$S_3 \times S_3$	36	$[2, 3, 3]$
6	11	$2^3 : S_3$	48	$[2, 3, 3]$
6	11	$2^3 : S_3$	48	$[2, 3, 4]$
8	45	$2^4 : S_3 : S_3$	576	$[3, 4, 4, 3]$

Table: Examples of transitive imprimitive string C-groups of degree n and rank $n/2 + 1$ for $n \leq 9$.

Corollary

Suppose $G = A_n$ of degree n. Let (G, S) be a string C-group with $S = \{\rho_0, \ldots, \rho_{r-1}\}$. If $r \geq n/2 + 2$, all subgroups G_i must be intransitive.

String C-group representations of alternating groups

The "Aveiro" theorem:

Theorem (Cameron, Fernandes, Leemans, Mixer, 2017)

The rank of A_n is 3 if $n = 5$; 4 if $n = 9$; 5 if $n = 10$; 6 if $n = 11$ and $\lfloor \frac{n-1}{2} \rfloor$ if $n \geq 12$. Moreover, if $n = 3, 4, 6, 7$ or 8, the group A_n is not a string C-group.

The proof of this result takes 39 pages and uses induction in some parts.
- 8 pages to refine our result on transitive groups and get to prove that all the maximal parabolic subgroups must be intransitive if the conjecture is false.
- 10 pages to handle the case where we assume there exists a 2-fracture graph.
- 21 pages to handle the case where we assume no 2-fracture graph exists.

Back to symmetric groups

Compiling the previous results, we get the following.

Corollary

If G is a transitive group of degree n having a string C-group of rank $r \geq (n+3)/2$, then G is necessarily the symmetric group S_n.

String C-groups of high rank

S_n	Rk $n-1$	Rk $n-2$	Rk $n-3$	Rk $n-4$	Rk $n-5$	Rk $n-6$
S_5	1	4				
S_6	1	4	2			
S_7	1	1	7	35		
S_8	1	1	11	36	68	
S_9	1	1	7	7	37	129
S_{10}	1	1	7	13	52	203
S_{11}	1	1	7	9	25	43
S_{12}	1	1	7	9	40	75
S_{13}	1	1	7	9	35	41
S_{14}	1	1	7	9	35	54
S_{15}	1	1	7	9	35	48
S_{16}	1	1	7	9	35	48

Table: The number of pairwise nonisomorphic string C-groups of rank $n-k$ for S_n with $1 \leq k \leq 6$ and $5 \leq n \leq 16$.

String C-groups of high rank

Let $\mathcal{S}(n, r)$ be the set of all string C-group representations of rank r for S_n. Define a relation $\sim$ on $\mathcal{S}(n, r) \times \mathcal{S}(n, r)$ by saying that for any elements $P, Q \in \mathcal{S}(n, r)$, $P \sim Q$ if and only if P is isomorphic to Q or to the dual of Q. The relation $\sim$ is an equivalence relation.

String C-groups of high rank

Let $\Sigma^\kappa(n) = \mathcal{S}(n, n-\kappa)/\sim$. The results of Fernandes-Leemans 2011 and Fernandes-Leemans-Mixer 2018 give the following sequence.

$$
\begin{aligned}
|\Sigma^1(n)| &= 1 \text{ for } n \geq 5 \\
|\Sigma^2(n)| &= 1 \text{ for } n \geq 7 \\
|\Sigma^3(n)| &= 7 \text{ for } n \geq 9 \\
|\Sigma^4(n)| &= 9 \text{ for } n \geq 11
\end{aligned}
$$

In addition, relying on computational results, it was conjectured that

$$
\begin{aligned}
|\Sigma^5(n)| &= 35 \text{ for } n \geq 13, \\
|\Sigma^6(n)| &= 48 \text{ for } n \geq 15, \text{ and} \\
|\Sigma^7(n)| &= 135 \text{ for } n \geq 17.
\end{aligned}
$$

String C-groups of high rank

The Brussels Theorem:

Theorem (Cameron-Fernandes-Leemans 2024)

For each fixed integer $\kappa \geq 1$, there exists an integer c_κ such that, for all $n \geq 2\kappa + 3$, $|\Sigma^\kappa(n)| = c_\kappa$.

String C-groups of high rank

- This theorem and the tools used in its proof, in particular the rank and degree extension, imply that if one knows the string C-groups of rank $(n+3)/2$ for S_n with n odd, one can construct from them all string C-groups of rank $(n+3)/2+k$ for S_{n+k} for any positive integer k.
- The classification of the string C-groups of rank $r \geq (n+3)/2$ for S_n is thus reduced to classifying string C-groups of rank r for S_{2r-3}.