Commuting probability of skew left braces

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Motivation

- Commuting probability of groups:

$$Pr(G) = \frac{|\{(x,y) \in G^2 : xy = yx\}|}{|G|^2}.$$

- Introduced by Erdős–Turán and developed by Gustafson.
- Measures how close a structure is to being abelian.
- Goal:
  - Introduce an analogous notion for skew braces.
  - Investigate bounds and structural consequences.

This talk is built on the results from:
S. Mondal and M.K. Yadav, Commuting probability of skew left braces, arXiv:2603.16771.

Why Skew Braces?

With genesis from the literature, mainly the work of Etingof-Schedler-Soloviev (1999),

- Rump introduced left brace (2007).
- Guarnieri–Vendramin introduced skew left braces (2017).
- The subject is connected to:
  - Yang–Baxter equation
  - Regular subgroups of a holomorph
  - Hopf–Galois theory
  - Radical rings
  - Quantum algebra

Definition of a Skew Brace

A skew (left) brace is a set B with two group operations

$$(B, +), \qquad (B, \circ)$$

satisfying

$$a \circ (b + c) = a \circ b - a + a \circ c,$$

where $-c$ is the inverse of c in $(B, +)$.

- A skew brace $(B, +, \circ)$ is said to be *trivial* if $+$ and $\circ$ coincide.
- A skew (left) brace $(B, +, \circ)$ is said to be (left) brace if $(B, +)$ is an abelian group.

Examples

For any group $(G, +)$, $(G, +, +_{op})$ is a skew brace with $x +_{op} y = y + x$.

Let $(F_2, \cdot) = \langle x, y \rangle$ be a free group, θ the automorphism of F_2:

$$\theta : x \leftrightarrows y$$

and $H_0 \leq F_2$ is the kernel of the homomorphism

$$F_2 \to \mathbb{Z}_2 = \langle \theta \rangle,$$

$x, y \mapsto \theta$. Then $(F_2, \cdot, \circ)$ is a skew brace, where '$\circ$' is defined by the rule

$$a \circ b = \begin{cases} ab, & \text{if } a \in H_0,\ b \in F_2, \\ a\theta(b), & \text{if } a \notin H_0,\ b \in F_2 \end{cases}$$

and $(F_2, \circ)$ is a free group of rank 2.

Key Maps

Define a map $\lambda_a : (B, +) \to (B, +)$, given by

$$\lambda_a(b) = -a + (a \circ b).$$

Then

$$\lambda_a \in \mathrm{Aut}(B, +)$$

and

$$\lambda : (B, \circ) \to \mathrm{Aut}(B, +)$$

is a group homomorphism.
Define

$$a * b = \lambda_a(b) - b,$$

a brace commutator.

Left ideal and brace homomorphism

A sub-group $(I, +)$ of a skew brace B is said to be a *left ideal* of B if $\lambda_a(y) \in I$ for all $a \in B$ and $y \in I$. A left ideal I of B is said to be an *ideal* if $(I, +)$ and $(I, \circ)$, respectively, normal subgroups of $(B, +)$ and $(B, \circ)$.

Equivalent interesting definitions in terms of $*$-operation are available.

Let B_1 and B_2 be two skew braces. A map $f : B_1 \to B_2$ is said to be a *brace homomorphism* if

$$f(a + b) = f(a) + f(b) \quad \text{and} \quad f(a \circ b) = f(a) \circ f(b)$$

for all $a, b \in B_1$.

A one-to-one and onto brace homomorphism from a skew brace B_1 to B_2 is called a brace isomorphism.

Brace Centralizers

For $x \in B$ define

$$Cb_B(x) = \{b \in B : x * b = b * x = [x, b]^+ = 1\}.$$

For a finite skew brace B and its subset S, we define centraliser of S as follows:

$$Cb_B(S) := \cap_{x \in S} Cb_B(x).$$

Define

$$\mathrm{Ann}(B) = Cb_B(B).$$

Properties:

- $\mathrm{Ann}(B) \subseteq Cb_B(x)$.
- $Cb_B(x)$ is a subgroup of $(B, \circ)$ (Theorem by Colazzo-Ferrara-Trombetti (2025)).

Commuting Probability

For a finite skew brace B,

$$\mathrm{Pb}(B) = \frac{|\{(x, y) \in B^2 : x * y = y * x = [x, y]^+ = 1\}|}{|B|^2},$$

where $[x, y]^+$ denotes the commutator of the elements x, y in $(B, +)$.

Interpretation:
Probability that two randomly chosen elements commute simultaneously with respect to all brace operations.

$\mathrm{Pb}(B) = 1$ if and only if B is a trivial brace (abelian group).

Centralizer Formula

Analogous to group theory,

$$\mathrm{Pb}(B) = \frac{\sum_{x \in B} |Cb_B(x)|}{|B|^2}.$$

This formula is the basic computational tool which we have used throughout the paper.

Can one connect it to the character theory of skew braces?

Basic Bounds

For a finite non-trivial skew brace B, let

$$d = |B/\mathrm{Ann}(B)|.$$

Theorem.

$$\frac{2d-1}{d^2} \leq \mathrm{Pb}(B) \leq \frac{d+1}{2d}.$$

Consequences:

- Probability is controlled by $|B/\mathrm{Ann}(B)|$.
- Sharp finite bounds are obtained.

Lower bound is attained when $d = p$, a prime divisor of $|B|$.

Exact Upper Bounds

Now we obtain exact upper bounds for $\mathrm{Pb}(B)$ for all finite skew braces B.

Theorem 1

Let $(B, +, \circ)$ *be a finite skew brace. Then* $\mathrm{Pb}(B) = 1$, $\mathrm{Pb}(B) = \frac{3}{4}$ *or* $\mathrm{Pb}(B) \leqslant \frac{5}{8}$.

Proof.

Let B be any finite skew brace. Let $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be a function defined by $f(x) = \frac{x+1}{2x}$. Then the derivative of f is

$$f'(x) = \frac{-1}{x^2} \ (\leqslant 0).$$

Thus f is a monotone decreasing function. If $d = 1$, that is, B is a trivial brace, then $\mathrm{Pb}(B) = 1$. If $d = 2, 3$, then, by preceding theorem and variants of it, we get $\mathrm{Pb}(B) = \frac{3}{4}$ and $\mathrm{Pb}(B) = \frac{5}{9}$, respectively. Note that for $d = 4$, we obtain $\mathrm{Pb}(B) \leqslant \frac{5}{8}$. Since f is a monotone decreasing function and $\frac{5}{9} \leqslant \frac{5}{8}$, it follows that $\mathrm{Pb}(B) \leqslant \frac{5}{8}$ for all $d \geqslant 3$, which completes the proof. □

Characterization of Pb$(B) = 3/4$

From the bound

$$\frac{2d-1}{d^2} \leq \mathrm{Pb}(B) \leq \frac{d+1}{2d},$$

we get:

Theorem

$$\mathrm{Pb}(B) = \frac{3}{4}$$

if and only if

$$|B/\mathrm{Ann}(B)| = 2.$$

Equivalently,

$$B/\mathrm{Ann}(B) \cong \mathbb{Z}_2.$$

Characterization of $\mathrm{Pb}(B) = 5/8$

Theorem
If, for a skew brace B,

$$\mathrm{Pb}(B) = \frac{5}{8},$$

then

$$(B/\mathrm{Ann}(B), +) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

and

$$(B/\mathrm{Ann}(B), \circ) \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Unlike group theory result, the converse of the preceding result is not true. There are numerous examples which establish this fact. One can use GAP to see that the skew braces B with GAP ids $(8, 45)$, $(8, 46)$, $(24, 664 - 666)$ are some counter examples to the converse of the preceding result.

Sub-braces and Quotients

For a sub-brace H of a skew brace B,

$$\mathrm{Pb}(B) \leq \mathrm{Pb}(H).$$

For an ideal N of B,

$$\mathrm{Pb}(B) \leq P_b(N)\ \mathrm{Pb}(B/N).$$

Applications: These inequalities are useful for inductive arguments.

Large Probability Implies Structure

Question:
Does a large commuting probability force strong algebraic structure?

Answer:

Yes

Nilpotent Skew Brace

For two sub-skew braces H and K of skew brace B, we define

$$H * K = \langle h * k : h \in H, k \in K \rangle^+,$$

a subgroup of $(B, +)$.

Set $\Gamma_1(B) := B$ and define

$$\Gamma_n(B) := \langle B * \Gamma_{n-1}(B), \Gamma_{n-1}(B) * B, [B, \Gamma_{n-1}(B)]^+ \rangle^+,$$

where $[B, \Gamma_{n-1}(B)]^+$ denotes the subgroup of $(B, +)$ generated by the set $\{[a, u]^+ : a \in B, u \in \Gamma_{n-1}(B)\}$.

$\Gamma_n(B)$ is an ideal of B and $\Gamma_{n+1}(B) \leq \Gamma_n(B)$ for all integers $n \geq 1$.

A skew brace B is said to be *nilpotent* (also called centrally nilpotent) if there exists an integer n such that $\Gamma_{n+1}(B) = \{0\}$.

Nilpotency Theorem

Theorem
If, for a skew brace B,

$$\mathrm{Pb}(B) > \frac{65}{128},$$

then B is nilpotent.

This parallels classical results for groups. In group theory:

Theorem
If, for a finite group G, $Pr(G) > \frac{1}{2}$, then G is nilpotent.

What happens in the interval $(\frac{1}{2}, \frac{65}{128}]$ for skew braces?

Isoclinism

Generalization of Hall's isoclinism from groups to skew braces (Letourmy-Vendramin).

For a skew brace B, the following maps are well defined:

$$\phi_+^B : (B/\operatorname{Ann} B)^2 \to \Gamma_2(B), \qquad (\bar{a}, \bar{b}) \mapsto [a, b]^+,$$
$$\phi_*^B : (B/\operatorname{Ann} B)^2 \to \Gamma_2(B), \qquad (\bar{a}, \bar{b}) \mapsto a * b,$$

where $\bar{a}, \bar{b} \in B/\operatorname{Ann}(B)$. We say that two skew braces A and B are *isoclinic* if there exist brace isomorphisms $\psi : A/\operatorname{Ann} A \to B/\operatorname{Ann} B$ and $\theta : \Gamma_2(A) \to \Gamma_2(B)$ such that the following diagram commutes:

$$\begin{array}{ccccc}
\Gamma_2(A) & \xleftarrow{\phi_+^A} & (A/\operatorname{Ann} A)^2 & \xrightarrow{\phi_*^A} & \Gamma_2(A) \\
\theta \big\downarrow & & \big\downarrow \psi\times\psi & & \big\downarrow \theta \\
\Gamma_2(B) & \xleftarrow{\phi_+^B} & (B/\operatorname{Ann} B)^2 & \xrightarrow{\phi_*^B} & \Gamma_2(B).
\end{array}$$

Invariance Under Isoclinism

Brace isoclinism is an equivalence relation.

Theorem
If A and B are isoclinic, then

$$\mathrm{Pb}(A) = \mathrm{Pb}(B).$$

Hence commuting probability depends only on the isoclinism class.

Further more,

$$|\Gamma_2(A) \cap \mathrm{Ann}(A)| = |\Gamma_2(B) \cap \mathrm{Ann}(B)|.$$

This condition has nice consequences.

Infinite Skew Braces

We say that x has *finite centraliser index in B* if $[(B, \circ) : Cb_B(x)]$, the index of $Cb_B(x)$ in $(B, \circ)$ is finite.

Let $\mathrm{FCI}(B)$ denote the set of all elements of B having finite centraliser index in B. The skew brace B is said to be *FCI* if $B = \mathrm{FCI}(B)$.

For a skew brace B, $\mathrm{FCI}(B)$ is a subgroup of $(B, \circ)$. Moreover, if B is two sided, symmetric or λ-homomorphic, then $\mathrm{FCI}(B)$ is a sub-skew brace of B.

For a $(B, +, \circ)$ be a two-sided brace (note that just brace). Then $\mathrm{FCI}(B)$ is an ideal of B.

Compact topological skew braces

A skew brace $(B, +, \circ)$ is called a *topological skew brace* if both the groups $(B, +)$ and $(B, \circ)$ are topological groups admitting a common topology.

Thus, all four maps $\tau_1^+ : (B, +) \times (B, +) \to (B, +)$, $\tau_1^\circ : (B, \circ) \times (B, \circ) \to (B, \circ)$, $\tau_2^+ : (B, +) \to (B, +)$ and $\tau_2^\circ : (B, \circ) \to (B, \circ)$, respectively, given by $\tau_1^+(a, b) = a + b$, $\tau_1^\circ(a, b) = a \circ b$, $\tau_2^+(a) = -a$ and $\tau_2^\circ(a) = a^{-1}$ are continuous.

The concepts of compact and Hausdorff topological skew brace $(B, +, \circ)$ are defined by considering these properties with respect to a given common topology for $(B, +)$ and $(B, \circ)$.

Let $(B, +, \circ)$ be a topological skew brace. Then the map $\tau : B \times B \to B$, given by $\tau(a, b) = a * b$, is continuous.

Infinite Skew Braces

Move to compact Hausdorff topological skew braces.

Define

$$Z := \{(a, b) \in B^2 : \tau(a, b) = \gamma^{\circ}(a, b) = \gamma^{+}(a, b) = 1\},$$

where $\gamma^{\circ}(a, b) = [a, b]^{\circ}$ and $\gamma^{+}(a, b) = [a, b]^{+}$.

Z is a closed subset of $B \times B$, where $B \times B$ is a topological space with the product topology.

It follows by

$$Z = \tau^{-1}(1) \cap (\gamma^{\circ})^{-1}(1) \cap (\gamma^{+})^{-1}(1).$$

Moreover, the sets $Cb_B(x)$ and $\mathrm{Ann}(B)$ are also closed subsets of B.

Definition

$B := (B, +, \circ)$ a compact Hausdorff topological skew brace.

$(B, +)$ is a compact Hausdorff topological group, and therefore, there is a unique probability Haar measure space $((B, +), \mathcal{M}, \mu)$. [It turns out that $((B, \circ), \mathcal{M}, \mu)$ is a probability Haar measure space.]

The product measure $\mu \times \mu$ on the product space $B \times B$ is a probability measure.

Z, being closed in $B \times B$, $(\mu \times \mu)(Z)$ is well defined.

The *commuting probability* of the skew brace B, denoted by $\mathrm{Pb}(B)$, is defined to be

$$\mathrm{Pb}(B) := (\mu \times \mu)(Z).$$

Main Infinite Results

For a Hausdorff topological skew brace, $\mathrm{FCI}(B)$ is a Borel set.

Theorem

Let $(B, +, \circ)$ be a compact skew brace. Then the following statements are equivalent:

- $\mathrm{Pb}(B) > 0$.
- $\mathrm{FCI}(B)$ is open in B.
- $[(B, \circ) : \mathrm{FCI}(B)] < \infty$.

Gap Theorem in the Infinite Case

For non-trivial compact skew braces,

$$\mathrm{Pb}(B) \leq \frac{3}{4}.$$

Moreover,

$$\left(\frac{5}{8}, 1\right)$$

contains only one commuting probability:

$$\frac{3}{4}.$$

Like groups, 0 is attained by $\mathrm{Pb}(B)$. The example we produce is an almost trivial skew brace on a group for which commuting probability is 0.

Conclusions

- Introduced commuting probability for skew braces.
- Established sharp bounds.
- Characterized probabilities $\frac{3}{4}$ and $\frac{5}{8}$.
- Proved nilpotency criterion.
- Established isoclinism invariance.
- Extended theory to compact topological skew braces.

Extended References

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Several recent articles by P. Shumyatsky and co-authors.

Thank You