Equivalent definitions of Nichols algebras

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Definition

A *coalgebra* (over k) is a triple (C, Δ, ε) with C a vector space, $\Delta \colon C \to C \otimes C$, and $\varepsilon \colon C \to k$ satisfying that

$$
\begin{array}{ccc}
C & \xrightarrow{\;\Delta\;} & C \otimes C \\
\Delta \downarrow & & \downarrow \mathrm{id}_C \otimes \Delta \\
C & \xrightarrow[\Delta \otimes \mathrm{id}_C]{} & C \otimes C \otimes C,
\end{array}
$$

$$
\begin{array}{ccccc}
C \otimes k & \xleftarrow{\mathrm{id}_C \otimes \varepsilon} & C \otimes C & \xrightarrow{\varepsilon \otimes \mathrm{id}_C} & k \otimes C \\
 & {}_\sim\searrow & \uparrow \Delta & \swarrow_\sim & \\
 & & C. & &
\end{array}
$$

Example

If (A, μ, η) is a finite-dimensional algebra, then A^* is a coalgebra as

$$
\Delta \colon A^* \xrightarrow{\mu^*} (A \otimes A)^* \xrightarrow{\sim} A^* \otimes A^*,
$$
$$
\varepsilon \colon A^* \xrightarrow{\eta^*} k^* \xrightarrow{\sim} k.
$$

Definition

A vector space V is *graded* if it admits subspaces $V(n)$ such that

$$V = \bigoplus_{n \geq 0} V(n).$$

A *morphism of graded vector spaces* $V \to W$ is a linear map
$f \colon V \to W$ such that $f(V(n)) \subseteq W(n)$ for all $n \geq 0$.

Definition

A coalgebra (C, Δ, ε) is *graded* if C is, and Δ and ε are graded
maps. This is,

$$\Delta(C(n)) \subseteq \bigoplus_{i=0}^{n} C(i) \otimes C(n-i),$$

$$\varepsilon(C(n)) = \begin{cases} k, & \text{if } n = 0, \\ 0, & \text{else.} \end{cases}$$

Definition

A graded coalgebra C is *connected* if $C(0) \cong k$.

Proposition

A connected coalgebra admits a unique group-like element 1_C, this is, a non-zero $x \in C$ such that $\Delta(x) = x \otimes x$.
Moreover, $1_C \in C(0)$.

<u>Proof.</u>
If $x \in C(n)$ is group-like, then $\Delta(x) = x \otimes x \in C(n) \otimes C(n)$, yet $\Delta(x) \in \bigoplus_{i=0}^{n} C(i) \otimes C(n-i)$, hence $n = 0$.
Let $x \in C(0)$ be non-zero, then as $\Delta(x) \in C(0) \otimes C(0)$, there is a unique non-zero $\lambda \in k$ such that $\Delta(x) = \lambda x \otimes x$. Hence λx is the unique group-like element of C. $\qquad\square$

Definition

Let C be a connected coalgebra, the *primitive elements of C* are

$$\mathcal{P}(C) = \{x \in C \mid \Delta(x) = x \otimes 1_C + 1_C \otimes x\}.$$

Proposition

Let C be a connected coalgebra, then $C(1) \subseteq \mathcal{P}(C)$.

<u>Proof.</u> If $x \in C(1)$ then
$\Delta(x) = 1_C \otimes y + z \otimes 1_C \in C(0) \otimes C(1) + C(1) \otimes C(0)$. Thus by
applying $\mathrm{id} \otimes \varepsilon$ respectively $\varepsilon \otimes \mathrm{id}$ it follows that $z = y = x$. $\qquad\square$

Definition

A *strictly graded coalgebra* C is a connected coalgebra such that
$\mathcal{P}(C) \subseteq C(1)$, and thus $\mathcal{P}(C) = C(1)$.

If (C, Δ, ε) is a graded coalgebra, we define for $i_1 + \cdots + i_r = n$

$$\Delta_{i_1, \ldots, i_r} \colon C(n) \xrightarrow{\Delta^{r-1}} C^{\otimes r} \xrightarrow{\pi_{i_1} \otimes \cdots \otimes \pi_{i_r}} C(i_1) \otimes \cdots \otimes C(i_r),$$
$$\Delta_{1^n} = \Delta_{1,1,\ldots,1},$$
$$I_C(n) = \ker(\Delta_{1^n}) \subseteq C(n).$$

Theorem

Let C be a connected coalgebra, then $I_C = \bigoplus_{n \geq 2} I_C(n)$ is a graded coideal of C. Moreover,

- *I_C is the largest coideal of C contained in $\bigoplus_{n \geq 2} C(n)$*
- *I_C is the unique coideal I of C contained in $\bigoplus_{n \geq 2} C(n)$ such that C/I is strictly graded.*

In particular, $\mathcal{B}(C) = C/I_C$ is a strictly graded coalgebra.

<u>Proof.</u> Heckenberger, Schneider [4, Theorem 1.6.13].   □

Braided
vector spaces

Coalgebra
structure

Nichols
algebra
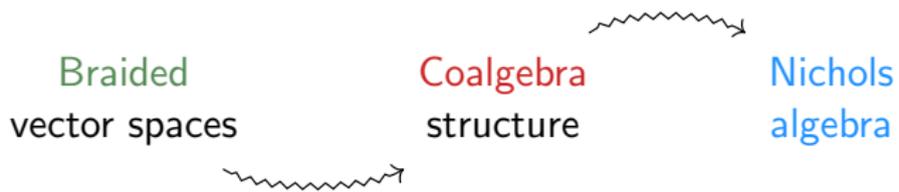
Definition

A *braided vector space* (V, c) is a vector space V, equipped with a linear isomorphism $c \in \mathrm{End}(V \otimes V)$ such that

$$(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c).$$

A *braided linear map* $f \colon (V, c) \to (W, d)$ is a linear map $f \colon V \to W$ such that

$$(f \otimes f)c = d(f \otimes f).$$

We denote the category of braided vector spaces as BrVec.

Definition

A *graded (co-/bi-/Hopf) algebra* is a graded vector space B with additional algebraic structure, such that the structure morphisms are graded.

Explicitly, for a bialgebra B

$$B(n)B(m) \subseteq B(n+m), \qquad 1_B \in B(0),$$

$$\Delta(B(n)) \subseteq \bigoplus_{i=0}^{n} B(i) \otimes B(n-i), \quad \varepsilon(B(n)) = \begin{cases} k, & \text{if } n = 0, \\ \{0\}, & \text{else.} \end{cases}$$

Definition

A *braided graded (co-/bi-/Hopf) algebra* is a braided and graded vector space (B, d) with additional algebraic structure, such that the structure morphisms are graded and commute with its braiding, and moreover

$$d(B(i) \otimes B(j)) = B(j) \otimes B(i) \text{ for all } i, j \geq 0.$$

Theorem

Let (V, c) be a braided vector space, then we may construct the free braided graded Hopf algebra $T(V, c) = (T(V), \bar{c})$, this is the graded vector space $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$ with the concatenation, and comultiplication defined on $V = T^1(V)$

$$\Delta(x) = x \otimes 1 + 1 \otimes x.$$

This satisfies the universal property that for any braided graded bialgebra (B, d) and $f \colon (V, c) \to (B, d)$ braided, there is a unique braided graded bialgebra morphism extending f;

Definition

The Nichols algebra of (V, c) is the Nichols algebra of the connected coalgebra $T(V, c)$;

$$\mathcal{B}(V, c) = \frac{T(V, c)}{I_V} = k \oplus V \bigoplus_{n \geq 2} \frac{V^{\otimes n}}{\ker(\Delta_{1^n})}.$$

$I_V = \bigoplus_{n \geq 2} \ker(\Delta_{1^n})$ is a braided and graded subspace of $T(V, c)$, and hence $T(V, c)/I_V$ a graded braided vector space.

Theorem

Let (V, c) be a braided vector space, then I_V is a braided graded Hopf ideal of $T(V, c)$. In particular, $\mathcal{B}(V, c)$ is a strictly graded braided Hopf algebra.

<u>Proof.</u> One verifies that $T(V)I_V$ and $I_V T(V)$ are coideals of $T(V)$, whence I_V is a braided graded biideal. Likewise, $S(I_V)$ is a coideal of $T(V)$, proving the claim. $\qquad\square$

Braided symmetrizers

> ## Definition
>
> Let (V, c) be a braided vector space, the *braided symmetrizers* are defined as $(n \geq 2)$
>
> $$\Omega^2 = \mathrm{id} + c,$$
> $$\Omega^{n+1} = (\mathrm{id} \otimes \Omega^n)(\mathrm{id} + c_1 + c_1 c_2 + \cdots + c_1 c_2 \cdots c_n).$$

Theorem

Let (V, c) be a braided vector space, then for $n \geq 2$

$$\Omega^n = \Delta_{1^n}.$$

In particular,
$\mathcal{B}(V, c) = T(V, c) / \ker(\Omega) = k \oplus V \bigoplus_{n \geq 2} V^{\otimes n} / \ker(\Omega^n).$

Sketch of proof. We utilize that

▶ $\Delta_{1^n} = (\Delta_{1^i} \otimes \Delta_{1^{n-i}}) \Delta_{i, n-i}$,

▶ for $x_1 \otimes \cdots \otimes x_n \in V^{\otimes n}$,

$$\Delta(x_1 \otimes \cdots \otimes x_n) = \prod_{i=1}^{n} (x_i \otimes 1 + 1 \otimes x_i).$$

The result follows by induction. $\qquad\qquad\square$

Approximations

Definition

Let (V, c) be a braided vector space and (B, d) a braided graded Hopf algebra. Then (B, d) is a *pre-Nichols algebra of* (V, c) if

1. $B(0) = k$,
2. $B(1) \cong V$ as braided vector space,
3. $B(1)$ generates the algebra B.

Proposition

(B, d) *is a pre-Nichols algebra if and only if there exists a braided graded Hopf ideal I of $T(V, c)$ contained in $\bigoplus_{n \geq 2} T^n(V)$ such that $(B, d) \cong T(V, c)/I$.*

Definition

Let (V, c) be a braided vector space and (B, d) a braided graded Hopf algebra. Then (B, d) is a *post-Nichols algebra* of (V, c) if

1. $B(0) = k$,
2. $B(1) \cong V$ as braided vector space,
3. B is strictly graded, this is $\mathcal{P}(B) = B(1)$.

Theorem

$(B, d) \cong \mathcal{B}(V, c)$ *if and only if* (B, d) *is a pre-Nichols and post-Nichols algebra of* (V, c).

<u>Proof.</u> $\mathcal{B}(V, c)$ is pre- and post-Nichols. If (B, d) is pre-Nichols then $(B, d) \cong T(V, c)/I$ for some braided graded Hopf ideal I contained in $\bigoplus_{n \geq 2} T^n(V)$. Since this is moreover strictly graded as it is post-Nichols, $I = I_V$ and $(B, d) \cong \mathcal{B}(V, c)$. □

Bilinear forms

Definition

Let V, W be k-vector spaces, and $\langle \, , \, \rangle \colon V \times W \to k$ a bilinear form. The *extended form of* $\langle \, , \, \rangle$ is the unique linear map

$$(\, , \,) \colon T(V) \otimes T(W) \to k$$

satisfying that

1. $(1, 1) = 1$,

2. $(T^n(V), T^m(W)) = 0$ for all $n \neq m$,

3. For all $v_1 \otimes \cdots \otimes v_n \in T^n(V)$, $w_1 \otimes \cdots \otimes w_n \in T^n(W)$

$$(v_1 \otimes \cdots \otimes v_n, w_1 \otimes \cdots \otimes w_n) = \prod_{i=1}^{n} \langle v_{n-i+1}, w_i \rangle.$$

Lemma (A)

i) If $\langle\,,\,\rangle$ is non-degenerate, then $(\,,\,)$ is non-degenerate.

ii) If $\langle\,,\,\rangle$ is non-degenerate, V and W are finite-dimensional, and (V, c) is a braided vector space, so too is (W, d) for $d\colon W \otimes W \to W \otimes W$ defined by

$$(v, d(w)) = (c(v), w)$$

for $v \in V \otimes V, w \in W \otimes W$.

Theorem (B)

1. If $(c(v), w) = (v, d(w))$, there is a unique well-defined linear map $[\ ,\ ]\colon T(V) \otimes T(W) \to k$ such that
   - $[1, 1] = 1$,
   - $[T^n(V), T^m(W)] = 0$ for $n \neq m$,
   - For $v \in T^n(V)$, $w \in T^n(W)$,

   $$[v, w] = (\Omega^n(v), w) = (v, \Omega^n(w))\,.$$

2. If $\langle\ ,\ \rangle$ is non-degenerate and $(c(v), w) = (v, d(w))$, then

   $$\mathcal{B}(V, c) = T(V)/\ker([\bullet, T(W)]),$$
   $$\mathcal{B}(W, d) = T(W)/\ker([T(V), \bullet]).$$

<u>Proof.</u> Heckenberger, Schneider [4, Theorem 7.2.3]. $\square$

Proposition

Let (V, c) be a finite-dimensional braided vector space, then

1. the evaluation $\langle \, | \, \rangle \colon V^* \otimes V \to k$ is non-degenerate,
2. (V^*, c^*) is a braided vector space,

Thus

$$\mathcal{B}(V^*, c^*) = T(V^*)/\ker([\bullet, T(V)]),$$
$$\mathcal{B}(V, c) = T(V)/\ker([T(V^*), \bullet]).$$

<u>Proof.</u> The evaluation $\langle \, | \, \rangle$ is indeed non-degenerate, and hence (V^*, c^*) is a braided vector space by Lemma (A). By construction the conditions of Theorem (B) are satisfied. $\square$

Skew derivations

Definition

Let $C = \bigoplus_{n \geq 0} C(n)$ be a graded coalgebra. For any linear functional $f \in C(1)^*$ we define the linear maps

$$\partial_f^l \colon C \to C \colon x \mapsto (f \otimes \mathrm{id})(\pi_1 \otimes \mathrm{id})\Delta,$$
$$\partial_f^r \colon C \to C \colon x \mapsto (\mathrm{id} \otimes f)(\mathrm{id} \otimes \pi_1)\Delta.$$

Definition

Let C be a graded coalgebra. A linear subspace $I \subseteq C$ is said to be ∂^l-*invariant* (respectively ∂^r-*invariant*) if $\partial_f^l(I) \subseteq I$ (respectively $\partial_f^r(I) \subseteq I$) for all $f \in C(1)^*$.

Given a graded coalgebra C, denote $C^+ = \bigoplus_{n \geq 1} C(n)$.

Proposition

Let C be a connected graded coalgebra.

1. *If C is strictly graded and $x \in C^+$, then $x = 0$ if and only if $\partial^l_f(x) = 0$ for all $f \in C(1)^*$.*

2. *I_C is the largest ∂^l-invariant subspace of C^+.*

<u>Proof.</u> Heckenberger, Schneider [4, Proposition 7.3.3]. □

Corollary

Let C be a connected coalgebra, and $x \in C^+$. Then $\pi(x) \in \mathcal{B}(C)$ is non-zero if and only if there exists a linear functional $f \in C(1)^*$ such that $\partial^l_f(x) \notin I_C$.

Nichols algebra functor

Definition

The category of *strictly graded braided Hopf algebras* consists of

- objects $(H, c, \mu, \eta, \Delta, \varepsilon, S)$ being strictly graded braided Hopf algebras,

- morphisms being braided and graded Hopf algebra morphisms.

We denote this category by $\mathfrak{C}$.

Theorem

Let (V,c) and (W,d) be braided vector spaces and $f\colon (V,c) \to (W,d)$ a braided linear map, then there is a unique morphism of strictly graded braided Hopf algebras

$$\mathcal{B}(f)\colon \mathcal{B}(V,c) \to \mathcal{B}(W,d)$$

such that the following diagram commutes;

Proposition

Let $(V, c), (W, d)$, and (R, e) be braided vector spaces, then

1. $\mathcal{B}(\mathrm{id}_{(V,c)}) = \mathrm{id}_{\mathcal{B}(V,c)}$,
2. *for all braided linear maps $f \colon (V, c) \to (W, d)$ and $g \colon (W, d) \to (R, e)$, $\mathcal{B}(g \circ f) = \mathcal{B}(g) \circ \mathcal{B}(f)$.*

Corollary

The Nichols algebra functor

$$\mathcal{B} \colon \mathrm{BrVec} \to \mathfrak{C}$$

is a well-defined functor.

For any strictly graded braided Hopf algebra (H, d), we denote

$$\mathfrak{R}(H, d) = (H(1), \tilde{d}), \quad \text{for } \tilde{d} = d|_{H(1) \otimes H(1)}.$$

If $f \colon (H, d) \to (Q, t)$ is a morphism of such, then we denote

$$\mathfrak{R}(f) = f|_{H(1)} \colon \mathfrak{R}(H, d) \to \mathfrak{R}(Q, t).$$

Lemma

Let $(H, d), (Q, t)$, and (R, s) be strictly graded braided Hopf algebras, then

1. $\mathfrak{R}(\mathrm{id}_{(H,d)}) = \mathrm{id}_{\mathfrak{R}(H,d)}$,

2. *for any morphisms of strictly graded braided Hopf algebras $f \colon (H, d) \to (Q, t)$ and $g \colon (Q, t) \to (R, s)$ it holds that*

$$\mathfrak{R}(g \circ f) = \mathfrak{R}(g) \circ \mathfrak{R}(f).$$

Lemma

Let (V, c) be a braided vector space and (H, d) a strictly graded braided Hopf algebra. Then for any braided linear map $f \colon (V, c) \to (H(1), \tilde{d})$ there exists a unique morphism of strictly graded braided Hopf algebras

$$\tilde{f} \colon \mathcal{B}(V, c) \to (H, d)$$

extending f, this is such that $i_H f = \tilde{f} p_V i_V$. In other words, such that the following diagram commutes;

$$
\begin{array}{ccc}
(V, c) & \xrightarrow{ f } & (H(1), \tilde{d}) \\
{\scriptstyle i_V} \downarrow & & \uparrow \\
(T(V), \bar{c}) & & \Big\downarrow {\scriptstyle i_H} \\
{\scriptstyle p_V} \downarrow & & \\
\mathcal{B}(V, c) & \xrightarrow[\tilde{f}]{} & (H, d).
\end{array}
$$

Theorem

Let (V, c) be a braided vector space, (H, d) a strictly graded braided Hopf algebra, and define

$$\varphi_{(V,c),(H,d)} \colon \operatorname{Hom}_{\mathrm{BrVec}}((V,c), \mathfrak{R}(H,d)) \to \operatorname{Hom}_{\mathfrak{C}}(\mathcal{B}(V,c), (H,d))$$
$$: f \mapsto \tilde{f}.$$

Then

1. $\varphi_{(V,c),(H,d)}$ is a bijection of sets,

2. $\varphi_{(V,c),-} \colon \operatorname{Hom}_{\mathrm{BrVec}}((V,c), \mathfrak{R}(\bullet)) \to \operatorname{Hom}_{\mathfrak{C}}(\mathcal{B}(V,c), \bullet)$ is a natural transformation,

3. $\varphi_{-,(H,d)} \operatorname{Hom}_{\mathrm{BrVec}}(\bullet, \mathfrak{R}(H,d)) \to \operatorname{Hom}_{\mathfrak{C}}(\mathcal{B}(\bullet), (H,d))$ is a natural transformation.

In conclusion, $\mathcal{B} \dashv \mathfrak{R}$ form a pair of adjoint functors.

The category BrVec admits coproducts, namely

$$(V, c) \coprod (W, d) = (V \oplus W, c_{V \oplus W}).$$

Corollary

Let (V, c) be a braided vector space, if U, W are braided subspaces of V such that $V = U \oplus W$, then

$$\mathcal{B}(V, c) \cong \mathcal{B}(U, c|_{U \otimes U}) \coprod \mathcal{B}(W, c|_{W \otimes W}).$$

<u>Problem.</u> Describe the coproduct in $\mathfrak{C}$.

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