# Quantum Yang–Baxter equation, Yetter–Drinfeld braces, and semi-abelian categories

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# Quantum Yang–Baxter equation

• It first appeared in [C.N. Yang, *Some exact results for the many-body problem in one dimension with repulsive delta-function interaction*, 1967] and, independently, in [R.J. Baxter, *Partition function of the eight-vertex lattice model*, 1972].

Given a vector space $V$, a morphism $c : V \otimes V \to V \otimes V$ is a solution of the *quantum Yang–Baxter equation* if the following equality holds:

$$(c \otimes \mathrm{Id}_V)c_{13}(\mathrm{Id}_V \otimes c) = (\mathrm{Id}_V \otimes c)c_{13}(c \otimes \mathrm{Id}_V),$$

where $c_{13} = (\mathrm{Id}_V \otimes \tau_{V,V})(c \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes \tau_{V,V})$ and $\tau_{V,V}$ is the canonical flip.

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The morphism $c$ satisfies the quantum Yang–Baxter equation if and only if $\tau_{V,V}c$ satisfies the *braid equation*:

$$(c \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes c)(c \otimes \mathrm{Id}_V) = (\mathrm{Id}_V \otimes c)(c \otimes \mathrm{Id}_V)(\mathrm{Id}_V \otimes c).$$

A pair $(V, c)$ where $c : V \otimes V \to V \otimes V$ satisfies the braid equation is called a *braided vector space*.

# Braided monoidal categories

The notion of *monoidal category* $(\mathcal{M}, \otimes, \mathbf{1})$ appeared in [J. Bénabou, *Catégories avec multiplication*, 1963]. One can describe monoids (algebras) $\mathsf{Mon}(\mathcal{M})$ in $\mathcal{M}$ and comonoids (coalgebras) $\mathsf{Comon}(\mathcal{M})$ in $\mathcal{M}$.

## Definition (A. Joyal, R. Street, 1986)

A monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$ is *braided* if, for all $X, Y$ in $\mathcal{M}$, there is an isomorphism $\sigma_{X,Y} : X \otimes Y \to Y \otimes X$ (*braiding*), natural in $X$ and $Y$, such that:

$$
\begin{array}{ccc}
A \otimes B \otimes C & \xrightarrow{\sigma_{A,B} \otimes \mathrm{Id}_C} & B \otimes A \otimes C \\
\Big\downarrow{\scriptstyle \sigma_{A,B \otimes C}} & & \swarrow{\scriptstyle \mathrm{Id}_B \otimes \sigma_{A,C}} \\
B \otimes C \otimes A & &
\end{array}
\qquad
\begin{array}{ccc}
A \otimes B \otimes C & \xrightarrow{\mathrm{Id}_A \otimes \sigma_{B,C}} & A \otimes C \otimes B \\
\Big\downarrow{\scriptstyle \sigma_{A \otimes B,C}} & & \swarrow{\scriptstyle \sigma_{A,C} \otimes \mathrm{Id}_B} \\
C \otimes A \otimes B & &
\end{array}
$$

If $\sigma_{X,Y}^{-1} = \sigma_{Y,X}$ for all $X, Y$ in $\mathcal{M}$, then it is said *symmetric monoidal*.
We denote a braided monoidal category by $(\mathcal{M}, \otimes, \mathbf{1}, \sigma)$.

## Examples (Symmetric monoidal categories)

1. $(\mathsf{Set}, \times, *, \tau)$ where $\times$ is the cartesian product, $\{*\}$ is a singleton and $\tau_{X,Y} : (x, y) \mapsto (y, x)$ for any $X, Y$ in $\mathsf{Set}$.

2. $(\mathsf{Vec}_{\Bbbk}, \otimes_{\Bbbk}, \Bbbk, \tau)$ where $\otimes_{\Bbbk}$ is the $\Bbbk$-tensor product and $\tau_{X,Y} : x \otimes y \mapsto y \otimes x$, for any $X, Y$ in $\mathsf{Vec}_{\Bbbk}$.

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- Given $(\mathcal{M}, \otimes, \mathbf{1}, \sigma)$ and $M, N, O$ in $\mathcal{M}$, the *braid equation* holds:

$$(\sigma_{N,O} \otimes \mathrm{Id}_M)(\mathrm{Id}_N \otimes \sigma_{M,O})(\sigma_{M,N} \otimes \mathrm{Id}_O) = (\mathrm{Id}_O \otimes \sigma_{M,N})(\sigma_{M,O} \otimes \mathrm{Id}_N)(\mathrm{Id}_M \otimes \sigma_{N,O})$$

If there is a forgetful functor $U : \mathcal{M} \to \mathsf{Vec}_{\Bbbk}$, $(M, \sigma_{M,M})$ is a braided vector space for any $M \in \mathcal{M}$.

# Bimonoids and Hopf monoids

Given $(\mathcal{M}, \otimes, \mathbf{1}, \sigma)$, one can define the category $\mathsf{Bimon}(\mathcal{M})$ of bimonoids in $\mathcal{M}$ and its full subcategory $\mathsf{Hopf}(\mathcal{M})$ of Hopf monoids in $\mathcal{M}$.

## Definition

Let $(\mathcal{M}, \otimes, \mathbf{1}, \sigma)$ be a braided monoidal category. A *bimonoid* in $\mathcal{M}$ is a datum $(B, m, u, \Delta, \varepsilon)$ where:

1) $(B, m : B \otimes B \to B, u : \mathbf{1} \to B)$ is in $\mathsf{Mon}(\mathcal{M})$,
2) $(B, \Delta : B \to B \otimes B, \varepsilon : B \to \mathbf{1})$ is in $\mathsf{Comon}(\mathcal{M})$,
3) $\Delta$, $\varepsilon$ are in $\mathsf{Mon}(\mathcal{M})$ (equivalently, $m$ and $u$ are in $\mathsf{Comon}(\mathcal{M})$), i.e.,

$$\Delta m = (m \otimes m)(\mathrm{Id} \otimes \sigma_{B,B} \otimes \mathrm{Id})(\Delta \otimes \Delta), \ \Delta u = u \otimes u, \ \varepsilon m = \varepsilon \otimes \varepsilon, \ \varepsilon u = \mathrm{Id}_{\mathbf{1}}.$$

A morphism of bimonoids is just a morphism of monoids and comonoids.

Given $H$ in $\mathsf{Bimon}(\mathcal{M})$, the set $\mathrm{Hom}(H, H)$ is a monoid with the *convolution product*: given $f, g : H \to H$ in $\mathcal{M}$, set $f * g := m_H(f \otimes g)\Delta_H$. The unit is $u_H\varepsilon_H$.

## Definition

A *Hopf monoid* $H$ in $\mathcal{M}$ is a bimonoid in $\mathcal{M}$ with a morphism $S : H \to H$ (called *antipode*) which is a convolution inverse of $\mathrm{Id}_H$.

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**Examples**:

- $\mathsf{Hopf}(\mathsf{Set}) = \mathsf{Grp}$,
- $\mathsf{Hopf}(\mathsf{Vec}_{\Bbbk}) = \mathsf{Hopf}_{\Bbbk}$ (the category of $\Bbbk$-Hopf algebras).

# Yetter–Drinfeld modules

Let $H$ be a Hopf algebra. The category ${}^H_H\mathcal{YD}$ of (left) Yetter–Drinfeld modules is defined in the following way:

1) Objects are data $(V, \rightharpoonup, \rho)$, where $(V, \rightharpoonup : H \otimes V \to V)$ is a left $H$-module, $(V, \rho : V \to H \otimes V, v \mapsto v_{-1} \otimes v_0)$ is a left $H$-comodule and the following compatibility condition holds true:

$$(h \rightharpoonup v)_{-1} \otimes (h \rightharpoonup v)_0 = h_1 v_{-1} S(h_3) \otimes (h_2 \rightharpoonup v_0).$$

2) Morphisms are left $H$-linear maps which are also left $H$-colinear.

- $({}^H_H\mathcal{YD}, \otimes, \Bbbk)$ is a monoidal category. Given $X, Y$ in ${}^H_H\mathcal{YD}$, $X \otimes Y$ is in ${}^H_H\mathcal{YD}$ with action and coaction

  $$h \rightharpoonup_{\otimes} (x \otimes y) := (h_1 \rightharpoonup_X x) \otimes (h_2 \rightharpoonup_Y y), \quad \rho_{\otimes}(x \otimes y) := x_{-1}y_{-1} \otimes x_0 \otimes y_0$$

  and $\Bbbk$ is in ${}^H_H\mathcal{YD}$ with $h \rightharpoonup k := \varepsilon(h)k$ and $\rho(k) = 1_H \otimes k$.

- $({}^H_H\mathcal{YD}, \otimes, \Bbbk, \sigma)$ is a (pre-)braided monoidal category with

  $$\sigma_{X,Y} : X \otimes Y \to Y \otimes X, \; x \otimes y \mapsto (x_{-1} \rightharpoonup_Y y) \otimes x_0$$

  If $H$ has bijective antipode, $\sigma_{X,Y}$ is bijective.

# Yetter–Drinfeld braces and matched pairs of actions

Solutions of the QYBE are provided via cocommutative Hopf braces [I. Angiono, C. Galindo, L. Vendramin, *Hopf braces and Yang–Baxter operators*, 2017].

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### Definition (D. Ferri, A.S.)

A **Yetter–Drinfeld brace** $(H, \cdot, \bullet, 1, \Delta, \varepsilon, S, T)$ is the datum of

1) A Hopf algebra $H^{\bullet} = (H, \bullet, 1, \Delta, \varepsilon, T)$,
2) An object $H^{\cdot} = (H, \cdot, 1, \Delta, \varepsilon, S)$ in $\mathsf{Hopf}({}^{H^\bullet}_{H^\bullet}\mathcal{YD})$ with action and coaction

$$a \rightharpoonup b := S(a_1) \cdot (a_2 \bullet b), \qquad \mathrm{Ad}_L : a \mapsto a_1 \bullet T(a_3) \otimes a_2$$

such that the following equalities are satisfied:

$$a \bullet (b \cdot c) = (a_1 \bullet b) \cdot S(a_2) \cdot (a_3 \bullet c) \quad (1)$$
$$(a_1 \rightharpoonup b_1) \otimes T(a_2 \rightharpoonup b_2) \bullet a_3 \bullet b_3 = (a_3 \rightharpoonup b_3) \otimes T(a_1 \rightharpoonup b_1) \bullet a_2 \bullet b_2.$$

A morphism of Yetter–Drinfeld braces $f : (H, \cdot, \bullet) \to (K, \cdot, \bullet)$ is a morphism of coalgebras and algebras with respect to $\cdot$ and $\bullet$. We denote this category by $\mathcal{YD}\mathsf{Br}$.

- $\mathcal{YD}\mathsf{Br}_{\mathrm{coc}} = \mathsf{HBr}_{\mathsf{coc}}$: $H^{\bullet}$ and $H^{\cdot}$ are Hopf algebras such that (1) holds.

*Question*: Do Yetter–Drinfeld braces produce solutions of the quantum Yang–Baxter equation?

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## Definition

Let $(H, \bullet, 1, \Delta, \varepsilon, T)$ be a Hopf algebra. A **matched pair of actions** $(H, \rightharpoonup, \leftharpoonup)$ is given by a pair of actions $\rightharpoonup, \leftharpoonup : H \otimes H \to H$ which are coalgebra morphisms s.t.

$$a \bullet b = (a_1 \rightharpoonup b_1) \bullet (a_2 \leftharpoonup b_2). \tag{2}$$

A morphism of matched pairs of actions $f : (H, \rightharpoonup_H, \leftharpoonup_H) \to (K, \rightharpoonup_K, \leftharpoonup_K)$ is a morphism of Hopf algebras $f : H \to K$ such that:

$$f(a \rightharpoonup_H b) = f(a) \rightharpoonup_K f(b), \quad f(a \leftharpoonup_H b) = f(a) \leftharpoonup_K f(b).$$

The category of matched pairs of actions will be denoted by $\mathsf{MP}$.

Any matched pair of actions $(H, \rightharpoonup, \leftharpoonup)$ satisfies

1) $a \rightharpoonup (b \bullet c) = (a_1 \rightharpoonup b_1) \bullet ((a_2 \leftharpoonup b_2) \rightharpoonup c)$,
2) $(a \bullet b) \leftharpoonup c = (a \leftharpoonup (b_1 \rightharpoonup c_1)) \bullet (b_2 \leftharpoonup c_2)$,
3) $(a_1 \rightharpoonup b_1) \otimes (a_2 \leftharpoonup b_2) = (a_2 \rightharpoonup b_2) \otimes (a_1 \leftharpoonup b_1)$,

so it is a matched pair of Hopf algebras $(H, H)$ in the sense of [S. Majid, 1990].

More precisely, $H \otimes H$ is a Hopf algebra with the tensor coalgebra structure and algebra structure defined by

$$(a \otimes h)(b \otimes g) = a \bullet (h_1 \rightharpoonup b_1) \otimes (h_2 \leftharpoonup b_2) \bullet g, \qquad 1 = 1_H \otimes 1_H.$$

It is called *double cross product* and is denoted by $H \bowtie H$.

Matched pairs of actions are such that $m_\bullet : H \bowtie H \to H$ is a morphism of algebras.

## Theorem (D. Ferri, A.S.)

*The following functors provide an isomorphism of categories:*

- $F : \mathsf{MP} \to \mathcal{YD}\mathsf{Br}$, $((H, \bullet, 1, \Delta, \varepsilon, T), \rightharpoonup, \leftharpoonup) \mapsto (H, \cdot, \bullet, 1, \Delta, \varepsilon, S, T)$ *where*

$$a \cdot b := a_1 \bullet (T(a_2) \rightharpoonup b), \qquad S(a) := a_1 \rightharpoonup T(a_2).$$

- $G : \mathcal{YD}\mathsf{Br} \to \mathsf{MP}$, $(H, \cdot, \bullet, 1, \Delta, \varepsilon, S, T) \to ((H, \bullet, 1, \Delta, \varepsilon, T), \rightharpoonup, \leftharpoonup)$ *where*

$$a \rightharpoonup b := S(a_1) \cdot (a_2 \bullet b), \qquad a \leftharpoonup b := T(a_1 \rightharpoonup b_1) \bullet a_2 \bullet b_2.$$

Under the assumption of cocommutativity, we recover the result in [I. Angiono, C. Galindo, L. Vendramin, 2017].

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• Yetter–Drinfeld braces (equivalently, matched pairs of actions) produce solutions to the braid equation $c : x \otimes y \mapsto (x_1 \rightharpoonup y_1) \otimes (x_2 \leftharpoonup y_2)$ [J.A. Guccione, J.J. Guccione, C. Valqui, 2024].

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• $c$ is involutive if and only if $\cdot$ is braided commutative [Y. Li, *Matched pairs and Yang–Baxter operators*, 2025].

# Coquasitriangular Hopf algebras and Yetter–Drinfeld braces

Let $(H, \mathcal{R} : H \otimes H \to \Bbbk)$ be a coquasitriangular Hopf algebra. This corresponds to a braiding on $\mathfrak{M}^H$.

Given the braided vector space $(H, \sigma_{H,H})$, where

$$\sigma_{H,H} : H \otimes H \to H \otimes H, \ a \otimes b \mapsto \mathcal{R}^{-1}(a_1 \otimes b_1) b_2 \otimes a_2 \mathcal{R}(a_3 \otimes b_3)$$

one obtains a matched pair of actions $(\rightharpoonup, \leftharpoonup)$ on $H$ in the following way:

$$a \rightharpoonup b := (\mathrm{Id} \otimes \varepsilon)\sigma_{H,H}(a \otimes b) = \mathcal{R}^{-1}(a_1 \otimes b_1) b_2 \mathcal{R}(a_2 \otimes b_3),$$

$$a \leftharpoonup b := (\varepsilon \otimes \mathrm{Id})\sigma_{H,H}(a \otimes b) = \mathcal{R}^{-1}(a_1 \otimes b_1) a_2 \mathcal{R}(a_3 \otimes b_2),$$

so that $\sigma_{H,H}(a \otimes b) = (a_1 \rightharpoonup b_1) \otimes (a_2 \leftharpoonup b_2)$.

### Theorem (D. Ferri, A.S.)

*Let* $(H, \bullet, 1, \Delta, \varepsilon, T, \mathcal{R})$ *be a coquasitriangular Hopf algebra. Then,* $(H, \cdot, \bullet, 1, \Delta, \varepsilon, S, T)$ *is a Yetter–Drinfeld brace where:*

$$a \cdot b = \mathcal{R}^{-1}(a_1 \bullet T(a_3) \otimes b_1) b_2 \bullet a_2, \qquad S(a) = T(a_3)\mathcal{R}(a_1 \otimes T(a_2) \bullet a_4)$$

*and the action is given by* $a \rightharpoonup b := b_2 \mathcal{R}^{-1}(a \otimes b_1 \bullet T(b_3))$.

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• The object $(H, \cdot, 1, \Delta, \varepsilon, S)$ in $\mathsf{Hopf}({}^{H^\bullet}_{H^\bullet}\mathcal{YD})$ coincides with the **transmutation** of the Hopf algebra $(H, \bullet, 1, \Delta, \varepsilon, T)$ in the sense of [S. Majid, *Transmutation theory and rank for quantum braided groups*, 1993].

# The Yetter–Drinfeld brace $(E(n), \cdot, \bullet)$

The Hopf algebra $(E(n), \bullet, 1, \Delta, \varepsilon, T)$ is generated as an algebra by $g, x_i$ for $i = 1, \dots, n$, with relations

$$g^2 = 1, \;\; x_i^2 = 0, \;\; x_i g = -g x_i, \;\; x_i x_j = -x_j x_i, \quad \text{for all } i = 1, \dots, n,$$

and $\Delta(g) = g \otimes g$, $\Delta(x_i) = x_i \otimes 1 + g \otimes x_i$, $T(g) = g$, $T(x_i) = x_i g$.

The object $(E(n), \cdot, 1, \Delta, \varepsilon, S)$ in $\mathsf{Hopf}({}^{E(n)^\bullet}_{E(n)^\bullet}\mathcal{YD})$ is generated as an algebra by $g, x_i$ with relations

$$g \cdot g = 1, \quad x_i \cdot x_j = -x_j \cdot x_i + 2A_{ij}(1-g), \quad x_i \cdot g = g \cdot x_i$$

and $S$ is defined as $S(g) = g$, $S(x_i) = -x_i \cdot g$, while the action $\rightharpoonup$ as

| $\rightharpoonup$ | $1$ | $g$ | $x_j$ | $x_j g$ |
|---|---|---|---|---|
| $1$ | $1$ | $g$ | $x_j$ | $x_j g$ |
| $g$ | $1$ | $g$ | $-x_j$ | $-x_j g$ |
| $x_i$ | $0$ | $0$ | $A_{ij}(1-g)$ | $A_{ij}(g-1)$ |
| $x_i g$ | $0$ | $0$ | $A_{ij}(g-1)$ | $A_{ij}(1-g)$ |

# The Yetter–Drinfeld brace $(\mathrm{SL}_q(2), \cdot, \bullet)$

Let $q \in \mathbb{C}^\times$. The $\mathbb{C}$-Hopf algebra $(\mathrm{SL}_q(2), \bullet, 1, \Delta, \varepsilon, T)$ is generated as an algebra by $a, b, c, d$, modulo the following relations:

$$\begin{aligned} ba &= qab, & ca &= qac, & db &= qbd, \\ dc &= qcd, & bc &= cb, & ad - da &= (q^{-1} - q)bc, \\ & & da - qbc &= 1. & & \end{aligned}$$

It is a Hopf algebra with

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ \varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}.$$

The object $(\mathrm{SL}_q(2), \cdot, 1, \Delta, \varepsilon, S)$ in $\mathsf{Hopf}({}^{\mathrm{SL}_q(2)^\bullet}_{\mathrm{SL}_q(2)^\bullet}\mathcal{YD})$ is generated as an algebra by $a, b, c, d$ modulo the relations

$$\begin{aligned} a \cdot c &= q^{-2} c \cdot a, & b \cdot d - d \cdot b &= (q^2 - 1) a \cdot b, \\ a \cdot b &= q^2 b \cdot a, & b \cdot c - c \cdot b &= (q^2 - 1)(a \cdot a - a \cdot d), \\ a \cdot d &= d \cdot a, & c \cdot d - d \cdot c &= (1 - q^2) c \cdot a \\ & & ad - q^{-2} cb &= 1, \end{aligned}$$

while $S$ is defined by

$$\begin{aligned} S(a) &= q^{-2} d + (1 - q^{-2}) a, & S(b) &= -q^{-2} b, \\ S(c) &= -q^{-2} c, & S(d) &= a \end{aligned}$$

and the action $\rightharpoonup$ by

| $\rightharpoonup$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $a$ | $q^{-1} b$ | $qc$ | $d$ |
| $b$ | $(1 - q^{-2}) b$ | $0$ | $(q - q^{-1})(d - a)$ | $(1 - q^2) b$ |
| $c$ | $0$ | $0$ | $0$ | $0$ |
| $d$ | $a$ | $qb$ | $q^{-1} c$ | $d$ |

# Semi-abelian categories

## Definition (G. Janelidze, L. Márki, W. Tholen, 2002)

A category $\mathcal{C}$ is *semi-abelian* when:

1) it is *pointed*, i.e. it has zero object (initial and terminal object),
2) it is *(Barr)-exact*:
   - it is *regular*: it has finite limits and any morphism factorizes as a regular epimorphism (coequalizer) followed by a monomorphism, and regular epimorphisms are stable under pullbacks,
   - any equivalence relation in $\mathcal{C}$ is a kernel pair,
3) it is *(Bourn)-protomodular*: the Split Short Five Lemma holds in $\mathcal{C}$,
4) it admits binary coproducts.

**Examples**: groups, rings (not necessarily unitary), Lie algebras, commutative $C^*$-algebras.

## Cocommutative Hopf braces

We consider the category $\mathsf{HBr}_{\mathrm{coc}}$ of cocommutative Hopf braces ($= \mathcal{YD}\mathsf{Br}_{\mathrm{coc}}$).

**Definition (I. Angiono, C. Galindo, L. Vendramin, 2017)**

A *cocommutative Hopf brace* is a datum $(H, \cdot, \bullet, 1, \Delta, \varepsilon, S, T)$ where:

- $H^{\cdot} := (H, \cdot, 1, \Delta, \varepsilon, S)$ and $H^{\bullet} := (H, \bullet, 1, \Delta, \varepsilon, T)$ are cocommutative Hopf algebras;
- the following compatibility condition is satisfied:

$$a \bullet (b \cdot c) = (a_1 \bullet b) \cdot S(a_2) \cdot (a_3 \bullet c).$$

A morphism of (cocommutative) Hopf braces $f : (H, \cdot, \bullet) \to (K, \cdot, \bullet)$ is a Hopf algebra map with respect to both the two Hopf algebra structures.

## Cocommutative Hopf braces

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**Definition (I. Angiono, C. Galindo, L. Vendramin, 2017)**

A *cocommutative Hopf brace* is a datum $(H, \cdot, \bullet, 1, \Delta, \varepsilon, S, T)$ where:

- $H^{\cdot} := (H, \cdot, 1, \Delta, \varepsilon, S)$ and $H^{\bullet} := (H, \bullet, 1, \Delta, \varepsilon, T)$ are cocommutative Hopf algebras;
- the following compatibility condition is satisfied:

$$a \bullet (b \cdot c) = (a_1 \bullet b) \cdot S(a_2) \cdot (a_3 \bullet c).$$

A morphism of (cocommutative) Hopf braces $f : (H, \cdot, \bullet) \to (K, \cdot, \bullet)$ is a Hopf algebra map with respect to both the two Hopf algebra structures.

• The category $\mathsf{SKB}$ of skew braces [L. Guarnieri, L. Vendramin, *Skew braces and the Yang–Baxter equation*, 2017] is semi-abelian [D. Bourn, A. Facchini, M. Pompili, *Aspects of the category SKB of skew braces*, 2023].

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*Question*: what can we say about $\mathsf{HBr}_{\mathrm{coc}}$?

## Categorical properties of $\mathsf{HBr}_{\mathrm{coc}}$

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- The category $\mathsf{HBr}_{\mathrm{coc}}$ is complete [A.L. Agore, *Constructing Hopf braces*, 2019]. The forgetful functor $F : \mathsf{HBr}_{\mathrm{coc}} \to \mathsf{Hopf}_{\Bbbk,\mathrm{coc}}$ preserves limits. In particular, $\times = \otimes$, so $\mathsf{HBr}_{\mathrm{coc}}$ is cartesian.

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- Since $F$ also reflects isomorphisms, the category $\mathsf{HBr}_{\mathrm{coc}}$ is protomodular as so is $\mathsf{Hopf}_{\Bbbk,\mathrm{coc}}$, i.e. the Split Short Five Lemma holds ($f$ and $g$ iso imply $\phi$ iso):

$$\begin{array}{ccccc}
\mathsf{Ker}(\pi) & \xrightarrow{k} & A & \underset{\gamma}{\overset{\pi}{\rightleftarrows}} & H \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle g} \\
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- The category $\mathsf{HBr}_{\mathrm{coc}}$ has binary coproducts [A.L. Agore, A. Chirvăsitu, *On the category of Hopf braces*, 2025].

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We are interested in the regularity of $\mathsf{HBr}_{\text{coc}}$.

## Kernels in $\mathsf{HBr}_{\mathrm{coc}}$

### Definition

A Hopf sub-brace $(B, \cdot, \bullet)$ of a Hopf brace $(A, \cdot, \bullet)$ is **normal** if:

1) $a_1 \cdot b \cdot S(a_2) \in B$, i.e. $B^{\cdot}$ is a normal Hopf subalgebra of $A^{\cdot}$,
2) $a_1 \bullet b \bullet T(a_2) \in B$, i.e. $B^{\bullet}$ is a normal Hopf subalgebra of $A^{\bullet}$,
3) $A \rightharpoonup B \subseteq B$, where $a \rightharpoonup b := S(a_1) \cdot (a_2 \bullet b)$.

### Proposition (M. Gran, A.S.)

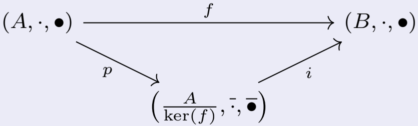
*Let $(B, \cdot, \bullet)$ be a Hopf sub-brace of $(A, \cdot, \bullet)$. The following are equivalent:*

1) $(B, \cdot, \bullet)$ *is a normal Hopf sub-brace;*
2) $(B, \cdot, \bullet)$ *is the kernel of a morphism in* $\mathsf{HBr}_{\mathrm{coc}}$.

# Regularity of $\mathsf{HBr}_{\mathrm{coc}}$

## Theorem (M. Gran, A.S.)

1) *The regular epi-mono factorization of a morphism* $f : (A, \cdot, \bullet) \to (B, \cdot, \bullet)$ *in* $\mathsf{HBr}_{\mathrm{coc}}$ *is*




2) *In* $\mathsf{HBr}_{\mathrm{coc}}$*, regular epimorphisms coincide with surjective morphisms and monomorphisms with injective morphisms.*

3) *Surjective morphisms in* $\mathsf{HBr}_{\mathrm{coc}}$ *are stable under pullbacks along injective morphisms, so the category* $\mathsf{HBr}_{\mathrm{coc}}$ *is regular.*

Since the direct image of a kernel in $\mathsf{HBr}_{\mathrm{coc}}$ is a kernel in $\mathsf{HBr}_{\mathrm{coc}}$, we have:

### Theorem (M. Gran, A.S.)

*The category* $\mathsf{HBr}_{\mathrm{coc}}$ *is semi-abelian.*

• We recover the semi-abelianness of $\mathsf{Hopf}_{\Bbbk,\mathrm{coc}}$ since the latter is isomorphic to the full subcategory of $\mathsf{HBr}_{\mathrm{coc}}$ given by the *trivial* cocommutative Hopf braces $(H, \cdot, \cdot)$.

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- The category $\mathsf{HBr}_{\mathrm{coc}}$ is also *strongly protomodular*.

In a semi-abelian category $\mathcal{C}$, an object $C$ is *abelian* if and only if $\langle \mathrm{Id}_C, \mathrm{Id}_C \rangle : C \to C \times C$ is a kernel. The category $\mathsf{Ab}(\mathcal{C})$ is abelian.

### Proposition

*The abelian category* $\mathsf{Ab}(\mathsf{HBr}_{\mathrm{coc}})$ *is given by trivial cocommutative Hopf braces* $(H, \cdot, \cdot)$ *such that* $\cdot$ *is abelian, hence it is isomorphic to* $\mathsf{Hopf}_{\mathbb{k},\mathrm{com},\mathrm{coc}}$.

# A torsion theory in $\mathsf{HBr}_{\mathrm{coc}}$

In any pointed category $\mathcal{C}$ a *torsion theory* $(\mathcal{T}, \mathcal{F})$ is a pair of full replete subcategories of $\mathcal{C}$ satisfying:

- for any $X$ in $\mathcal{C}$ there is a short exact sequence

  $$0 \longrightarrow T \longrightarrow X \longrightarrow F \longrightarrow 0$$

  where $T \in \mathcal{T}$ and $F \in \mathcal{F}$;

- the only morphism from a $T \in \mathcal{T}$ to an $F \in \mathcal{F}$ is the zero morphism.

A torsion theory is called *hereditary* if $\mathcal{T}$ is closed under subobjects.

Consider the following subcategories of $\mathsf{HBr}_{\text{coc}}$:

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2) The category $\mathsf{PHBR}_{\mathrm{coc}}$ whose objects are given by the cocommutative Hopf braces whose "underlying" Hopf algebras are universal enveloping algebras of *post-Lie algebras* [B. Vallette, *Homology of generalised partition posets*, 2007].

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A post-Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \rightharpoonup)$ is the datum of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ and a linear map $\rightharpoonup : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ such that the following equalities hold:

i) $x \rightharpoonup [y,z] = [x \rightharpoonup y, z] + [y, x \rightharpoonup z]$,

ii) $\Big([x,y] + (x \rightharpoonup y) - (y \rightharpoonup x)\Big) \rightharpoonup z = \Big(x \rightharpoonup (y \rightharpoonup z)\Big) - \Big(y \rightharpoonup (x \rightharpoonup z)\Big)$.

## Theorem (M. Gran, A.S)

*Let $\Bbbk$ be an algebraically closed field of characteristic 0.*

1) *The pair* $(\mathsf{PHBR}_{\mathrm{coc}}, \mathsf{SKB})$ *is an hereditary torsion theory in* $\mathsf{HBr}_{\mathrm{coc}}$.
   *For any* $(H, \cdot, \bullet)$ *in* $\mathsf{HBr}_{\mathrm{coc}}$ *we have a short exact sequence*

$$(U(P(H)), \cdot, \bullet) \longrightarrow (H, \cdot, \bullet) \longrightarrow (\Bbbk G(H), \cdot, \bullet)$$

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2) $\mathsf{SKB}$ *is a Birkhoff subcategory of* $\mathsf{HBr}_{\mathrm{coc}}$, *i.e. a full replete subcategory which is reflective*

$$\mathsf{SKB} \underset{U}{\overset{\mathsf{F}}{\underset{\perp}{\leftrightarrows}}} \mathsf{HBr}_{\mathrm{coc}}$$

   *and closed under subobjects and regular quotients in* $\mathsf{HBr}_{\mathrm{coc}}$.
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3) *Since* $\mathsf{F}$ *preserves finite limits,* $\mathsf{SKB}$ *is a localization of* $\mathsf{HBr}_{\mathrm{coc}}$*.*

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# Thank you for your attention!