

Modules, Extensions, and (Co)homology of Symmetric Quandles

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- Racks and Quandles
- Symmetric Racks (resp. Quandles)
- Symmetric Rack (resp. Quandle) Modules
- Abelian extensions of symmetric racks (resp. Quandles)
- Generalized Cohomology of Symmetric racks (resp. Quandles)

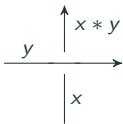
Knots and Links

- A **link** (or **knot**) L with m -components is an embedding of $\bigsqcup_m \mathbb{S}^1$ (or \mathbb{S}^1) into \mathbb{S}^3 .

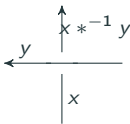


- Two links L_1 and L_2 are said to be **equivalent** if L_1 can be transformed into L_2 via an **ambient isotopy**.
- For a regular projection of L , then we consider the **link diagram** $D(L)$ to capture over/under information.

For each crossing of an oriented knot diagram, we set

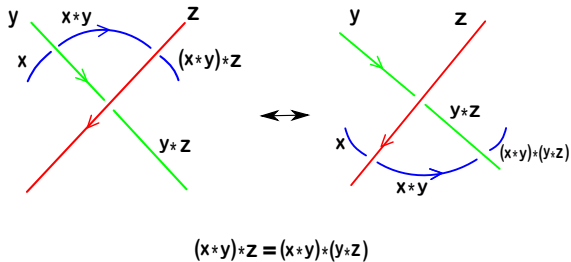
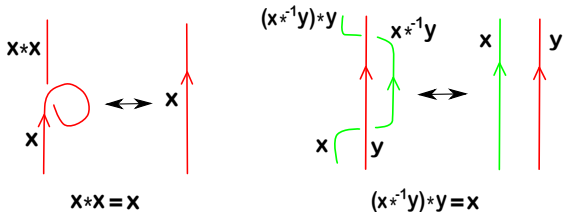


(a) Positive crossing



(b) Negative crossing

Reidemeister moves and Quandle axioms



Definition

A **quandle** is a set X with a binary operation $*$ satisfying:

1. $x * x = x$ for all $x \in X$,
2. For all $x, y \in X$, there exists a unique $z \in X$ such that $x = z * y$,
3. $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in X$.

Alternatively, X is a quandle if and only if for each $x \in X$, the map $S_x : X \rightarrow X$ given by $S_x(y) = y * x$ is an automorphism of X fixing x .

A **rack** is a set X with $*$ which satisfies axioms (2) and (3).

Why racks and quandles?

- Racks naturally produce non-degenerate set-theoretic solutions of the Yang–Baxter equation:

$$R : X \times X \rightarrow X \times X, \quad R(x, y) = (y, x * y).$$

- Fundamental quandles are remarkably almost complete knot invariants: they classify oriented knots up to orientation reversal (Joyce–Matveev, 1982).
- Quandle cohomology gives rise to powerful cocycle invariants for oriented knots, links and surface-links.
- However, classical quandle cocycle invariants require **orientability**.

Question

Can one define cocycle invariants for **unoriented or non-orientable** knots and surface-links?

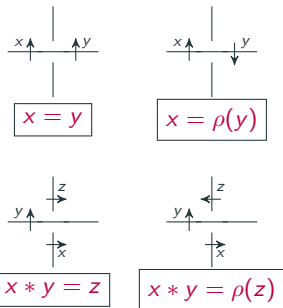
Symmetric Racks and Symmetric Quandles

Definition

Let $(X, *)$ be a rack (resp. quandle). A map $\rho_X : X \rightarrow X$ is called a **good involution** if:

- ρ_X is an involution, i.e., $\rho_X^2 = \text{id}_X$
- $\rho_X(x * y) = \rho_X(x) * y$
- $x * \rho_X(y) = x *^{-1} y$

for all $x, y \in X$. The pair (X, ρ_X) is called a **symmetric rack (resp. quandle)**.



⁰S. Kamada and K. Oshiro, *Homology groups of symmetric quandles and cocycle invariants of links and surface-links*, Trans. Amer. Math. Soc. 362 (2010), no.10, 5501–5527.

Examples of Symmetric Quandles

- For a codimension-two manifold pair (W, L) , let $Q(L)$ denote the set of homotopy classes $[(D, a)]$, where D is an oriented meridian disk of L and a is a path in $W \setminus L$ from ∂D to a base point $p \in W \setminus L$. Define

$$[(D_1, a_1)] * [(D_2, a_2)] = [(D_1, a_1 \circ a_2^{-1} \circ \partial D_2 \circ a_2)].$$

This gives $Q(L)$ the structure of a quandle. Let $\rho : Q(L) \rightarrow Q(L)$ is given by $\rho([(D, a)]) = [(-D, a)]$, where $-D$ is D with opposite orientation. The pair $(Q(L), \rho)$ is called the **fundamental symmetric quandle** of L .

- Let Q be a quandle, and let Q_1 and Q_2 represent two copies of Q . For an element $x \in Q$, we denote the corresponding elements in Q_1 and Q_2 by x_1 and x_2 , respectively. Define $D(Q)$ as the disjoint union of Q_1 and Q_2 . For any elements $x, y \in Q$, we set:

$$x_i * y_1 = (x * y)_i \quad \text{and} \quad x_i * y_2 = (x *^{-1} y)_i,$$

for $i = 1, 2$. Consequently, the set $D(Q)$ forms a quandle. The involution $\rho : D(Q) \rightarrow D(Q)$ that interchanges x_1 and x_2 for every $x \in Q$ constitutes a good involution.

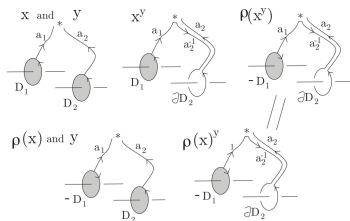


Figure 2: A meridian disk with path

Motivation from Classical Algebra

Recall that Kamada and Oshiro introduced symmetric quandle (co)homology to define cocycle invariants for **unoriented** and **non-orientable** links and surface-links.

However, their theory uses only **trivial coefficients**.

(Co)homology Theory	Coefficients
Group (co)homology	G -modules
Lie algebra (co)homology	\mathfrak{g} -modules
Hochschild (co)homology	bimodules

What should the coefficient objects be for symmetric quandle (co)homology?

Groups	Symmetric Quandles
G	(X, ρ)
G -module	?
Semidirect product	?
Extensions	?
$H^2(G, M)$?

Goal: complete the right-hand column.

Definition

Let \mathcal{C} be a category with pullbacks and $A \in \mathcal{C}$. A **Beck module** over A is an abelian group object in the slice category \mathcal{C}/A .

Trunks

A **trunk**¹ \mathbb{T} is an object analogous to a category, with objects and morphisms $\text{Hom}_{\mathbb{T}}(A, B)$, and some commutative squares called **preferred squares**.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow h \\ C & \xrightarrow{i} & D \end{array}$$

Given two trunks S and T , a **trunk map** $F : S \rightarrow T$ is a map that assigns to each object A of S , an object $F(A)$ of T , and to every morphism $f : A \rightarrow B$ of S , a morphism $F(f) : F(A) \rightarrow F(B)$ of T , such that a preferred square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow h \\ C & \xrightarrow{i} & D \end{array} \rightsquigarrow \begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ F(g) \downarrow & & \downarrow F(h) \\ F(C) & \xrightarrow{F(i)} & F(D) \end{array}$$

Example

For any category C , there is a well-defined trunk $T(C)$ with the same objects and morphisms as C , and whose preferred squares are the commutative diagrams in C .

¹R. Fenn, C. Rourke, B. Sanderson, *Trunks and classifying spaces*, Appl. Categ. Structures 3 (1995), no. 4, 321–356.

Trunk of a Symmetric Rack (resp. Quandle)

Given a symmetric rack (resp. quandle) (X, ρ_X) , we defined a trunk $T(X, \rho_X)$:

- For each $x \in X$, $T(X, \rho_X)$ has precisely one object,
- For each $x, y \in X$, there are morphisms $\gamma_x : x \rightarrow \rho_X(x)$, $\text{id}_x : x \rightarrow x$, $\alpha_{x,y} : x \rightarrow x * y$ and $\beta_{x,y} : y \rightarrow x * y$ such that the following squares are commutative:

$$\begin{array}{ccc} x & \xrightarrow{\alpha_{x,y}} & x * y \\ \alpha_{x,z} \downarrow & & \downarrow \alpha_{x*y,z} \\ x * z & \xrightarrow{\alpha_{x*z,y*z}} & (x * y) * z \end{array}$$

$$\begin{array}{ccc} y & \xrightarrow{\beta_{x,y}} & x * y \\ \alpha_{y,z} \downarrow & & \downarrow \alpha_{x*y,z} \\ y * z & \xrightarrow{\beta_{x*z,y*z}} & (x * y) * z \end{array}$$

$$\begin{array}{ccc} x & \xrightarrow{\alpha_{x,y}} & x * y \\ \gamma_x \downarrow & & \downarrow \gamma_{x*y} \\ \rho_X(x) & \xrightarrow{\alpha_{\rho_X(x),y}} & \rho_X(x) * y \end{array}$$

$$\begin{array}{ccc} x & \xrightarrow{\beta_{x,y}} & x * y \\ \text{id}_x \downarrow & & \downarrow \gamma_{x*y} \\ \rho_X(x) & \xrightarrow{\beta_{\rho_X(x),y}} & \rho_X(x) * y \end{array}$$

$$\begin{array}{ccc} x & \xrightarrow{\gamma_x} & \rho_X(x) \\ \text{id}_x \downarrow & & \downarrow \gamma_{\rho_X(x)} \\ x & \xrightarrow{\text{id}_x} & x \end{array}$$

$$\begin{array}{ccc} x & \xrightarrow{\alpha_{x,\rho_X(y)}} & x * \rho_X(y) \\ \text{id}_x \downarrow & & \downarrow \alpha_{x*\rho_X(y),y} \\ x & \xrightarrow{\text{id}_x} & x \end{array}$$

Symmetric Rack (resp. Quandle) Modules

Thus, a trunk map $A : T(X, \rho_X) \rightarrow \text{Ab}$ yields abelian groups A_x and group homomorphisms $\phi_{x,y} : A_x \rightarrow A_{x*y}$, $\psi_{x,y} : A_y \rightarrow A_{x*y}$ and $\eta_x : A_x \rightarrow A_{\rho_X(x)}$ such that for all $x, y, z \in X$,

1. $\phi_{x*y,z} \phi_{x,y} = \phi_{x*z,y*z} \phi_{x,z}$,
2. $\phi_{x*y,z} \psi_{x,y} = \psi_{x*z,y*z} \phi_{y,z}$,
3. $\eta_{\rho_X(x)} \eta_x = \text{id}$,
4. $\eta_{x*y} \phi_{x,y} = \phi_{\rho_X(x),y} \eta_x$,
5. $\psi_{\rho_X(x),y} = \eta_{x*y} \psi_{x,y}$,
6. $\phi_{x*-1_{y,y}} \phi_{x,\rho_X(y)} = \text{id}$,

Notation: $\mathcal{F} = (A, \phi, \psi, \eta)$.

Definition

Let (X, ρ_X) be a symmetric rack. An (X, ρ_X) -**module** is a trunk map $\mathcal{F} = (A, \phi, \psi, \eta) : T(X, \rho_X) \rightarrow \text{Ab}$ such that $\phi_{x,y} : A_x \rightarrow A_{x*y}$ is an isomorphism and

$$(7) \quad \psi_{x*y,z}(a) = \phi_{x*z,y*z} \psi_{x,z}(a) + \psi_{x*z,y*z} \psi_{y,z}(a),$$

$$(8) \quad \phi_{x*-1_{y,y}} \psi_{x,\rho_X(y)}(\eta_y(b)) = -\psi_{x*\rho_X(y),y}(b),$$

hold for all $a \in A_z$, $b \in A_y$ and $x, y, z \in X$.

If (X, ρ_X) is a symmetric quandle, then \mathcal{F} satisfies $\phi_{x,x}(a) + \psi_{x,x}(a) = a$ for all $a \in A_x$ and $x \in X$.

Theorem

Let (X, ρ_X) be a symmetric rack. Then the category $\text{SRMod}_{(X, \rho_X)}$ of (X, ρ_X) -modules is equivalent to the category $\text{Ab}(\text{SR}|_{(X, \rho_X)})$ of abelian group objects in the slice category over (X, ρ_X) .

Idea of proof: Let $\mathcal{F} = (A, \phi, \psi, \eta)$ an (X, ρ_X) -module.

STEP 1: We define the **semi-direct product** $\mathcal{F} \rtimes X$ of \mathcal{F} and X to be the set $\{(a, x) \mid x \in X, a \in A_x\}$ equipped with the binary operation

$$(a, x) \tilde{*} (b, y) := (\phi_{x,y}(a) + \psi_{x,y}(b), x * y)$$

and the map $\rho_{\mathcal{F} \rtimes X} : \mathcal{F} \rtimes X \rightarrow \mathcal{F} \rtimes X$ defined by

$$\rho_{\mathcal{F} \rtimes X}((a, x)) = (\eta_x(a), \rho_X(x))$$

for all $a \in A_x$, $b \in A_y$ and $x, y \in X$.

STEP 2: We check that the semi-direct product $(\mathcal{F} \rtimes X, \rho_{\mathcal{F} \rtimes X})$ is a symmetric rack and prove that the natural projection map $p : (\mathcal{F} \rtimes X, \rho_{\mathcal{F} \rtimes X}) \rightarrow (X, \rho_X)$ has a canonical abelian group structure, given by

- $\mu : (\mathcal{F} \rtimes X) \times_X (\mathcal{F} \rtimes X) \rightarrow (\mathcal{F} \rtimes X)$ given by $((a_1, x), (a_2, x)) \mapsto (a_1 + a_2, x)$,
- $\sigma : X \rightarrow (\mathcal{F} \rtimes X)$ given by $x \mapsto (0, x)$,
- $\nu : (\mathcal{F} \rtimes X) \rightarrow (\mathcal{F} \rtimes X)$ given by $(a, x) \mapsto (-a, x)$,

This association gives rise to a functor $\mathcal{T} : \text{SRMod}_{(X, \rho_X)} \rightarrow \text{Ab}(\text{SR}|_{(X, \rho_X)})$.

STEP 3: Let (X, ρ_X) be a symmetric rack and $p : (Y, \rho_Y) \rightarrow (X, \rho_X)$ an abelian group object with the multiplication map m , the inverse map i , and the section s . Let (X, \star) and (Y, \star) be the underlying racks. Then there exists an (X, ρ_X) -module (R, ϕ, ψ, η) given by an induced abelian group structure on the fibre $R_x := p^{-1}(x)$ for each $x \in X$, where

$$\phi_{x,y} : R_x \rightarrow R_{x \star y} \text{ is given by } u \mapsto u \star s(y),$$

$$\psi_{x,y} : R_y \rightarrow R_{x \star y} \text{ is given by } v \mapsto s(x) \star v$$

and

$$\eta_x : R_x \rightarrow R_{\rho_X(x)} \text{ is given by } u \mapsto \rho_Y(u)$$

for all $u \in R_x$, $v \in R_y$ and $x, y \in X$. Furthermore, the association gives a functor $\mathcal{S} : \text{Ab}(\text{SR}|_{(X, \rho_X)}) \rightarrow \text{SRMod}_{(X, \rho_X)}$.

STEP 4: We show that the functors \mathcal{T} and \mathcal{S} are inverses to each other.

An (X, ρ_X) -module $\mathcal{F} = (A, \phi, \psi, \eta)$ is called **homogeneous** if the constituent groups are all isomorphic, that is, $A_x \cong A_y$ for all $x, y \in X$.

Examples

- Let A an abelian group whose each element has order 2 or 4. Then, $\mathcal{F} = (A, \phi, \psi, \eta)$ is an (X, ρ_X) -module, where $A_x = A$, $\phi_{x,y}(a) = -a$, $\psi_{x,y}(a) = 2a$ and $\eta_x(a) = -a$ for all $x \in X$ and $a \in A$.
- We have a well-defined right action of $G_{(X, \rho_X)}$ on (X, ρ_X) , given by $x \cdot e_y = x * y$ for all $x, y \in G_{(X, \rho_X)}$. Let $[x]$ denote the orbit of x under this action. Then, we have an (X, ρ_X) -module $\mathcal{F} = (A, \phi, \psi, \eta)$ with $A_x = A_{\rho_X(x)} := A_{[x]}$ and

$$\phi_{x,y} : A_{[x]} \rightarrow A_{[x*y]} \text{ defined by } a \mapsto f_{[x]}(a),$$

$$\psi_{x,y} : A_{[x]} \rightarrow A_{[x*y]} \text{ defined by } b \mapsto 0 \text{ and}$$

$$\eta_x : A_{[x]} \rightarrow A_{[\rho_X(x)]} \text{ defined by } c \mapsto g_{[x]}(c),$$

where $f_{[x]}, g_{[x]} : A_{[x]} \rightarrow A_{[x]}$ are group homomorphisms such that $f_{[x]}^2 = \text{id} = g_{[x]}^2$ and $f_{[x]}g_{[x]} = g_{[x]}f_{[x]}$ for all $x \in X$.

Abelian extensions of symmetric racks (quandles)

Definition

Let (X, ρ_X) be a symmetric rack (resp. quandle) and $\mathcal{F} = (A, \phi, \psi, \eta)$ an (X, ρ_X) -module. An **extension** of (X, ρ_X) by \mathcal{F} consists of:

1. A symmetric rack (resp. quandle) (E, ρ_E) with operation \star .
2. An epimorphism $f : (E, \rho_E) \rightarrow (X, \rho_X)$ inducing a partition $E = \cup_{x \in X} E_x$, where $E_x := f^{-1}(x)$.
3. For each $x \in X$, a left A_x -action on E_x given by $(a, s) \mapsto a \cdot s$, satisfying:
 - The A_x -action on E_x is free and transitive.
 - $(a \cdot s) \star t = \phi_{x,y}(a) \cdot (s \star t)$.
 - $s \star (b \cdot t) = \psi_{x,y}(b) \cdot (s \star t)$.
 - $\rho_E(a \cdot s) = \eta_x(a) \cdot \rho_E(s)$.

Two extensions $f_1 : (E_1, \rho_{E_1}) \rightarrow (X, \rho_X)$ and $f_2 : (E_2, \rho_{E_2}) \rightarrow (X, \rho_X)$ of (X, ρ_X) by \mathcal{F} are **equivalent** if there exists an isomorphism $\Phi : (E_1, \rho_{E_1}) \rightarrow (E_2, \rho_{E_2})$ such that:

1. $f_2 \circ \Phi = f_1$.
2. $\Phi(a \cdot s) = a \cdot \Phi(s)$ for all $s \in E_x$, $a \in A_x$, and $x \in X$.

Definition

Let (X, ρ_X) be a symmetric rack (or quandle), and $\mathcal{F} = (A, \phi, \psi, \eta)$ an (X, ρ_X) -module.

Given an extension $f : (E, \rho_E) \rightarrow (X, \rho_X)$ and a section $s : (X, \rho_X) \rightarrow (E, \rho_E)$, a **factor set** is a map:

$$\sigma : X \times X \rightarrow \bigsqcup_{x \in X} A_x, \quad \text{such that} \quad s(x) \star s(y) = \sigma_{x,y} \cdot s(x * y)$$

and a factor set is called a **coboundary** if there exists a map $v : X \rightarrow \bigsqcup_{x \in X} A_x$ such that

$$\sigma_{x,y} = \phi_{x,y}(v_x) + \psi_{x,y}(v_y) - v_{x*y} \quad \text{and} \quad \eta_x(v_x) = v_{\rho_X(x)}$$

for all $x, y \in X$, where $\sigma_{x,y} \in A_{x*y}$

Note: If $\sigma_{x,y} = 0$, then s is a symmetric rack homomorphism. Thus, σ represents the **obstruction** to s being a homomorphism.

Theorem

Let (X, ρ_X) be a symmetric rack and $\mathcal{F} = (A, \phi, \psi, \eta)$ an (X, ρ_X) -module. Let $\sigma : X \times X \rightarrow \bigsqcup_{x \in X} A_x$ be a map and define

$$E(\mathcal{F}, \sigma) = \{(a, x) \mid x \in X, a \in A_x\}.$$

The binary operation on $E(\mathcal{F}, \sigma)$ is given by:

$$(a, x) \tilde{*} (b, y) := (\phi_{x,y}(a) + \psi_{x,y}(b) + \sigma_{x,y}, x * y),$$

and the map $\rho_{E(\mathcal{F}, \sigma)} : E(\mathcal{F}, \sigma) \rightarrow E(\mathcal{F}, \sigma)$ is: $(a, x) \mapsto (\eta_x(a), \rho_X(x))$, for all $a \in A_x$, $b \in A_y$, and $x, y \in X$.

Then, $(E(\mathcal{F}, \sigma), \rho_{E(\mathcal{F}, \sigma)})$ is an extension of (X, ρ_X) by \mathcal{F} with factor set σ if and only if the following conditions hold:

$$(F1) \quad \sigma_{x*y, z} + \phi_{x*y, z}(\sigma_{x, y}) = \phi_{x*z, y*z}(\sigma_{x, z}) + \sigma_{x*z, y*z} + \psi_{x*z, y*z}(\sigma_{y, z})$$

$$(F2) \quad \sigma_{\rho_X(x), y} = \eta_{x*y}(\sigma_{x, y})$$

$$(F3) \quad \phi_{x*\rho_X(y), y}(\sigma_{x, \rho_X(y)}) = -\sigma_{x*\rho_X(y), y}$$

Moreover, if (X, ρ_X) is a symmetric quandle, then the additional condition holds:

$$(F4) \quad \sigma_{x, x} = 0 \text{ for all } x \in X.$$

Let $Z^2(X, \mathcal{F})$ be the set of factor sets $\sigma : X \times X \rightarrow \bigsqcup_{x \in X} A_x$. This forms an abelian group with the operation:

$$(\sigma + \tau)_{x,y} := \sigma_{x,y} + \tau_{x,y}$$

The subgroup of coboundaries is denoted $B^2(X, \mathcal{F})$, and the cohomology is:

$$\mathcal{H}_{\text{SR}}^2(X, \mathcal{F}) := Z^2(X, \mathcal{F}) / B^2(X, \mathcal{F}).$$

We prove the following theorem.

Theorem

Let (X, ρ_X) be a symmetric rack and $\mathcal{F} = (A, \phi, \psi, \eta)$ be an (X, ρ_X) -module. Then there is an abelian group $\mathcal{H}_{\text{SR}}^2(X, \mathcal{F})$ whose elements are in bijective correspondence with the set of equivalence classes of extensions of (X, ρ_X) by \mathcal{F} .

Generalized (co)homology for symmetric racks (quandles)

Definition

The **symmetric rack algebra** of a symmetric rack (X, ρ_X) is the associative \mathbb{Z} -algebra, denoted by $\mathbb{Z}(X, \rho_X)$, which is generated by the set $\{\phi_{x,y}^{\pm 1}, \psi_{x,y}, \eta_x \mid x, y \in X\}$ and admits the following defining relations:

$$(A1) \quad \phi_{x,y} \phi_{x,y}^{-1} = \phi_{x,y}^{-1} \phi_{x,y} = 1$$

$$(A2) \quad \phi_{x*y,z} \phi_{x,y} = \\ \phi_{x*z,y*z} \phi_{x,z}$$

$$(A3) \quad \phi_{x*y,z} \psi_{x,y} = \\ \psi_{x*z,y*z} \phi_{y,z}$$

$$(A4) \quad \eta_{\rho_X(x)} \eta_x = 1$$

$$(A5) \quad \phi_{\rho_X(x),y} \eta_x = \eta_{x*y} \phi_{x,y}$$

$$(A6) \quad \psi_{\rho_X(x),y} = \eta_{x*y} \psi_{x,y}$$

$$(A7) \quad \phi_{x*\rho_X(y),y} \phi_{x,\rho_X(y)} = 1$$

$$(A8) \quad \psi_{x*y,z} = \\ \phi_{x*z,y*z} \psi_{x,z} + \psi_{x*z,y*z} \psi_{y,z}$$

$$(A9) \quad \phi_{x*\rho_X(y),y} \psi_{x,\rho_X(y)} \eta_y = \\ -\psi_{x*\rho_X(y),y}$$

The **symmetric quandle algebra** of a symmetric quandle additionally satisfies: $\phi_{x,x} + \psi_{x,x} = 1$ for all $x, y, z \in X$.

Chain Complex for Symmetric Racks

Let (X, ρ_X) be a symmetric rack. For each $n \geq 0$, define:

$$C_n(X, \rho_X) := \mathbb{Z}(X, \rho_X)X^n$$

where $X^0 = \{p\}$ for a fixed $p \in X$.

For convenience, define: $[x_1 \cdots x_n] := (((x_1 * x_2) * x_3) * \cdots * x_n)$ for $(x_1, x_2, \dots, x_n) \in X^n$.

For $n \geq 2$, define $\partial_n : C_n(X, \rho_X) \rightarrow C_{n-1}(X, \rho_X)$ as:

$$\begin{aligned} \partial_n(x_1, \dots, x_n) &= \sum_{j=2}^n (-1)^j (-1)^n \phi_{[x_1 \cdots \widehat{x}_j \cdots x_n], [x_j \cdots x_n]}(x_1, \dots, \widehat{x}_j, \dots, x_n) \\ &\quad - \sum_{j=2}^n (-1)^j (-1)^n (x_1 * x_j, \dots, x_{j-1} * x_j, x_{j+1}, \dots, x_n) \\ &\quad + (-1)^n \psi_{[x_1 \widehat{x}_2 x_3 \cdots x_n], [x_2 x_3 \cdots x_n]}(x_2, \dots, x_n) \end{aligned}$$

The boundary map for $n = 1$ is given by: $\partial_1(x) = -\psi_{x * -1, p, p}(p)$.

Using conditions (A2), (A3), and (A8), it can be shown that:

$\partial_{n-1} \partial_n = 0$. for all $n \geq 1$. Hence, $(C_n(X, \rho_X), \partial)$ forms a chain complex.

Subcomplex of the Chain Complex

Define a submodule $D_n^{SR}(X, \rho_X)$ of $C_n(X, \rho_X)$ for $n \geq 1$. This is generated by $U_n \cup V_n$, where:

$$U_n = \{ \eta_{[x_1 \dots x_n]}(x_1, \dots, x_n) - (\rho_X(x_1), x_2, \dots, x_n) \mid (x_1, \dots, x_n) \in X^n \}$$

For $n \geq 2$, V_n is defined as:

$$V_n = \bigcup_{i=2}^n \{ \phi_{[x_1 \dots \hat{x}_i \dots x_n], [x_i \dots x_n]}(x_1 * x_i, \dots, x_{i-1} * x_i, \rho_X(x_i), x_{i+1}, \dots, x_n) \\ + (x_1, \dots, x_n) \mid (x_1, x_2, \dots, x_n) \in X^n \}$$

For $n = 1$, $V_1 = \emptyset$.

Then one can verify that $\partial_n(D_n^{SR}(X, \rho_X)) \subset D_{n-1}^{SR}(X, \rho_X)$ for each $n \geq 1$, i.e. $D_n^{SR}(X, \rho_X)$ is a subcomplex.

For quandles: Let (X, ρ_X) be a symmetric quandle. Define $D_n^{SQ}(X, \rho_X)$ to be the submodule of $C_n(X, \rho_X)$ generated by the $U_n \cup V_n \cup W_n$, where

$$W_n := \bigcup_{i=1}^{n-1} \{ (x_1, \dots, x_n) \in X^n \mid x_i = x_{i+1} \}.$$

Then $\partial_n(D_n^{SQ}(X, \rho_X)) \subset D_{n-1}^{SQ}(X, \rho_X)$ for each $n \geq 2$.

Homology and Cohomology of Symmetric Racks

Let (X, ρ_X) be a symmetric rack and A be a right $\mathbb{Z}(X, \rho_X)$ -module. Define the homology chain complex:

$$C_n^{SR}((X, \rho_X), A) := \frac{A \otimes_{\mathbb{Z}(X, \rho_X)} C_n(X, \rho_X)}{A \otimes_{\mathbb{Z}(X, \rho_X)} D_n^{SR}(X, \rho_X)}.$$

The induced boundary map ∂_n gives the homology groups: $H_n^{SR}((X, \rho_X), A)$.
If A is a left $\mathbb{Z}(X, \rho_X)$ -module, define the cochain complex:

$$C_{SR}^n((X, \rho_X), A) := \text{Hom}_{\mathbb{Z}(X, \rho_X)}(C_n^{SR}(X, \rho_X), A)$$

with the coboundary map $\delta^n : C_{SR}^n((X, \rho_X), A) \rightarrow C_{SR}^{n+1}((X, \rho_X), A)$.

The cohomology groups are given by: $H_{SR}^n((X, \rho_X), A)$.

Remark: Analogously, define (co)homology groups for symmetric quandles.

Our work:²

- Generalized the (co)homology introduced by Kamada and Oshiro³ Any abelian group A can be considered as trivial homogeneous (X, ρ_X) -module by setting $A_x = A$, $\phi_{x,y} = \text{id}_A$, $\psi_{x,y} = 0$ and $\eta_x = -\text{id}_A$ for all $x, y \in X$.
- Aim to capture richer invariants (for spatial graphs and low dimensional manifolds) beyond the original (co)homology (trivial action case).

²B. Karmakar, D. Saraf, & M. Singh, *Generalized (co)homology of symmetric quandles over homogeneous Beck modules*, *Journal of Pure and Applied Algebra* 229 (6) (2025), 107956, 31 pp.

³S. Kamada and K. Oshiro, *Homology groups of symmetric quandles and cocycle invariants of links and surface-links*, *Trans. Amer. Math. Soc.* 362 (2010), no.10, 5501–5527.

Summary

We developed for symmetric quandles the same module-extension-cohomology framework that exists for groups.

Groups	Symmetric Quandles
Group G	Symmetric quandle (X, ρ)
G -module M	(X, ρ) -module \mathcal{F}
Beck modules over G	Beck modules over (X, ρ)
Semidirect product $M \rtimes G$	Semidirect product $\mathcal{F} \rtimes X$
Extensions of G by M	Extensions of (X, ρ) by \mathcal{F}
$H^2(G, M)$ classifies extensions	$H_{SR}^2(X, \mathcal{F})$ classifies extensions
Group ring $\mathbb{Z}[G]$	Symmetric rack algebra $\mathbb{Z}(X, \rho)$
Group homology/cohomology	Symmetric rack (co)homology

Thank You!!