ON SOME BEAUTIFUL CHILDREN OF ABSTRACT ALGEBRA

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SOME HISTORICAL REMARKS

Galois and Lie

Évariste Galois (1811–1832)


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studied existence of formulas for zeros of polynomials via their symmetries (Galois group)

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Galois and Lie

Sophus Lie (1842-1899)

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studied solutions of differential equations via their symmetries (infinitesimal group / Lie algebra)

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Semisimple Lie algebras

simple Lie algebras: non-abelian, no proper quotients (like $\mathfrak{sl}_n(\mathbb{C})$)

full classification: Wilhelm Killing (1847 – 1923), Élie Cartan (1869 – 1951), Hermann Weyl (1885 – 1955)

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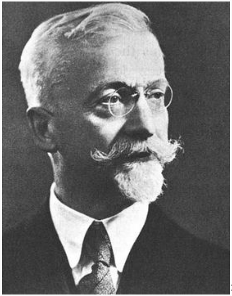*
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application in particle physics (standard model and its extensions)

Quantum groups, Hopf algebras

some physical models can not be explained with Lie algebras

Vladimir Drinfeld (1954 –, Fields medal 1990): semisimple Lie algebras admit Hopf algebra quantizations (quantum groups)


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some fundamental tools in Hopf algebra theory: braidings, braided Hopf algebras, Nichols algebras

NICHOLS ALGEBRAS

Nichols algebras

a **braided vector space** is a vector space V together with an automorphism c of $V \otimes V$ satisfying

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c).$$

Example: $v_1, \ldots, v_n$ basis of V, $c(v_i \otimes v_j) = q_{ij} v_j \otimes v_i$ with scalars $q_{ij} \neq 0$.

The tensor algebra $T(V)$ has a unique comultiplication $\Delta : T(V) \to T(V) \otimes T(V)$ with $\Delta(v) = 1 \otimes v + v \otimes 1$ for all $v \in V$.

$T(V)$ has a unique maximal coideal $I(V)$ in $\oplus_{n \geq 2} V^{\otimes 2}$.
$(\Delta(I(V)) \subseteq I(V) \otimes T(V) + T(V) \otimes I(V))$

The quotient $B(V) = T(V)/I(V)$ is the **Nichols algebra of** (V, c).

Nichols algebras

The Nichols algebra of V is a braided Hopf algebra.

If $c(v \otimes w) = w \otimes v$ for all $v, w \in V$, then $B(V)$ is the ring of polynomials in $\dim V$ indeterminates.

If $c(v \otimes w) = -w \otimes v$ for all $v, w \in V$, then $B(V)$ is the exterior algebra of V.

In other cases the structure of $B(V)$ is much more complicated.

SOME COMBINATORICS BEHIND NICHOLS ALGEBRAS

Root systems

Highly influential for semisimple Lie algebras: the **root system**.

Definition. Let *E* be a finite-dimensional Euclidean vector space with inner product denoted by $(\cdot, \cdot)$. A **root system** Φ in *E* is a finite set of non-zero vectors (called roots) that satisfy the following conditions:

- The roots span E.
- The only scalar multiples of a root $\alpha \in \Phi$ that belong to Φ are α and $-\alpha$.
- For every root $\alpha \in \Phi$, the set Φ is closed under reflection through the hyperplane perpendicular to α.
- (Integrality) If α and β are roots in Φ, then the projection of β onto the line through α is an integer or half-integer multiple of α.

Nichols algebras are governed by a more general structure.

- it relies on roots in a free $\mathbb{Z}$-module with a bicharacter, rather than in a Euclidean space;
- the chambers of the hyperplane arrangement attached to the root system are not uniform; reflections on roots depend on these chambers

There is a complete classification of bicharacters admitting a finite (generalized) root system (I.H.)

Simplicial arrangements with integrality assumption are also classified (M. Cuntz, I.H.)

Without integrality conditions: more examples, no full classification.

https://faculty.washington.edu/moishe/branko/BG274%20Catalogue%20of%20simplicial%20arrangements.pdf

Nichols algebras and frieze patterns

$$
\begin{array}{ccccccccccccc}
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
& 1 & & 2 & & 1 & & 2 & & 1 & & 2 & \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1
\end{array}
$$

$$
\text{pattern:} \quad \begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array} \quad \Rightarrow \quad ad - bc = 1
$$

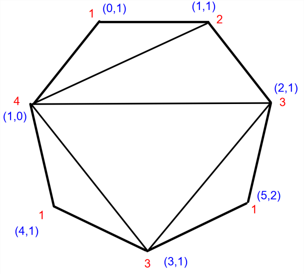
$$
\begin{array}{ccccccccccccccc}
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
& \color{red}{1} & & \color{red}{3} & & \color{red}{1} & & \color{red}{2} & & \color{red}{2} & & 1 & & 3 & \\
1 & & 2 & & 2 & & 1 & & 3 & & 1 & & 2 & & 2 \\
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}
$$

the red numbers form the quiddity cycle

Suppose that $\dim V = 2$. The arrangement of the bicharacter is a pencil of lines in a plane. It is determined by a finite sequence of positive integers (appearing in the matrices of reflections).

This is a quiddity sequence! The root system is contained in the frieze pattern.

Nichols algebras and frieze patterns



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1 1 1 1 1 1 1 1 1

1 2 3 1 3 1 4 1

1 5 2 2 2 3 3

2 3 3 1 5 2

1 4 1 2 3

1 1 1 1

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