

Galois theory and homology of abelian functor categories

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Algebra Seminar VUB, May 2026

Structure of the talk

1. Abelian categories
2. Torsion theories in abelian functor categories
3. Categorical Galois theory
4. Central extensions in abelian functor categories
5. Higher central extensions and generalized Hopf formulae for homology
6. Double central extensions, and second and third Hopf formulae in abelian functor categories

1. Abelian categories

Definition

A category \mathcal{A} is **abelian** if

- it is pointed, i.e., it has a zero object 0 ,
- it has binary products,
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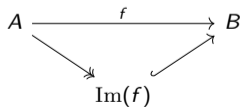
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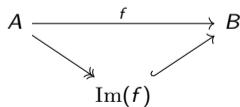
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- $\text{Mod}_R, \text{Ab}_t, \text{Ab}(\text{CH}), \text{Sh}(\text{Ab})$

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- The binary product $A \times B$ and the binary coproduct $A + B$ coincide in an abelian category, they are denoted by $A \oplus B$.
- Any abelian category \mathcal{A} is **Ab-enriched**, i.e., all $\text{Hom}_{\mathcal{A}}(A, B)$ have an abelian group structure, and composition of morphisms is bilinear. More precisely, given $f, g : A \rightarrow B$, their sum $f + g : A \rightarrow B$ is given by the composite

$$A \xrightarrow{(1_A, 1_A)} A \oplus A \xrightarrow{f \oplus g} B \oplus B \xrightarrow{[1_B, 1_B]} B.$$

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Proposition (Tierney's theorem)

A category is **abelian** if and only if it is **additive** and **(Barr-)exact**.

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A functor $F : \mathcal{A} \rightarrow \mathcal{B}$ between additive categories \mathcal{A}, \mathcal{B} is **additive** if it preserves finite biproducts.

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Example (Additive functor)

The left adjoint in an adjunction between additive categories is additive as it preserves colimits, in particular finite coproducts.

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Proof

Let

$$K \xrightarrow{k} A \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{s} \end{array} B$$

be a split short exact sequence in an additive category, i.e., $ps = 1_B$ and (K, k) is the kernel of p .

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Remark (Quasi-abelian categories)

The notion of **quasi-abelian category** is a slight generalization of that of abelian category. Quasi-abelian categories are 'nice' regular additive categories, that are not necessarily exact.

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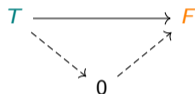
The notion of **quasi-abelian category** is a slight generalization of that of abelian category. Quasi-abelian categories are 'nice' regular additive categories, that are not necessarily exact. Examples of quasi-abelian categories are given by any abelian category, the categories of torsion-free abelian groups, of topological abelian groups and of Banach spaces.

2. Torsion theories in abelian functor categories

Definition

Let \mathcal{C} be a pointed category. A **torsion theory** $(\mathcal{T}, \mathcal{F})$ in \mathcal{C} consists of two replete full subcategories \mathcal{T} (**torsion subcategory**) and \mathcal{F} (**torsion-free subcategory**) of \mathcal{C} satisfying:

- Any morphism from an object $T \in \mathcal{T}$ to an object $F \in \mathcal{F}$ is 0.



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Slogan

We think of \mathcal{C} as 'complicated category', which can be 'decomposed' into two subcategories \mathcal{T}, \mathcal{F} that are 'easier' to understand. 'Local' information about \mathcal{T}, \mathcal{F} can be 'glued' together into 'global' information about \mathcal{C} via short exact sequences.

Example (Torsion theory)

$(\text{Ab}_t, \text{Ab}_{\text{tf}})$ is a torsion theory in Ab :

- Let $f : A \rightarrow B$ be a group homomorphism from a torsion abelian group to a torsion-free abelian group. Then, for any $x \in A$, there exists $n \in \mathbb{N}$ such that $nx = 0$. Hence:

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- Given an abelian group A , we have the short exact sequence

$$0 \longrightarrow T(A) \hookrightarrow A \twoheadrightarrow A/T(A) \longrightarrow 0,$$

where $T(A)$ is the subgroup of all torsion elements in A .

Proposition

Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory in a pointed category \mathcal{C} .

- 1 The inclusion functor $U : \mathcal{F} \rightarrow \mathcal{C}$ has a left adjoint $F : \mathcal{C} \rightarrow \mathcal{F}$ such that any component $\eta_X : X \rightarrow UF(X)$ of its unit is a normal epimorphism, i.e., \mathcal{F} is a normal epi-reflective subcategory of \mathcal{C} .

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- 2 The inclusion functor $V : \mathcal{T} \rightarrow \mathcal{C}$ has a right adjoint $T : \mathcal{C} \rightarrow \mathcal{T}$ such that any component $\varepsilon_X : VT(X) \rightarrow X$ of its counit is a normal monomorphism, i.e., \mathcal{T} is a normal mono-coreflective subcategory of \mathcal{C} .

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Proposition

Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory in a pointed category \mathcal{C} . The left adjoint $F : \mathcal{C} \rightarrow \mathcal{F}$ is semi-left exact, i.e., it preserves pullbacks of the form

$$\begin{array}{ccc} X \times_{UF(X)} U(Y) & \xrightarrow{p_2} & U(Y) \\ p_1 \downarrow & & \downarrow U(f) \\ X & \xrightarrow{\eta_X} & UF(X), \end{array}$$

where $f : Y \rightarrow F(X)$ is any morphism in \mathcal{F} .

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Proposition (E. (2025))

\mathcal{F} is a torsion-free subcategory of $\mathcal{A}^{\mathbb{T}}$.

Let T be an object in \mathbb{T} , and $F : \mathbb{T} \rightarrow \mathcal{A}$ be a functor.

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
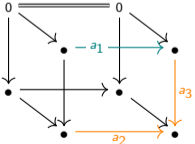
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\mathcal{F}	\mathcal{A}	2- $\text{Arr}(\mathcal{A})$	3- $\text{Arr}(\mathcal{A})$
$\text{Obj}(\mathcal{T})$	$\bullet \xrightarrow{a} \bullet$ a epimorphism	 a epimorphism	 a_1 epimorphism a_2, a_3 jointly epimorphic

Proposition (E. (2025))

\mathcal{F} is a torsion-free subcategory of $\mathcal{A}^{\mathbb{T}}$.

Proof

1 Let $\alpha : F \Rightarrow G$ with $F \in \mathcal{T}$ and $G \in \mathcal{F}$. Consider:

$$\begin{array}{ccc} \bigoplus_{t \in [T]} F(d(t)) & \xrightarrow{[F(t)]_{t \in [T]}} & F(T) \\ \bigoplus_{t \in [T]} \alpha_{d(t)} \downarrow & & \downarrow \alpha_T \\ \bigoplus_{t \in [T]} G(d(t)) & \xrightarrow{[G(t)]_{t \in [T]}} & G(T) \end{array}$$

$$\left. \begin{array}{l} [F(t)]_{t \in [T]} \text{ is an epi,} \\ G(d(t)) = 0 \text{ for all } t \in [T] \end{array} \right\} \Rightarrow \alpha = 0$$

2 Let $F \in \mathcal{A}^{\mathbb{T}}$. We have to show that there exist $T(F) \in \mathcal{T}$ and $F(F) \in \mathcal{F}$ such that:

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- $F(F) \in \mathcal{F}$ since $1_T \in [T]$ for any $T \notin \mathcal{S}$.
- $T(F) \in \mathcal{T}$ since φ is an epi.

3. Categorical Galois theory

Categorical Galois theory introduced by G. Janelidze in the 1980's both generalizes classical Galois theory and central extensions (of groups).

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Definition

A **Galois structure** $\Gamma = (\mathcal{C}, \mathcal{F}, F, U, \mathcal{E}, \mathcal{L})$ consists of an adjunction

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{U} \end{array} \mathcal{F}$$

and classes \mathcal{E}, \mathcal{L} of morphisms in \mathcal{C}, \mathcal{F} , respectively, satisfying certain conditions.

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Slogan

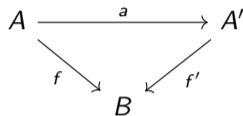
We think of \mathcal{C} as 'complicated category' whose class \mathcal{E} of 'extensions' we would like to better understand, and of \mathcal{F} as category which (or at least its class \mathcal{L} of 'extensions') is 'easier' to understand.

For the following, let us fix a Galois structure $\Gamma = (\mathcal{C}, \mathcal{F}, F, U, \mathcal{E}, \mathcal{L})$.

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Let B be an object in \mathcal{C} .

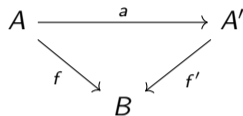
- \mathcal{C}/B : slice category of \mathcal{C} over B whose objects are morphism $f : A \rightarrow B$ in \mathcal{C} and whose morphisms are given by commutative triangles:



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We have the adjunction

$$\mathcal{E}(B) \begin{array}{c} \xrightarrow{F^B} \\ \perp \\ \xleftarrow{U^B} \end{array} \mathcal{Z}(F(B))$$

$$\begin{array}{ccc} A & & F(A) \\ \downarrow f & \mapsto & \downarrow F(f) \\ B & & F(B) \end{array}$$

$$\begin{array}{ccc} B \times_{UF(B)} U(X) & \longrightarrow & U(X) \\ U^B(\varphi) \downarrow & & \downarrow U(\varphi) \\ B & \xrightarrow{\eta_B} & UF(B) \end{array}$$

Definition

The Galois structure $\Gamma = (\mathcal{C}, \mathcal{F}, F, U, \mathcal{E}, \mathcal{L})$ is **admissible** if, for all B in \mathcal{C} , the right adjoint $U^B : \mathcal{L}(F(B)) \rightarrow \mathcal{E}(B)$ is fully faithful.

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In this case, we can view $\mathcal{L}(F(B))$ as full subcategory of $\mathcal{E}(B)$.

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Proposition

Let the right adjoint $U : \mathcal{F} \rightarrow \mathcal{C}$ in $\Gamma = (\mathcal{C}, \mathcal{F}, F, U, \mathcal{E}, \mathcal{L})$ be fully faithful. TFAE:

- 1 Γ is admissible.
- 2 F preserves all pullbacks of the form

$$\begin{array}{ccc} B \times_{UF(B)} U(X) & \xrightarrow{p_2} & U(X) \\ p_1 \downarrow & & \downarrow U(\varphi) \\ B & \xrightarrow{\eta_B} & UF(B), \end{array}$$

where φ is in \mathcal{L} .

Definition

Let $\Gamma = (\mathcal{C}, \mathcal{F}, F, U, \mathcal{E}, \mathcal{L})$ be an admissible Galois structure, and $f : A \rightarrow B$ be an extension, i.e. a morphism in \mathcal{E} .

- 1 f is a **trivial extension** if it lies in the essential image of $U^B : \mathcal{L}(F(B)) \rightarrow \mathcal{E}(B)$, or equivalently, if its corresponding naturality square is a pullback.

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Remark

In many cases, every extension is monadic, and an extension $f : A \rightarrow B$ is central if and only if ρ_1 in the following pullback is a trivial extension:

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Example (Admissible Galois structure)

$\Gamma = (\text{Grp}, \text{Ab}, \text{ab}, \text{U}, \text{Surj}(\text{Grp}), \text{Surj}(\text{Ab}))$ is an admissible Galois structure.

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Let $f : A \rightarrow B$ be a surjective group homomorphism. TFAE:

- f 'categorically' central with respect to Γ .
- f is 'algebraically' central, i.e., $[\text{Ker}(f), A] = \{1\}$.

4. Central extensions in abelian functor categories

Recall

$\mathcal{A}^{\mathbb{T}}$ is the category of functors from a finite category \mathbb{T} to an abelian category \mathcal{A} , \mathcal{S} is a replete full subcategory of \mathbb{T} , and \mathcal{F} is the full subcategory of $\mathcal{A}^{\mathbb{T}}$ with objects the functors $F : \mathbb{T} \rightarrow \mathcal{A}$ with $F(T) = 0$ for all $T \notin \mathcal{S}$.

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- If $F, G \in \mathcal{F}$, then α is in $\text{NEpi}(\mathcal{F})$ if and only if it is in $\text{NEpi}(\mathcal{A}^{\mathbb{T}})$.

Proposition (E. (2025))

Let $\alpha : F \Rightarrow G$ be in $\text{NEpi}(\mathcal{S}^{\mathbb{T}})$. TFAE:

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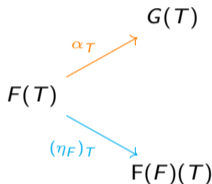
$$\begin{array}{ccc} & & G(T) \\ & \nearrow \alpha_T & \\ F(T) & & \\ & \searrow (\eta_F)_T & \\ & & F(F)(T) \end{array}$$

If $T \notin \mathcal{S}$, this amounts to α_T being an isomorphism.

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Example

Let us consider the adjunction

$$\text{Arr}(\mathcal{A}) \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{U} \end{array} \mathcal{A}$$

with unit

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$$\begin{array}{ccc} \mathbb{Z} & \xlongequal{\quad} & \mathbb{Z} \\ \parallel & & \downarrow \\ \mathbb{Z} & \longrightarrow & 0 \end{array}$$

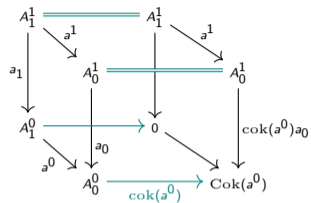
in Ab is central but not trivial.

Example

Let us consider the adjunction

$$\text{Arr}^2(\mathcal{A}) \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{U} \end{array} 2\text{-Arr}(\mathcal{A})$$

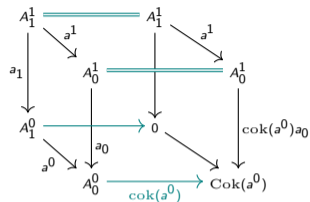
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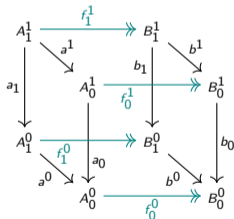
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An extension



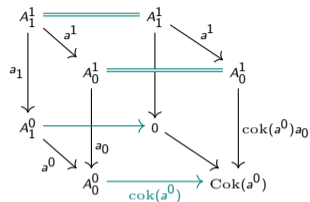
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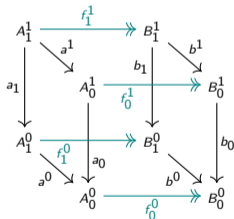
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$$\text{Arr}^2(\mathcal{A}) \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{U} \end{array} 2\text{-Arr}(\mathcal{A}) \quad \text{with unit}$$



An extension



is:

- trivial if and only if f_1^0 is an isomorphism, and $f_0^0, \text{cok}(a^0)$ are jointly monomorphic,
- central if and only if f_1^0 is an isomorphism.

Lemma

Let \mathcal{A} be an abelian category.

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- 2 Let

$$\begin{array}{ccc} \text{Ker}(h) & \xrightarrow{\bar{i}} & \text{Ker}(f) \\ \text{ker}(h) \downarrow & & \downarrow \text{ker}(f) \\ P & \xrightarrow{i} & A \\ h \downarrow & & \downarrow f \\ C & \xrightarrow{g} & B \end{array}$$

be a commutative diagram. Then the lower square is a pullback if and only if $\bar{i} : \text{Ker}(h) \rightarrow \text{Ker}(f)$ is an isomorphism.

Proposition (E. (2025); Everaert, Gran (2010))

Let $\alpha : F \Rightarrow G$ be in $\text{NEpi}(\mathcal{A}^{\mathbb{T}})$. TFAE:

- 1 α is a central extension.
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"3. \Leftrightarrow 2.":

$$\text{Ker}(\alpha) \text{ in } \mathcal{F} \quad \Leftrightarrow \quad \text{Ker}(\alpha)(T) = 0 \text{ for all } T \notin \mathbb{S}$$

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"3. \Rightarrow 1.":

Let $\text{Ker}(\alpha)$ be in \mathcal{F} .

$$\begin{array}{ccccc}
 \text{Ker}(\text{UF}(p_1)) & \longleftarrow \text{Ker}(p_1) & \xrightarrow{\cong} & \text{Ker}(\alpha) & \\
 \text{ker}(\text{UF}(p_1)) \downarrow & & & & \downarrow \text{ker}(\alpha) \\
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 \text{UF}(F) & \xleftarrow{\eta_F} & F & \xrightarrow{\alpha} & G
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Let $\text{Ker}(\alpha)$ be in \mathcal{F} .

Remember that $F : \mathcal{A}^{\mathbb{T}} \rightarrow \mathcal{F}$, as a left adjoint between additive categories, is additive and hence preserves split short exact sequences.

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- $\Rightarrow \eta_{\text{Ker}(p_1)}$ is an iso
- $\Rightarrow p_1$ is a trivial extension
- $\Rightarrow \alpha$ is a central extension

5. Higher central extensions and generalized Hopf formulae for homology

Observation (Janelidze (1991))

Remember that $\Gamma = (\text{Grp}, \text{Ab}, \text{ab}, \text{U}, \text{Surj}(\text{Grp}), \text{Surj}(\text{Ab}))$ is a Galois structure. A surjective group homomorphism $f : A \rightarrow B$ is a central extension if and only if $[\text{Ker}(f), A] = \{1\}$.

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Are there classes $\text{Surj}(\text{Grp})^1, \text{Surj}(\text{Ab})^1$ of morphisms in $\text{Ext}(\text{Grp}), \text{CExt}(\text{Grp})$ such that

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is a Galois structure?

Remember that, if $f : A \rightarrow B$ is a free presentation of B , then

$$H_1(B, \mathbb{Z}) = \frac{B}{[B, B]}, \quad H_2(B, \mathbb{Z}) = \frac{[A, A] \cap \text{Ker}(f)}{[\text{Ker}(f), A]}.$$

For certain classes $\text{Surj}(\text{Grp})^1, \text{Surj}(\text{Ab})^1$ of morphisms in $\text{Ext}(\text{Grp}), \text{CExt}(\text{Grp}),$
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A morphism in $\text{Surj}(\text{Grp})^1$

$$\begin{array}{ccc} A' & \xrightarrow{a} & A \\ f' \downarrow & & \downarrow f \\ B' & \xrightarrow{b} & B \end{array}$$

is central with respect to Γ_1 if and only if

$$[\text{Ker}(f'), \text{Ker}(a)] = \{1\} \text{ and } [\text{Ker}(f') \cap \text{Ker}(a), A'] = \{1\}.$$

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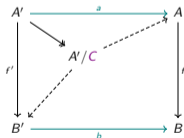
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$$\text{Ext}^2(\text{Grp}) \begin{array}{c} \xrightarrow{\text{ab}_2} \\ \perp \\ \xleftarrow{U_2} \end{array} \text{CExt}^2(\text{Grp})$$



Here $C = [\text{Ker}(f'), \text{Ker}(a)] \cdot [\text{Ker}(f') \cap \text{Ker}(a), A']$.

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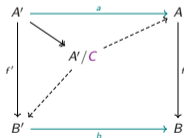
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Remember that, if (1) is a double free presentation of B , then

$$H_3(B, \mathbb{Z}) = \frac{[A', A'] \cap \text{Ker}(f') \cap \text{Ker}(a)}{[\text{Ker}(f'), \text{Ker}(a)] \cdot [\text{Ker}(f') \cap \text{Ker}(a), A']}$$

Categorical framework (Everaert, Gran, Van der Linden (2008))

Let

$$\Gamma = (\mathcal{C}, \mathcal{F}, F, U, \mathcal{E}, \mathcal{L})$$

be a 'good' Galois structure (e.g. \mathcal{C} semi-abelian, \mathcal{F} Birkhoff subcategory of \mathcal{C} , $U : \mathcal{F} \rightarrow \mathcal{C}$ inclusion functor with left adjoint $F : \mathcal{C} \rightarrow \mathcal{F}$, $\mathcal{E} = \text{NEpi}(\mathcal{C})$, $\mathcal{L} = \text{NEpi}(\mathcal{F})$).

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For any object $f : A \rightarrow B$ in $\text{Ext}(\mathcal{C})$, there exists a normal subobject

$$[f]_1 \in \mathcal{C}$$

of A such that $U_1 F_1(f : A \rightarrow B) = (A/[f]_1 \rightarrow B)$.

$$\Gamma = (\mathcal{C}, \mathcal{F}, F, U, \mathcal{E}, \mathcal{Z})$$

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For any object

$$\begin{array}{ccc} A' & \xrightarrow{a} & A \\ f' \downarrow & & \downarrow f \\ B' & \xrightarrow{b} & B \end{array}$$

in $\text{Ext}^2(\mathcal{C})$, there exists a normal subobject

$$[(a, b)]_2 \in \mathcal{C}$$

of A' such that

$$U_2 F_2((a, b)) = \begin{array}{ccc} A' / [(a, b)]_2 & \longrightarrow & A \\ \downarrow & & \downarrow f \\ B' & \xrightarrow{b} & B \end{array}$$

Definition

Let \mathcal{E} be a class of morphisms in a category \mathcal{C} .

- 1 An object P of \mathcal{C} is \mathcal{E} -projective if, given a morphism $f : A \rightarrow B$ in \mathcal{E} , any morphism $g : P \rightarrow B$ factors through f .

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- 3 A double \mathcal{E} -projective presentation of an object B of \mathcal{C} is given by a morphism

$$\begin{array}{ccc} P_2 & \longrightarrow & P_1 \\ \downarrow & & \downarrow \\ P_{\{1\}} & \longrightarrow & B \end{array}$$

in \mathcal{E}^1 , where $P_2, P_1, P_{\{1\}}$ are \mathcal{E} -projective objects.

Definition

Let $\Gamma = (\mathcal{C}, \mathcal{F}, F, U, \mathcal{E}, \mathcal{L})$ be a 'good' Galois structure, and B be an object in \mathcal{C} .

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In particular, $H_1(B, \mathcal{F})$, $H_2(B, \mathcal{F})$, $H_3(B, \mathcal{F})$ only depend on B , and lie in \mathcal{F} .

6. Double central extensions, and second and third Hopf formulae in abelian functor categories

Observe that $\Gamma = (\mathcal{A}^{\mathbb{T}}, \mathcal{F}, F, U, \text{NEpi}(\mathcal{A}^{\mathbb{T}}), \text{NEpi}(\mathcal{F}))$ is a 'good' Galois structure. Thus, we have the 'good' Galois structure

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We have seen that $\alpha : F \rightarrow G$ in $\text{NEpi}(\mathcal{A}^{\mathbb{T}})$ is central if and only if $\alpha_T : F(T) \rightarrow G(T)$ is an isomorphism for any $T \notin \mathbb{S}$, or equivalently, $\text{Ker}(\alpha) \in \mathcal{F}$.

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$$H_2(G, \mathcal{F}) = \frac{[F] \cap \text{Ker}(\alpha)}{[\text{Ker}(\alpha)]}.$$

Example

Consider $\text{Arr}(\mathcal{A}) \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{U} \end{array} \mathcal{A}$, and

$$\begin{array}{ccccc} A_1 & \xrightarrow{\ker((\eta_a)_1)} & A_1 & \xrightarrow{(\eta_a)_1} & 0 \\ \downarrow & & \downarrow a & & \downarrow \\ \text{Im}(a) & \xrightarrow{\ker((\eta_a)_1)} & A_0 & \xrightarrow{(\eta_a)_0} & \text{Cok}(a). \end{array}$$

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Consider projective presentation:

$$\begin{array}{ccccc}
 \text{Ker}(\rho_1) & \xrightarrow{\quad} & P_1 & \xrightarrow{\rho_1} & B_1 \\
 q|_{\text{Ker}(\rho_1)} \downarrow & & \downarrow q & & \downarrow b \\
 \text{Ker}(\rho_0) & \xrightarrow{\quad} & P_0 & \xrightarrow{\rho_0} & B_0
 \end{array}$$

Thus: $H_2(b, \mathcal{A}) = \frac{\text{Im}(q) \cap \text{Ker}(\rho_0)}{\text{Im}(q|_{\text{Ker}(\rho_1)})}$

Proposition (E. (2025))

Let

$$\begin{array}{ccc} F' & \xrightarrow{f} & F \\ \alpha' \downarrow & & \downarrow \alpha \\ G' & \xrightarrow{g} & G \end{array} \quad (2)$$

be in $\text{NEpi}(\mathcal{A}^{\mathbb{T}})^1$.

- 1** (2) is central with respect to $\Gamma_1 = (\text{Ext}(\mathcal{A}^{\mathbb{T}}), \text{CExt}(\mathcal{A}^{\mathbb{T}}), F_1, U_1, \text{NEpi}(\mathcal{A}^{\mathbb{T}})^1, \text{NEpi}(\mathcal{F})^1)$ if and only if, for any $T \notin \mathbb{S}$, α'_T and f_T are jointly monic, or equivalently, $\text{Ker}(\alpha') \cap \text{Ker}(f) \in \mathcal{F}$.

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2 It holds that

$$[(f, g)]_2 = [\text{Ker}(\alpha') \cap \text{Ker}(f)].$$

3 If (2) is a double $\text{NEpi}(\mathcal{A}^{\mathbb{T}})$ -projective presentation of G , then

$$H_3(G, \mathcal{F}) = \frac{[F'] \cap \text{Ker}(\alpha') \cap \text{Ker}(f)}{[\text{Ker}(\alpha') \cap \text{Ker}(f)]}.$$

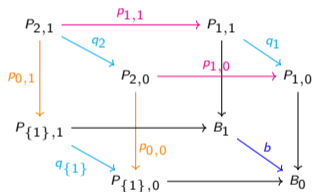
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Example

Consider $\text{Arr}(\mathcal{A}) \xrightarrow{\text{F}} \mathcal{A} \xleftarrow{\text{U}} \text{Arr}(\mathcal{A})$. Recall that $([a])_0 = \text{Im}(a)$.

Consider double projective presentation:



$$\text{Thus: } H_3(b, \mathcal{A}) = \frac{\text{Im}(q_2) \cap \text{Ker}(p_{0,0}) \cap \text{Ker}(p_{1,0})}{\text{Im}(q_2|_{\text{Ker}(p_{0,1})} \cap \text{Ker}(p_{1,1}})}$$

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