On the total degree of a finite group

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1_G	trivial character

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We:	is there $1 < H < G$ with $1_H^G = \tau_G$?

context: call $H < G$ **rich** in G if $[1_H^G, \chi] \geq 1$ for all $\chi \in \mathrm{Irr}(G)$

special case $1_H^G = \tau_G$: H is "just rich" in G

Examples

$$T(G) = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1) \leq \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 = |G| \leq \left(\sum_{\chi \in \mathrm{Irr}(G)} \chi(1) \right)^2 = T(G)^2$$

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- $G = S_3 = D_6$:

$$\begin{array}{r|rrr} 1_G & 1 & 1 & 1 \\ \chi_2 & 1 & 1 & -1 \\ \chi_3 & 2 & -1 & 0 \\ \hline \tau_G & 4 & 1 & 0 \end{array}$$

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$$T(G) = \begin{cases} q^2 & ; \quad 2 \mid q \\ q(q+1)/2 + 1 & ; \quad q \equiv 1 \pmod 4 \\ q(q+1)/2 & ; \quad q \equiv 3 \pmod 4 \end{cases}$$

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- $G = A_5 \cong \mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5)$: $\qquad T(G) = 16$

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  Proof:

$$\begin{aligned} T(G) - T(H) &= ((\tau_G)_H - \tau_H)(1) \\ &= \sum_{\psi \in \mathrm{Irr}(H)} ([\psi, (\tau_G)_H] - 1)\,\psi(1) \\ &> 0 \end{aligned}$$

(Note: $[1_H, (\tau_G)_H] = [1_H^G, \tau_G] \geq 2$)

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(i) G is simple, $|G| < n^2$,
if $n = 100$ then $|G|$ is prime or
$G \in \{A_5, \mathrm{PSL}(2,7), A_6, \mathrm{PSL}(2,8), \mathrm{PSL}(2,11), \mathrm{PSL}(2,13)\}$

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(ii) G has a minimal normal subgroup N that is solvable,
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N is an irreducible module for G/N,
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(iii) G has a minimal normal subgroup $N \cong S^k = S \times S \times \cdots \times S$
for S nonabelian simple with $T(N) = T(S)^k < n$,
S as above: $k = 1$,
if not case (ii) then $G \leq \mathrm{Aut}(S)$

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- After finitely many steps:
  work list becomes empty, and **result list** is complete.

$T(G)$	$\#G$	$\|G_{\max}\|$	$G_{\max}$
2	1	2	2
3	1	3	3
4	3	6	S_3
5	1	5	5
6	5	12	A_4
7	1	7	7
8	7	20	$5:4$
9	3	21	$7:3$
10	7	24	S_4
11	1	11	11
12	16	42	$7:6$
13	1	13	13
14	7	72	$3^2:Q_8$
15	5	55	$11:5$
16	23	72	$3^2:8$
17	1	17	17
18	19	78	$13:6$
19	1	19	19
20	35	110	$11:10$
21	2	57	$19:3$
22	13	72	$(S_3 \times S_3):2$, $(3 \times A_4):2$, $(2^2:9):2$
23	1	23	23
24	55	156	$13:12$

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25	2	25	5^2, 25
26	8	144	$3^2:QD_{16}$
27	7	171	$19:9$
28	61	168	$\mathrm{PSL}(3,2)$
29	1	29	29
30	34	240	$2^4:15$
31	1	31	31
32	126	300	$5^2:(3:4)$
33	7	253	$23:11$
34	10	120	$5:S_4$
35	3	203	$29:7$
36	123	600	$5^2:\mathrm{SL}(2,3)$
37	1	37	37
38	13	216	$3^3:Q_8$
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43	1	43	43
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  If $N \trianglelefteq G$ then $T(G) \geq T(G/N) + T(N) - 1$

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- Large groups G with given $T(G)$ are often Frobenius groups.

Frobenius groups

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Proposition:
If $T(G) = p^2$ for a prime p then G is abelian or
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p	$p \cdot (p^2 - p + 1)$	G
2	$2 \cdot 3$	S_3
3	$3 \cdot 7$	$7{:}3$
5	$5 \cdot 21$	
7	$7 \cdot 43$	$43{:}7$
11	$11 \cdot 111$	
13	$13 \cdot 157$	$157{:}13$

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46	19	360	A_6
48	400	600	$5^2:24$
49	3	301	$43:7$
50	25	432	$((3^2:Q_8):3):2$
51	7	243	$(9:9):3$, $(9:9):3$, $(9 \times 3).3^2$
52	257	702	$3^3:26$
54	72	666	$37:18$
55	1	55	55
56	799	812	$29:28$
57	1	57	57
58	25	784	$7^2:Q_{16}$
60	309	1176	$7^2:\mathrm{SL}(2,3)$
62	12	992	$2^5:31$
63	13	903	$43:21$
64	943	784	$7^2:16$

$T(G)$	$\#G$	$\lvert G_{\max}\rvert$	$G_{\max}$
65	3	689	$53:13$
66	62	2352	$7^2:(\mathrm{SL}(2,3).2)$
68	256	500	$(5^2:5):4$
69	7	1081	$47:23$
70	43	610	$61:10$
72	1288	1332	$37:36$
74	11	216	$(3^2:3):D_8$,
			$(3^2:3):D_8$,
			$(3^2:3):Q_8$
75	10	915	$61:15$
76	551	1200	$5^2:(24:2)$
77	3	737	$67:11$
78	56	2352	$7^2:(3\times Q_{16})$
80	2507	1640	$41:40$
81	16	657	$73:9$
82	22	720	$3^2:((5\times Q_8):2)$,
			$\mathrm{PGL}(2,9)$
84	947	1806	$43:42$
85	2	405	$3^4:5$

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86	15	960	$2^4:A_5$
87	13	1711	$59:29$
88	11384	1474	$67:22$
90	161	4032	$2^6:(7:9)$
91	4	1027	$79:13$
92	746	2162	$47:46$
93	5	1053	$(3^3:13):3$
94	35	648	$(2^2:((9\times 3):3)):2$,
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			$(2^2:((9\times 3):3)):2$,
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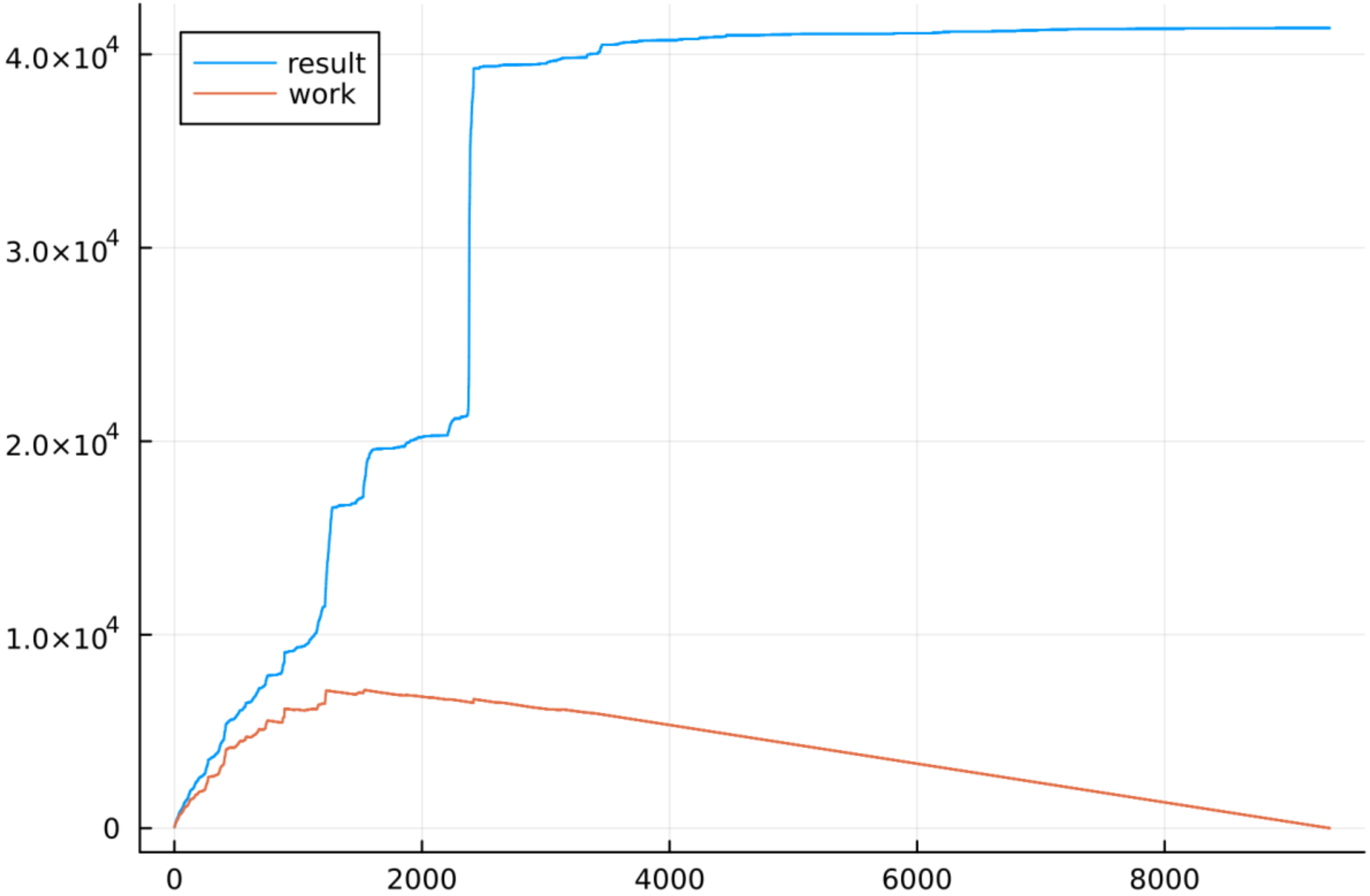
			$(2^2:(3^2.3^2)):2$
95	1	95	95
96	18652	3240	$3^4:(5:8)$
98	32	1344	$2^3:\mathrm{PSL}(3,2)$,
			$2^3.\mathrm{PSL}(3,2)$
99	35	2211	$67:33$
100	660	1620	$3^4:20$

Numbers of 2-groups of small total degree

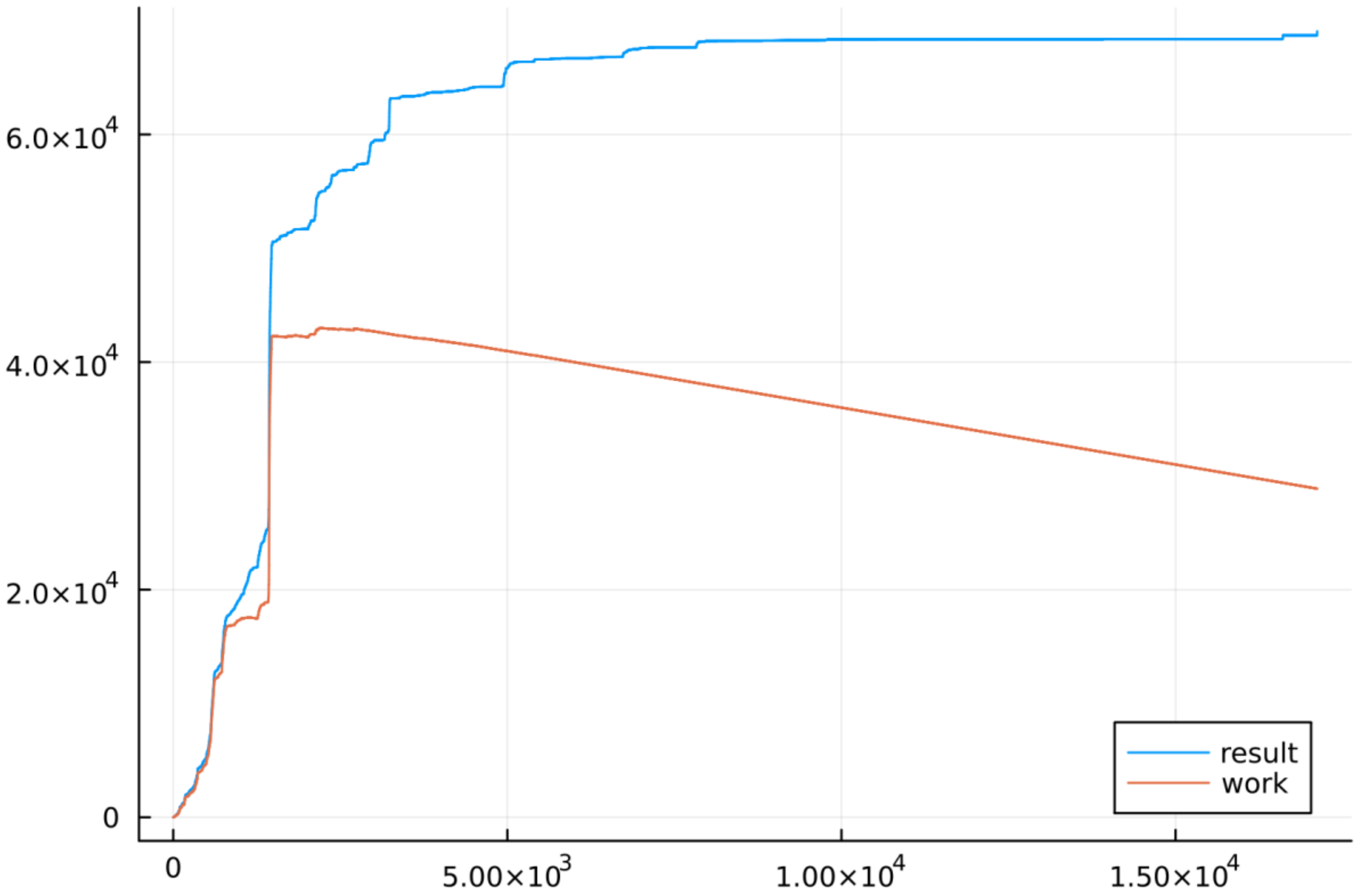
T	2^1	2^2	2^3	2^4	2^5	2^6	2^7
2	1						
4		2					
6			2				
8			3				
10				3			
12				6			
14							
16				5	5		
18					3		
20					21		
22							
24					15	6	
26							
28						29	
30							
32					7	72	4
34						3	
36						48	5
38							
40						67	30
42							
44							27
46							
48						31	224
50							

T	2^6	2^7	2^8	2^9
52		196		
54				
56		655	16	
58				
60		102	10	
62				
64	11	565	208	
66		3		
68		32	54	
70				
72		231	554	
74				
76			350	
78				
80		179	1828	36
82				
84			576	8
86				
88			10893	61
90				
92			498	38
94				
96		60	17501	366
98				
100			266	70

Classify $T(G) \leq 100$



Try to classify $T(P) \leq 120$ for 2-groups P



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- Test candidates G for isomorphism (cheap if `IdGroup` is available; **not** for order 512)
- Compute $T(G)$ (cheap if G is supersolvable)

What do we have to compute?

- Given F, compute the irreducible modules N
- Given F and N, compute extensions $1 \to N \to G \to F \to 1$ (up to equivalence)
- Test candidates G for isomorphism (cheap if `IdGroup` is available; **not** for order 512)
- Compute $T(G)$ (cheap if G is supersolvable)
- Ideally, exclude candidates N for F **a priori**

A lower bound for $T(G)$

Let p be a prime, $e \in \{0, 1\}$, $|G| = p^{2m+e}$.

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Idea of the proof:

$T(G|\lambda)p^{m+e-1} \geq |G/Z|$ for $\lambda \in \mathrm{Irr}(Z)$.

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(The bound is sharp only for small $|G|$.)

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  - and some with more complicated structure ...

Thank you for your attention!