

Pointed Hopf algebras over Chevalley groups

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Pointed Hopf algebras over finite simple groups

Work in progress. N. A., G. Carnovale, István Heckenberger (Marburg), Leandro Vendramin (Brussels).

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I. Overview.

Let $\mathbb{k}$ be an algebraically closed field of characteristic 0.

Problem: Classify (or characterize) finite dimensional Hopf algebras.

First stage: Classify (or characterize) finite dimensional *pointed* Hopf algebras.

A Hopf algebra is pointed if every simple comodule is one-dimensional. Group algebras, enveloping algebras, quantized enveloping algebras of Lie (super)algebras and their finite-dimensional versions are pointed.

Basic invariants of a pointed Hopf algebra H are:

- the group $G(H) = \{x \in H^\times : \Delta(x) = x \otimes x\}$ (group-like elements),
- the infinitesimal braiding V (a Yetter-Drinfeld module over $\mathbb{k}G(H)$, (i.e., a module over the Drinfeld double $D(\mathbb{k}G(H))$))
- the Nichols algebra $\mathcal{B}(V) = \bigoplus_{n \geq 0} \mathcal{B}^n(V)$, a connected Hopf algebra in ${}_{\mathbb{k}G(H)}^{\mathbb{k}G(H)}\mathcal{YD}$.

Actually, $\mathcal{B}(V) \# \mathbb{k}G(H) \hookrightarrow \text{gr } H$ (*the graded Hopf algebra associated to the coradical filtration*).

These invariants form the core of the method for the study of (pointed) Hopf algebras proposed in [A-Schneider 1998, 2000].

The question must be addressed by considering different types of groups: Classify (or characterize) finite dimensional *pointed* Hopf algebras H with $G(H)$ in some class.

- $G(H)$ **abelian**.

Theorem. [A-Schneider 2010]. The classification is known for $G(H)$ abelian such that $(|G(H)|, 210) = 1$.

Without this hypothesis, there is a characterization [Heckenberger 2006, Angiono 2011, Angiono-García Iglesias 2017] following [AS].

- $G(H)$ solvable.

- [A-Heckenberger-Schneider 2007; using results by Fomin-Kirillov, Milinski-Schneider, A-Graña and A-Zhang]. The classification for $\mathbb{S}_3$ is achieved.
- [Heckenberger-Vendramin 2017]. V (the infinitesimal braiding) decomposable (semisimple but not simple), G not necessarily solvable.
- [A-Heckenberger-Vendramin 2024]. V simple, G solvable (there are essentially 10 finite-dimensional Nichols algebras *not coming from the abelian groups*).
- For example, $\dim H$ odd $\rightsquigarrow$ essentially like in the abelian case.

- $G(H)$ alternating.

- [A-Fantino 2007].

Any finite-dimensional complex pointed Hopf algebra H with $G(H) \simeq \mathbb{A}_5$ is isomorphic to the group algebra $\mathbb{k}\mathbb{A}_5$.

- [A-Fantino-Graña-Vendramin 2011].

Let $n \geq 5$. Any finite-dimensional complex pointed Hopf algebra H with $G(H) \simeq \mathbb{A}_n$ is isomorphic to the group algebra $\mathbb{k}\mathbb{A}_n$.

Definition. [A-Fantino-Graña-Vendramin 2011].

A finite group *collapses* if any finite-dimensional complex pointed Hopf algebra H with $G(H) \simeq G$ has $H \simeq \mathbb{k}G$.

- $G(H)$ simple non-abelian.

Conjecture. [A-Fantino-Graña-Vendramin 2011].

A finite simple non-abelian group G *collapses*

(Any finite-dim. complex pointed Hopf alg. H with $G(H) \simeq G$ has $H \simeq \mathbb{k}G$.)

Classification Theorem of finite simple groups.

- C_p cyclic groups of prime order.
- A_n , $n \geq 5$, alternating groups.
- Finite groups of Lie type.
- 27 sporadic groups.

Finite groups of Lie type:

- Chevalley groups.

"Algebraic groups over finite fields", split versions, e.g., $PSL_2(q) := SL_2(q)/Z$, $q = p^n$.

- Steinberg groups.

"Algebraic groups over finite fields", non-split versions, e.g., $PSU_3(q) := SU_3(q)/Z$.

- Suzuki-Ree groups.

"Algebraic groups over finite fields", degenerate versions in small characteristic.

Currently, we know that the following simple groups collapse:

- ◇ [A-Fantino-Graña-Vendramin 2011] The alternating groups $\mathbb{A}_n$, $n \geq 5$.
- ◇ [A-Fantino-Graña-Vendramin 2011] The sporadic groups, including the Tits group, other than the Fischer group Fi_{22} , the Baby Monster B and the Monster M .
- ◇ [A-Carnovale-García 2017] The groups $\mathbf{PSL}_2(q)$, where either $q \equiv 1 \pmod{4}$ or $q > 2$ is even.
- ◇ [Carnovale-Costantini 2021] The Suzuki–Ree groups.
- ◇ [A-Carnovale-García 2024] The groups $\mathbf{PSL}_{n+1}(q)$, ($n \geq 3$), $\mathbf{PSL}_3(q)$ ($q > 2$), and $\mathbf{PSp}_{2n}(q)$ ($n \geq 3$).

Theorem. [A-Carnovale, March 2026].

The finite simple Chevalley groups, other than $\mathrm{PSL}_2(q)$ with $q \equiv 3 \pmod{4}$ (and than $\mathrm{PSL}_3(2) \simeq \mathrm{PSL}_2(7)$), collapse.

Because of the Classification Theorem of Finite Simple Groups, we see that the Conjecture remains open only for

- $\mathrm{PSL}_2(q)$ with $q \equiv 3 \pmod{4}$,
- the Steinberg groups,
- the Fischer group Fi_{22} , the Baby Monster B and the Monster M (all three settled, N.A., G. C., I. H., L. V., work in progress).

II. The framework.

Definition. A braided vector space is a pair (V, c) where V is a vector space and $c : V \otimes V \rightarrow V \otimes V$ is a linear automorphism that satisfies the braid equation (equivalent to the qYBE):

$$(c \otimes \text{id})(\text{id} \otimes c)(c \otimes \text{id}) = (\text{id} \otimes c)(c \otimes \text{id})(\text{id} \otimes c).$$

If (V, c) is a b. v. s., then the braid group $\mathbb{B}_n$ acts on $V^{\otimes n}$ by

$$\sigma_i \longmapsto \text{id}_{V^{\otimes(i-1)}} \otimes c \otimes \text{id}_{V^{\otimes(n-i-1)}}$$

Most applications of the qYBE arise from these representations.

There is a group epimorphism $\mathbb{B}_n \twoheadrightarrow \mathbb{S}_n$, $\sigma_i \mapsto s_i = (i, i+1)$.

In 1986, Drinfeld found a method to construct braided vector spaces from Hopf algebras (with bijective antipode).

Definition. Let G be a finite group. A Yetter-Drinfeld module over the group algebra $\mathbb{k}G$ is a vector space V provided with

- a (linear) G -action $\cdot : G \times V \rightarrow V$,

- a grading $V = \bigoplus_{g \in G} V_g$,

such that $g \cdot V_h = V_{ghg^{-1}}$, $\forall h, g \in G$.

Then V is a braided vector space with braiding $c : V \otimes V \rightarrow V \otimes V$

$$c(v \otimes w) = g \cdot w \otimes v, \quad v \in V_g, w \in V.$$

Facts.

- The category ${}_{\mathbb{k}G}^{\mathbb{k}G}\mathcal{YD}$ of Yetter-Drinfeld modules over G is semisimple.
- The simple objects in ${}_{\mathbb{k}G}^{\mathbb{k}G}\mathcal{YD}$ are $M(\mathcal{O}, \rho) := \text{Ind}_{G^g}^G \rho$ where $\mathcal{O} \subset G$ is a conjugacy class and ρ is an irreducible representation of the isotropy group G^g for a fixed $g \in \mathcal{O}$.

Definition/Lemma. [A.-Schneider 1998] Given $V \in {}_{\mathbb{k}G}^{\mathbb{k}G}\mathcal{YD} \exists!$ (up to $\simeq$) graded Hopf algebra $R = \bigoplus_{n \geq 0} R^n$ in ${}_{\mathbb{k}G}^{\mathbb{k}G}\mathcal{YD}$ such that

- $R^0 = \mathbb{k}$,
- $R^1 \simeq V$ in ${}_{\mathbb{k}G}^{\mathbb{k}G}\mathcal{YD}$,
- $R^1 = \{x : \Delta(x) = x \otimes 1 + 1 \otimes x\}$,
- $R = \mathbb{k}\langle R^1 \rangle \leftarrow T(V)$.

The braided Hopf algebra R is called the **Nichols algebra** of V and is denoted by $\mathcal{B}(V) = R$.

Lemma. [AFGV 2011] Let G be a finite group. TFAE:

- G collapses.
- If $0 \neq V \in \mathbb{k}_G^G \mathcal{YD}$, then $\dim \mathcal{B}(V) = \infty$.
- If $V \in \mathbb{k}_G^G \mathcal{YD}$ is irreducible, then $\dim \mathcal{B}(V) = \infty$.

So we are reduced to

Problem: Given a finite simple group G and $V \in \mathbb{k}_G^G \mathcal{YD}$ irreducible, prove that

$$\dim \mathcal{B}(V) = \infty.$$

Now, $\mathcal{B}(V) \simeq T(V)/\mathcal{J}(V)$, where $\mathcal{J}(V)$ is the maximal graded Hopf ideal generated by elements in $\bigoplus_{n \geq 2} T^n(V)$.

Crucial question: determine $\mathcal{J}(V)$ (it depends only on c).

Drawback:

Although it turns out that the notion of Nichols algebra is equivalent to several other definitions (by Nichols, Rosso, Woronowicz, Lusztig...), there is no general way to compute the ideal $\mathcal{J}(V)$.

However, there is one favorable fact: there are criteria that guarantee that the dimension is infinite.

III. Racks and conjugacy classes.

Problem: Given a finite simple group G , $\mathcal{O} \subset G$ a conjugacy class and ρ an irrep of G^g (for a fixed $g \in \mathcal{O}$) prove that

$$\dim \mathcal{B}(M(\mathcal{O}, \rho)) = \infty.$$

Fact. If V is a braided vector space and $W \hookrightarrow V$ is a braided subspace, then

$$\mathcal{B}(W) \hookrightarrow \mathcal{B}(V), \quad \text{hence } \dim \mathcal{B}(W) \leq \dim \mathcal{B}(V).$$

Now, support of $M(\mathcal{O}, \rho) = \{g \in G : M(\mathcal{O}, \rho)_g \neq 0\} = \mathcal{O}$ is a *rack*.

So we consider braided subspaces of $M(\mathcal{O}, \rho)$ via subracks of $\mathcal{O}$.

A rack is a pair $(X, \triangleright)$, $X \neq \emptyset$ a set, $\triangleright : X \times X \rightarrow X$ such that

- $\varphi_x := x \triangleright _$ is a bijection for all $x \in X$,
- $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$ for all $x, y, z \in X$.

Main example: X a conjugacy class in G , $x \triangleright y := xyx^{-1}$.

- $\mathfrak{q} : X \times X \rightarrow \mathbb{k}^\times$ is a 2-cocycle $\iff \mathfrak{q}_{x \triangleright y, x \triangleright z} \mathfrak{q}_{x, z} = \mathfrak{q}_{x, y \triangleright z} \mathfrak{q}_{y, z}$ for all $x, y, z \in X$; e.g., any constant $X \times X \rightarrow \mathbb{k}^\times$ is a 2-cocycle.

Let $V = \mathbb{k}X$ and $c^{\mathfrak{q}} : V \otimes V \rightarrow V \otimes V$ be given by

$$c^{\mathfrak{q}}(e_x \otimes e_y) = \mathfrak{q}_{x, y} e_{x \triangleright y} \otimes e_x, \quad x, y \in X, .$$

Then: $\mathfrak{q}$ is a cocycle $\iff c^{\mathfrak{q}}$ is a solution of the braid equation.

Remark: This extends to $n > 1$: $\mathfrak{q} : X \times X \rightarrow \mathbf{GL}(n, \mathbb{k})$,
 $V = \mathbb{k}X \otimes \mathbb{k}^n$ and $c^{\mathfrak{q}} : V \otimes V \rightarrow V \otimes V$ given by

$$c^{\mathfrak{q}}(e_x v \otimes e_y w) = e_{x \triangleright y} \mathfrak{q}_{x,y}(w) \otimes e_x v, \quad x, y \in X, \quad v, w \in \mathbb{k}^n.$$

We say that a rack is *abelian* if $x \triangleright y = y$ for any $x, y \in X$.

Remark: Given a finite simple group G , $\mathcal{O} \subset G$ a conjugacy class and ρ an irrep of G^g , suppose that X is a subrack of $\mathcal{O}$. Then $V_X := \bigoplus_{g \in X} M(\mathcal{O}, \rho)_g$ is a braided subspace of $M(\mathcal{O}, \rho)$.

If X is abelian, then V_X is of diagonal type $\leftarrow$ comes from abelian groups $\leftarrow$ known by [Heckenberger 2006].

These arguments are collectively called *abelian techniques*.

Definition. A conjugacy class $\mathcal{O}$ of a finite group G is of type C when there are

- $H \leq G$ and
- $r, s \in H \cap \mathcal{O}$ such that

- $rs \neq sr$,
- $H = \langle \mathcal{O}_r^H, \mathcal{O}_s^H \rangle$,
- $\mathcal{O}_r^H \neq \mathcal{O}_s^H$ and
- $\min\{|\mathcal{O}_r^H|, |\mathcal{O}_s^H|\} > 2$ or $\max\{|\mathcal{O}_r^H|, |\mathcal{O}_s^H|\} > 4$.

Theorem. [A-Carnovale-García 2017] If a conjugacy class $\mathcal{O}$ is type C, then $\dim \mathcal{B}(\mathcal{O}, \rho) = \infty$ for any irrep ρ of G^g , fixed $g \in \mathcal{O}$.

IV. Nichols algebras over Chevalley groups.

Let G be a finite simple Chevalley group. For $g \in G$ there are three possibilities:

- ◇ g is unipotent,
- ◇ g is semisimple,
- ◇ g is neither unipotent nor semisimple. In this case, $g = g_s g_u$ (Chevalley-Jordan decomposition) $\rightsquigarrow$ it reduces to the unipotent case.

- ◇ The unipotent conjugacy classes are not difficult to describe. Using the knowledge of such classes in the corresponding algebraic group, we can decide when it is of type C; in the remaining cases, we apply abelian techniques. This was done in a series of papers [ACG I, . . . , ACG V, Carnovale-Costantini VI, ACG VII].
- ◇ The semisimple conjugacy classes are very difficult to deal with, specially those that are not split (e.g. not diagonal).

For them we had to introduce two new techniques, one using the main result of [AHV], the other reducing to verifying three Lie-theoretical conditions.

Example. Let $q = p^n$, p prime.

$$\text{Let } G = \mathbf{SL}_2(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{F}_q^{2 \times 2} : ad - bc = 1 \right\}.$$

A (finite) maximal torus = maximal subgroup consisting of semisimple elements = maximal subgroup whose order is coprime to p

Fact. G has up to conjugacy two finite maximal tori:

- The diagonal (or split) torus $\left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{F}_q^\times \right\}$.
- The non-split torus = $\text{Cent}_G \begin{pmatrix} 0 & b \\ 1 & d \end{pmatrix}$, where $T^2 - dT - b$ is irreducible in $\mathbb{F}_q[T]$.

Definition. [A-Carnovale] A finite rack Y is of *type* Ω if it has a solvable subrack X that does not belong to the list of racks supporting the 10 possible Nichols algebras in [AHV].

Theorem. [A-Carnovale]

Let Y be a finite rack of type Ω . Then Y collapses.

(This is a consequence of [AHV]).

We fix the following data:

- G is a finite group,
- $M \leq G$,
- $T \leq N_G(M)$ is an abelian subgroup and
- $r \in T$.

Theorem. [A-Carnovale] Assume that

- $M \leq G$ is quasi-simple (M is perfect and $M/Z(M)$ is simple.),
- $r \notin \text{Cent}_G(M)$,
- $(\mathcal{O}_r^G \cap T) \setminus ((\mathcal{O}_r^M \cap T) \cup \text{Cent}_G(M)) \neq \emptyset$.

Then $\mathcal{O}_r^G$ is of type C.

We then translate this to Lie group terms and investigate which semisimple orbits can be dealt with these ideas.

We fix

- a simple algebraic group $\mathbb{G}$ over $\mathbb{k} := \overline{\mathbb{F}_p}$,
- $w \in W$ (the Weyl group); $\mathcal{T}^{F_w}$ the corresponding finite torus.

Theorem Potente. [A-Carnovale] Let $\mathbb{M}$ be a w -suitable subgroup of $\mathbb{G}$ and $t \in \mathcal{T}^{F_w} \setminus e$. Assume that the following conditions hold:

$\pi([M, M])$ is quasi-simple (quasi-simplicity); □ Q

$[t, [M, M]_p] \neq e$ (non-centrality); □ N

$\text{Cent}_W(w\theta) \cdot t \setminus \left(\left(\mathbf{Z} \text{Cent}_{W_{\mathbb{M}}}(w\theta) \cdot t \right) \cup \text{Cent}_{\mathcal{T}}([M, M]_p) \right) \neq \emptyset$ □ W
 (orbit condition).

Then, $\mathcal{O}$ is of type C.

Summarizing, let $\mathcal{O}$ be a semisimple conjugacy class in a (finite) Chevalley group.

- First, we see if $\mathcal{O}$ satisfies the conditions of Theorem *Potente*. If so, $\mathcal{O}$ is of type C .
- If not, we see if $\mathcal{O}$ is of type Ω . If so, $\dim \mathcal{B}(M(\mathcal{O}, \rho)) = \infty$ for any ρ .
- If not, we apply abelian techniques (depend on ρ).

Luckily, it always works!

Thank you for your attention!

V. Simple racks.

Let X be a rack (a conjugacy class of a finite group) and consider morphisms of racks

$$\begin{array}{ccc} Z & & \\ \pi \downarrow & & \\ X & \xhookrightarrow{\iota} & Y \end{array} \quad \pi \text{ surjective, } \iota \text{ injective.}$$

If X is of type C, then Z and Y are of type C.

Definition. A rack X is simple if for any surjective map of racks $X \xrightarrow{\pi} Y$, either $|Y| = 1$ or π is an isomorphism.

A finite rack admits a surjective map onto a simple rack.

Theorem. [A-Graña (with help of Guralnick), Joyce]. Let X be a finite simple rack with $|X|$ elements. Then either

(I) $|X| = p^n$ where p is a prime and $n \in \mathbb{N}$. In this case, essentially X is an affine rack $(\mathbb{F}_q, T)$, where T acts irreducibly; or

(II) $|X|$ is divisible by at least two primes. In this case, there are a simple non-abelian group L , $t \in \mathbb{N}$, and $\theta \in \text{Aut } L$, such that X is a twisted conjugacy class of type (G, T) , where

$$G = L^t \quad \text{and} \quad T \in \text{Aut}(L^t) \text{ acts by}$$

$$T(\ell_1, \dots, \ell_t) = (\theta(\ell_t), \ell_1, \dots, \ell_{t-1}), \quad \ell_1, \dots, \ell_t \in L.$$

Furthermore, L and t are unique, and T only depends on its conjugacy class in $\text{Out}(L^t) = \text{Aut}(L^t)/\text{Inn}(L^t)$.

Problem: [A-Fantino-Graña-Vendramin 2011]

Determine the simple racks X for which there exist a finite group G and $V \in \mathbb{k}_G^G \mathcal{YD}$ such that $\dim \mathcal{B}(V) < \infty$ and $\sup V = X$.

Theorem. [Heckenberger-Meir-Vendramin 2022], [A-H-V 2024]

Let X be a simple rack such that $|X| = p^n = q$, p prime.

If there exist a finite group G and $V \in \mathbb{k}_G^G \mathcal{YD}$ s. t. $\dim \mathcal{B}(V) < \infty$ and $\sup V \simeq X$ as racks, then

$X \simeq \text{Aff}(\Gamma, \phi)$ with $\Gamma = \mathbb{F}_q$ and $\phi =$ multiplication by $d \in \mathbb{F}_q^\times$, where

$$(\mathbb{F}_q, d) \in \{(\mathbb{F}_3, 2), (\mathbb{F}_4, w), (\mathbb{F}_5, 2), (\mathbb{F}_5, 3), (\mathbb{F}_7, 3), (\mathbb{F}_7, 5)\}.$$

Here $w \in \mathbb{F}_4 \setminus \{0, 1\}$.

What happens with the simple racks of the second kind?

Conjugacy classes of finite simple non-abelian groups are simple racks. For many of them, it is still unknown whether they support a finite dimensional Nichols algebra.

We know just one simple rack X for which there exist a finite group G and $V \in {}^{\mathbb{k}G}_{\mathbb{k}G}\mathcal{YD}$ such that $\dim \mathcal{B}(V) < \infty$ and $\sup V = X$:

$X = \mathcal{O}_2^5 =$ transpositions of 5 elements, $G = \mathbb{S}_5$, there are two possible V 's (the rest was proved to be infinite-dimensional), one of them gives $\mathcal{B}(V) \simeq FK(5)$ (the other has the same finite dimension).

Problem: What happens with $X = \mathcal{O}_2^6 =$ transpositions of 6 elements, $G = \mathbb{S}_6$?

There are two possible V 's (the rest was proved to be infinite-dimensional); does one of them give $\mathcal{B}(V) \simeq FK(6)$? Finite dimension?

Thank you for your attention!