Idempotents in quandle rings

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- Let V be a vector space over a field $\mathbb{F}$. Let $R : V \otimes V \to V \otimes V$ be a $\mathbb{F}$-linear map and R_{ij} denotes the map $V \otimes V \otimes V \to V \otimes V \otimes V$ that acts as R on the (i, j)-th tensor factors and as the identity on the remaining third factor. Then R is called a solution to the quantum Yang-Baxter equation (QYBE) if it satisfies the equation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.$$

- The QYBE first appeared explicitly in the work of Yang [1967] and Baxter [1972]. Nevertheless, the equation had been lurking in several earlier works, known as the star-triangle relation or the triangle equation.
- The linear map $T : V \otimes V \to V \otimes V$ given by $T(v \otimes w) = w \otimes v$ is trivially a solution to the QYBE.
- We can see that a linear map $R : V \otimes V \to V \otimes V$ is a solution to the QYBE if and only if the map $S = T\ R$ satisfies the Yang-Baxter equation (YBE)

$$S_{12} S_{23} S_{12} = S_{23} S_{12} S_{23}.$$

- Since T is invertible, it follows that solutions to the QYBE are in bijection with solutions to the YBE.

- Let X be a set and $r : X \times X \to X \times X$ a map satisfying the braid relation
$$r_{12}\, r_{23}\, r_{12} = r_{23}\, r_{12}\, r_{23},$$
where $r_{ij} : X \times X \times X \to X \times X \times X$ acts as r on the (i, j)-th factors and as the identity on the remaining third factor. Then the pair (X, r) is called a set-theoretical solution to the YBE.
- If X is a basis for a vector space V, then a set theoretic solution (X, r) induces a solution to the YBE. In 1992, Vladimir Drinfeld proposed the program to classify all set-theoretical solutions to the YBE.
- In what follows, we will focus on a specific class of solutions that are intimately related to knots and links in the Euclidean 3-space.

- Racks and quandles are algebraic structures with a binary operation satisfying axioms modelled on the three Reidemeister moves of planar diagrams of knots and links in the 3-space.
- More precisely, a quandle is a non-empty set X with a binary operation $(x, y) \mapsto x * y$ satisfying the following axioms:
    1. Idempotency: $x * x = x$ for all $x \in X$.
    2. Invertibility: For any $x, y \in X$, there exists a unique $z \in X$ such that $x = z * y$.
    3. Self-Distributivity: $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in X$.
- A rack is a non-empty set X with a binary operation that satisfy the invertibility and the self-distributivity axiom.
- The invertibility and the self-distributivity axioms are equivalent to the map $S_y : X \to X$, given by $S_y(x) = x * y$, being an automorphism of X for each $y \in X$.
- Racks and quandles give bijective non-degenerate set-theoretical solutions to the YBE. In fact, if $(X, *)$ is a rack, then the map $r : X \times X \to X \times X$ given by $r(x, y) = (y, x * y)$ is such a solution to the YBE.

- A knot K is a smooth/piecewise linear embedding of a circle $\mathbb{S}^1$ into $\mathbb{R}^3$.
- Two knots K_1 and K_2 are said to be equivalent if K_1 can be transformed into K_2 via an ambient isotopy.
- More precisely, there exists a continuous mapping $H : \mathbb{R}^3 \times [0, 1] \to \mathbb{R}^3$ such that:

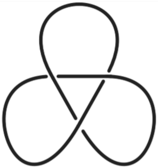
(i) for each $t \in [0, 1]$, the map $x \mapsto H(x, t)$ is a homeomorphism of $\mathbb{R}^3$ onto itself

(ii) $H(x, 0) = x$ for all $x \in \mathbb{R}^3$

(iii) $H(K_1, 1) = K_2$.

- Links and their equivalence is defined analogously.
- Roughly speaking, two knots/links are equivalent if we can turn one into the other simply by wiggling, and not by cutting and gluing.

Examples of knots and links



Trefoil

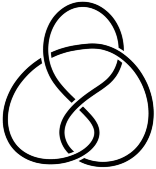
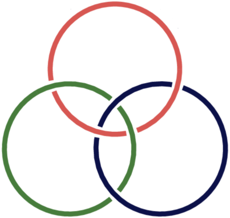


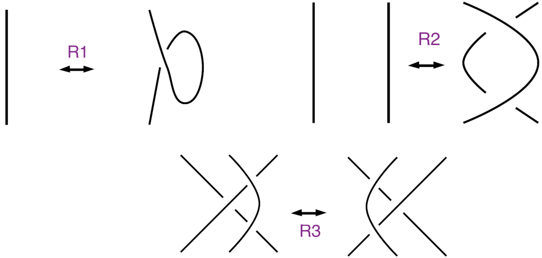
Figure eight

Hopf link



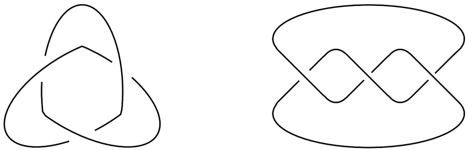
Borromean Rings

- Fortunately, there is an easier way to study knots/links, simply as 4-valent graphs.
- If K is a knot/link in $\mathbb{R}^3$, then its projection P on the xy-plane is called a regular projection if the preimage of every point in P consists of either one or two points of K.
- If P is a regular projection of a knot/link K, then we define the corresponding knot/link diagram $D(K)$ by redrawing P with a broken arc at each crossing (place with two preimages in K) to incorporate the over/under crossing information.
- Reidemeister proposed the following moves on knot/link diagrams:

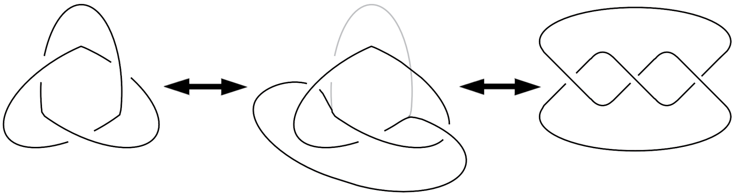


Theorem [Reidemeister, 1926]

Two knots/links are equivalent if and only if all of their link diagrams can be deformed into each other by a finite sequence of the three Reidemeister moves.

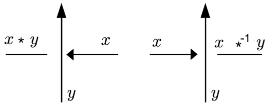


Two projections of the trefoil knot

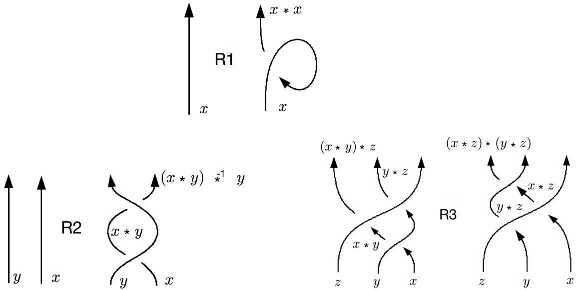


Getting from one projection to the other

- For each crossing of an oriented knot/link diagram, we set



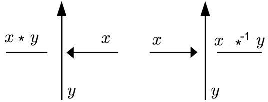
- The three quandle axioms are equivalent to the three Reidemeister moves of knot/link diagrams.



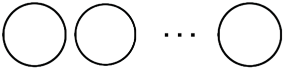
- If K is an oriented knot/link, then its fundamental quandle is defined as

$$Q(K) := \langle Arcs\ in\ D(K) \mid R \rangle,$$

where the set of relations R consists of expressions $x * y = z$ whenever the arc y passes over the double point separating x and z.



- If L is a trivial n-component link, then $Q(L)$ is the free quandle on n generators.



Theorem [Matveev/Joyce, 1982]

Let K' and K be two oriented knots. Then K' is equivalent to either K or $-K^*$ if and only if their fundamental quandles are isomorphic.

- Besides knot theory, quandles arise in various other settings.
- If G is a group, then the binary operation $x * y = yxy^{-1}$ turns G into a quandle $\mathrm{Conj}(G)$, called the conjugation quandle of G.
- If G is a group and $\varphi \in \mathrm{Aut}(G)$ an automorphism of G, then the binary operation $a * b = \varphi(ab^{-1})b$ gives a quandle structure on G.

  If G is abelian and φ is the inversion map, then the preceding quandle is called the core quandle and denoted by $\mathrm{Core}(G)$. In particular, if G is cyclic of order n, then it is the dihedral quandle R_n.
- Let k be any commutative ring and X a free k-module equipped with an antisymmetric bilinear form $\langle -, - \rangle : X \times X \to k$. Then X can be turned into a quandle when equipped with the binary operation $x * y = x + \langle x, y \rangle y$.
- If L is a nilpotent Lie algebra of class 3, then it can be turned into a quandle with the binary operation $x * y = x + [x, y]$.
- Let X be an algebraic variety, A a connected commutative algebraic group and $f : X \times X \to A$ a regular map with $f(x, x) = 0$ for all x. Then the set $X \times A$ has a quandle variety structure with quandle operation given by $(x, a) * (y, b) = (x, a + f(x, y))$.

- Let $(X, *)$ be a quandle and $\mathbb{K}$ an associative ring with unity $\mathbf{1}$. Let e_x be a unique symbol corresponding to each $x \in X$. Let $\mathbb{K}[X]$ be the set of all formal expressions of the form $\sum_{x \in X} \alpha_x e_x$, where $\alpha_x \in \mathbb{K}$ such that all but finitely many $\alpha_x = 0$.
- The set $\mathbb{K}[X]$ has a free $\mathbb{K}$-module structure with basis $\{e_x \mid x \in X\}$ and admits a product given by

$$\Big(\sum_{x \in X} \alpha_x e_x\Big)\Big(\sum_{y \in X} \beta_y e_y\Big) = \sum_{x,y \in X} \alpha_x \beta_y e_{x*y},$$

  where $x, y \in X$ and $\alpha_x, \beta_y \in \mathbb{K}$. This turns $\mathbb{K}[X]$ into a ring (rather a $\mathbb{K}$-algebra) called the quandle ring/algebra of X with coefficients in $\mathbb{K}$. The construction attempts to bring ring and representation theoretic techniques to the subject.
- Even though the coefficient ring $\mathbb{K}$ is associative, the quandle ring $\mathbb{K}[X]$ is non-associative when X is a non-trivial quandle. The quandle X can be identified as a subset of $\mathbb{K}[X]$ via the natural map $x \mapsto \mathbf{1}e_x = e_x$.

- Analogous to group rings, we define the augmentation map

$$\varepsilon : \mathbb{K}[X] \to \mathbb{K}$$

by setting

$$\varepsilon\Big(\sum_{x \in X} \alpha_x e_x\Big) = \sum_{x \in X} \alpha_x.$$

Clearly, ε is a surjective ring homomorphism, and $\Delta_{\mathbb{K}}(X) := \ker(\varepsilon)$ is a two-sided ideal of $\mathbb{K}[X]$, called the augmentation ideal of $\mathbb{K}[X]$.

- Recall that, a quandle X is called trivial if $x * y = x$ for all $x, y \in X$. The next result characterises trivial quandles in terms of their augmentation ideals.

Theorem [With Bardakov-Passi, 2019]

Let X be a quandle and $\mathbb{K}$ an associative ring with unity. Then the quandle X is trivial if and only if $\Delta_{\mathbb{K}}^2(X) = \{0\}$.

- Units in group rings play a fundamental role in the structure theory of group rings.
- In contrast, it turns out that idempotents are the most natural objects in quandle rings since each quandle element is, by definition, an idempotent of the quandle ring.
- In general, the computation of idempotents is an important problem in ring theory. It is well-known that integral group rings do not have non-trivial idempotents (Passman). In contrary, we shall see that integral quandle rings of many non-trivial quandles possess non-trivial idempotents.
- Let X be a quandle and $\mathbb{K}$ an integral domain with unity. A non-zero element $u \in \mathbb{K}[X]$ is called an idempotent if $u^2 = u$. Let

$$\mathcal{I}\big(\mathbb{K}[X]\big) = \big\{u \in \mathbb{K}[X] \mid u^2 = u\big\}.$$

  denote the set of all idempotents of $\mathbb{K}[X]$. It is clear that the basis elements $\{e_x \mid x \in X\}$ are idempotents of $\mathbb{K}[X]$, and we refer them as trivial idempotents.
- A non-trivial idempotent is an element of $\mathbb{K}[X]$ that is not of the form e_x for any $x \in X$.

- If X is a finite quandle having a subquandle Y with more than one element such that $|Y|$ is invertible in $\mathbb{K}$. Then $\mathbb{K}[X]$ has a non-trivial idempotent.
- In fact, a direct check shows that the element $u = \frac{1}{|Y|} \sum_{y \in Y} e_y$ is a non-trivial idempotent of $\mathbb{K}[X]$.
- If $u \in \mathbb{K}[X]$ is an idempotent, then $\varepsilon(u) = 0, 1$.

Proposition

If T is a trivial quandle, then $\mathcal{I}(\mathbb{K}[\mathrm{T}]) = e_{x_0} + \Delta_{\mathbb{K}}(\mathrm{T})$, where $x_0 \in \mathrm{T}$ is a fixed element.

- We exploit the idea of coverings [Eisermann, 2014] to determine idempotents for large families of quandles.
- A quandle homomorphism $p : X \to Y$ is called a quandle covering if it is surjective and $S_x = S_{x'}$ whenever $p(x) = p(x')$ for any $x, x' \in X$.
- Clearly, an isomorphism of quandles is a quandle covering, called a trivial covering.
- Some basic examples are:
    - A surjective group homomorphism $p : G \to H$ yields a quandle covering $\mathrm{Conj}(G) \to \mathrm{Conj}(H)$ if and only if $\ker(p)$ is a central subgroup of G.
    - A surjective group homomorphism $p : G \to H$ yields a quandle covering $\mathrm{Core}(G) \to \mathrm{Core}(H)$ if and only if $\ker(p)$ is a central subgroup of G of exponent two.
    - Let X be a quandle and F a non-empty set viewed as a trivial quandle. Consider $X \times F$ with the product quandle structure $(x, s) * (y, t) = (x * y, s)$. Then the projection $p : X \times F \to X$ given by $(x, s) \to x$ is a quandle covering.

- Let $p : X \to Y$ be a quandle covering, and $\mathcal{F}(Y)$ the set of all finite subsets of Y. For each $y \in Y$, let $\mathcal{F}(p^{-1}(y))$ be the set of all finite subsets of $p^{-1}(y)$, and denote a typical element of this set by I_y.
- Given elements x, y in a quandle X of finite type, we set

$$[e_x]_y := e_x + e_{x*y} + e_{x*y*y} + \cdots + e_{x \underbrace{*y*y*\cdots*y}_{(|S_y|-1)\ \text{times}}},$$

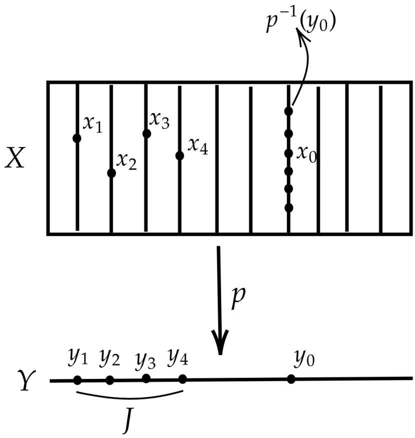
the sum of the basis elements in the S_y-orbit of e_x.

Theorem [With Elhamdadi-Nunez-Swain, 2023]

Let X be a quandle of finite type and $p : X \to Y$ a non-trivial quandle covering. If $\mathbb{K}[Y]$ has only trivial idempotents, then the set of idempotents of $\mathbb{K}[X]$ is

$$\mathcal{I}(K[X]) = \Bigg\{ \sum_{y \in J} \Big(\sum_{x \in I_y,\ \sum \alpha_x = 0} \alpha_x \, [e_x]_{x_0} \Big) + \Big(\sum_{x' \in I_{y_0},\ \sum \alpha_{x'} = 1} \alpha_{x'} \, e_{x'} \Big) \ \Big| \ J \in \mathcal{F}(Y),\ I_y \in \mathcal{F}(p^{-1}(y)),\ I_{y_0} \in \mathcal{F}(p^{-1}(y_0)),\ x_0 \in I_{y_0},\ y_0 \in Y,\ \alpha_x, \alpha_{x'} \in \mathbb{K} \Bigg\}.$$

Idea of the proof



- Let $p : X \to Y$ be a non-trivial quandle covering such that $\mathbb{K}[Y]$ has only trivial idempotents. Then every idempotent of $\mathbb{K}[X]$ has augmentation value 1. On the other hand, we can check that the idempotents of the mod 2 and the complex quandle ring of the dihedral quandle R_3 can have augmentation value 0 and 1 both.
- A computer-assisted computations for quandles of order less than six suggests the following conjecture.

Conjecture

Any non-zero idempotent of the integral quandle ring of a quandle has augmentation value 1.

- A direct check shows that integral quandle ring of R_3 and R_5 has only trivial idempotents. A computer check for quandles of order less than seven suggests the following conjecture.

Conjecture

The integral quandle ring of a semi-latin quandle has only trivial idempotents. In particular, the integral quandle ring of a finite latin quandle has only trivial idempotents.

- Let $X_i = \langle S_i \mid R_i \rangle$ be a collection of $n \geq 2$ quandles given in terms of presentations. Then their *free product* $X_1 \star X_2 \star \cdots \star X_n$ is the quandle defined by the presentation

$$X_1 \star X_2 \star \cdots \star X_n = \langle S_1 \sqcup S_2 \sqcup \cdots \sqcup S_n \mid R_1 \sqcup R_2 \sqcup \cdots \sqcup R_n \rangle.$$

- For example, the free quandle FQ_n of rank n can be seen as

$$FQ_n = \langle x_1 \rangle \star \langle x_2 \rangle \star \cdots \star \langle x_n \rangle,$$

the free product of n copies of trivial one element quandles $\langle x_i \rangle$.

Theorem [With Elhamdadi-Nunez-Swain, 2023]

Let FQ_n be the free quandle of rank $n \geq 1$. Then $\mathbb{Z}[FQ_n]$ has only trivial idempotents. The same assertion holds for the free quandle of countably infinite rank.

- The key idea is to use a length function on the left-normalised expressions of elements of free products of quandles.

- Let L be a link in $\mathbb{R}^3$, $Q(L)$ its fundamental quandle and X a quandle. Then the number of quandle homomorphisms $|\operatorname{Hom}(Q(L), X)|$ is an invariant of L, called the quandle coloring invariant.
- A link invariant which determines the quandle coloring invariant is called an enhancement. Further, an enhancement is called proper if there are examples in which the enhancement distinguishes links which have the same quandle coloring invariant.
- Let X and Y be quandles and $\operatorname{Hom}_{alg}\left(\mathbb{K}[X], \mathbb{K}[Y]\right)$ denotes the set of $\mathbb{K}$-algebra homomorphisms from $\mathbb{K}[X]$ to $\mathbb{K}[Y]$.

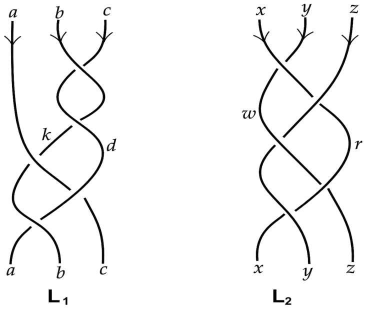
Theorem

If L is a link and X a quandle, then the pair

$$\left(|\operatorname{Hom}\left(Q(L), X\right)|, \quad |\operatorname{Hom}_{alg}\left(\mathbb{K}[Q(L)], \mathbb{K}[X]\right)|\right)$$

is a proper enhancement of the quandle coloring invariant $|\operatorname{Hom}(Q(L), X)|$.

- Consider the braid diagrams of links L_1 and L_2.



- Then $|\operatorname{Hom}(Q(L_1), R_6)| = 12 = |\operatorname{Hom}(Q(L_2), R_6)|$, but $|\operatorname{Hom}_{alg}(\mathbb{Z}[Q(L_1)], \mathbb{Z}[R_6])| \neq |\operatorname{Hom}_{alg}(\mathbb{Z}[Q(L_2)], \mathbb{Z}[R_6])|$.

Thank you for your attention

Dank u voor uw aandacht

Merci pour votre attention