From Cyclic Rota-Baxter algebras to pre-Calabi-Yau algebras and double Poisson algebras

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joint work

This talk is based on the following work with Kai Wang:

Y. Qin and K. Wang, From Cyclic Rota-Baxter algebras to Pre-Calabi-Yau algebras and double Poisson algebras, in preparation.

Plan

- I, Algebraic versions
- II, Infinity versions (or called homotopic versions)

I.1, Quantum Yang-Baxter equations and classical Yang-Baxter equations

Definition

Let V be a linear space. A matrix $R : V \otimes V \to V \otimes V$ is called a solution to **quantum Yang-Baxter equation** (or R-matrix), that is, if R satisfies the following:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \text{ in } V \otimes V \otimes V,$$

where the subscripts indicate which tensor factors are being utilized, for instance $R_{12} = R \otimes \mathrm{id}_V$ and $R_{23} = \mathrm{id}_V \otimes R$.

I.1, Quantum Yang-Baxter equations and classical Yang-Baxter equations

Definition (Belavin-Drinfel'd 1982)

Let $\mathfrak{g}$ be a Lie algebra and $r = \sum_i a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$. r is called a solution of **classical Yang-Baxter equation** (CYBE) in $\mathfrak{g}$ if $CYBE(r) = 0$, that is,

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \text{ in } U(\mathfrak{g}),$$

where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ and

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1; r_{13} = \sum_i a_i \otimes 1 \otimes b_i; r_{23} = \sum_i 1 \otimes a_i \otimes b_i.$$

r is said to be skew-symmetric if

$$r = \sum_i (a_i \otimes b_i - b_i \otimes a_i)$$

We also denote $r_{21} = \sum_i b_i \otimes a_i$.

A.A. Belavin and V.G. Drinfel'd, Solutions of the classical Yang - Baxter equation for simple Lie algebras, Functional Analysis and Its Applications. **16** (1982), 159–180.

I.1, The applications of classical Yang-Baxter equations

CYBE on matrix emerges from so called quasi-classical solutions to the quantum Yang-Baxter equation, in which R-matrix admits an asymptotic expansion in terms of an expansion parameter $\hbar$

$$R_{\hbar} = I + \hbar r + \mathcal{O}\left(\hbar^2\right),$$

where $r \in \mathrm{End}(V) \otimes \mathrm{End}(V)$ is a solution of CYBE.

I.2, Associative Yang-Baxter equations

Definition (Aguiar 2000)

Let A be an algebra and $r = \sum_i a_i \otimes b_i \in A \otimes A$. r is called a solution of **associative Yang-Baxter equation** (AYBE) in A if $AYBE(r) = 0$, that is,

$$r_{12} \cdot r_{13} - r_{23} \cdot r_{12} + r_{13} \cdot r_{23} = 0 \text{ in } A \otimes A \otimes A.$$

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Denote $AYBE'(r) := -\sigma_{(23)} AYBE(r) \sigma_{(23)}^{-1}$. When r is skew-symmetric ($r = -r_{21}$) , then $AYBE'(r)$ posses a cyclic formula:

$$AYBE'(r) = r_{12} \cdot r_{23} + r_{31} \cdot r_{12} + r_{23} \cdot r_{31},$$

and the AYBE implies the CYBE:

$$CYBE(r) = AYBE(r) + AYBE'(r).$$

M. Aguiar, Infinitesimal Hopf algebras, Contemp. Math. **267** (2000) 1-29.

I.2, AYBE on matrix algebras

Let V be a linear space. We have a isomorphism between matrices and double brackets:

$$\mathsf{End}(V) \otimes \mathsf{End}(V) \quad \cong \quad \mathsf{End}(V \otimes V)$$

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Let V be a linear space. We have a isomorphism between matrices and double brackets:

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Then the skew-symmetry for r is isomorphic to the double bracket satisfying:

$$\{\{a,b\}\} = -\sigma_{(12)}\{\{b,a\}\},\ \forall a,b\in V;$$

and the cyclic formula $AYBE'(r)$ is isomorphic to the following Jacobi identity:

$$\{\{-,\{\{-,-\}\}\}\}_L+\sigma_{(123)}\{\{-,\{\{-,-\}\}\}\}_L\sigma_{(123)}^{-1}+\sigma_{(123)}^2\{\{-,\{\{-,-\}\}\}\}_L\sigma_{(123)}^{-2},$$

where $\{\{-,-\}\}_L(x_1\otimes x_2\otimes x_3):=\{\{x_1,x_2\}\}\otimes x_3$.

I.3, Double Lie algebras

Definition (Schedler 2009)

A **double Lie algebra** is a linear space equipped with a double bracket

$$\{\{-,-\}\} : V \otimes V \to V \otimes V$$

satisfying the following identities for all $a, b, c \in V$

(i) Skew-symmetry:

$$\{\{a,b\}\} = -\sigma_{(12)}\{\{b,a\}\};$$

(ii) Jacobi identity:

$$\{\{-,\{\{-,-\}\}\}\}_L + \sigma_{(123)}\{\{-,\{\{-,-\}\}\}\}_L \sigma_{(123)}^{-1} + \sigma_{(123)}^{2}\{\{-,\{\{-,-\}\}\}\}_L \sigma_{(123)}^{-2} = 0,$$

where $\{\{-,-\}\}_L(x_1 \otimes x_2 \otimes x_3) := \{\{x_1,x_2\}\} \otimes x_3$.

Definition (Van den Bergh 2008)

A **double Poisson algebra** is an associative algebra $(A, \cdot)$ equipped with a double Lie algebra structure $\{\{-,-\}\}$ satisfying the Leibniz rule: for all $a, b, c \in A$

$$\{\{a, b \cdot c\}\} = \{\{a, b\}\} \cdot c + b \cdot \{\{a, c\}\}, \tag{1}$$

where

$$\{\{a, b\}\} \cdot c = \{\{a, b\}\}^{[1]} \otimes (\{\{a, b\}\}^{[2]} \cdot c),$$

$$b \cdot \{\{a, c\}\} = (b \cdot \{\{a, c\}\}^{[1]}) \otimes \{\{a, c\}\}^{[2]}.$$

M. Van den Bergh, Double Poisson algebras, Trans. Amer. Math. Soc. **360** (2008), no. 11, 5711–5769.

I.4, Double Poisson algebras

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$$b \cdot \{\{a, c\}\} = (b \cdot \{\{a, c\}\}^{[1]}) \otimes \{\{a, c\}\}^{[2]}.$$

Example (Van den Bergh 2008)

Put $A = k[t]$*. Up to automorphisms of* A*, the only double Poisson brackets on* A *are given by*

$$\{\{t, t\}\} = t \otimes 1 - 1 \otimes t$$

and

$$\{\{t, t\}\} = t^2 \otimes t - t \otimes t^2.$$

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Proposition

Let $(V, \{\{-, -\}\})$ *be a double Lie algebra. Then* $S(V)$ *is a Poisson algebra.*

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Proposition

Let $(V, \{\{-, -\}\})$ be a double Lie algebra. Then $S(V)$ is a Poisson algebra.

The above proposition suggests that the dual space of a double Lie algebra can be considered as a formal Poisson manifold.

M. Van den Bergh, Double Poisson algebras, Trans. Amer. Math. Soc. **360** (2008), no. 11, 5711–5769.

I.5, Rota-Baxter algebras

Definition

Let $(A, \cdot)$ be an algebra and M an A-module. A linear operator $T : M \to A$ is called a **relative Rota-Baxter operator on** M if it satisfies the following relation: $a, b \in M$,

$$T(a) \cdot T(b) = T\big(a \cdot T(b) + T(a) \cdot b\big). \tag{3}$$

In this case, the triple (A, M, T) is called a **relative Rota-Baxter algebra**. In particular, if we take $M = A$, $(A, \cdot, T)$ is called a **Rota-Baxter algebra**.

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Example

Let $f(x) \in \mathbf{C}(\mathbb{R})$ be a continuous function. Define integral as the Rota−Baxter operator

$$T(f)(x) = \int_0^x f(t)dt.$$

By the formula for integration by parts, we have

$$\int_0^x f(t)dt \int_0^x g(t)dt = \int_0^x f(t) \int_0^t g(v)dvdt + \int_0^x g(t) \int_0^t f(v)dvdt,$$

that is, Rota-Baxter operator.

I.5, Rota-Baxter algebras

Theorem (Aguiar 2000)

Let A be an algebra and $\mathbf{r} = \sum_i \mathbf{a}_i \otimes \mathbf{b}_i \in A \otimes A$ is a solution of AYBE'. Then the operator $T : A \to A$ given by

$$T(x) = \sum a_i x b_i$$

is a Rota-Baxter operator.

M. Aguiar, Pre-Poisson algebras, Lett. Math. Phys. **54** (2000) 263-277.

I.6, Rota-Baxter algebras on on matrix algebras

Let V be finite-dimensional. We define a nature nondegenerate bilinear form by trace:

$$< f, g > := \mathrm{tr}(f \circ g), \ \forall f, g \in \mathrm{End}(V).$$

Thus we have $\mathrm{End}(V) \cong \mathrm{End}(V)^{\vee}$, which induces the following isomorphisms:

$$\mathrm{End}(V \otimes V) \cong \mathrm{End}(V) \otimes \mathrm{End}(V) \cong \mathrm{End}(V) \otimes \mathrm{End}(V)^{\vee} \cong \mathrm{End}(\mathrm{End}(V)).$$

In this way, any double bracket

$$\{\{-,-\}\} : V \otimes V \to V \otimes V$$

can be uniquely determined by a linear operator

$$T : \mathrm{End}(V) \to \mathrm{End}(V).$$

Conversely, each linear operator T on $\mathrm{End}(V)$ corresponds the double bracket $\{\{-,-\}\}$, which can be expressed as follow:

$$\{\{a,b\}\} = \sum_{i=1}^{N} T^{\vee}(e^i)(a) \otimes e_i(b) = \sum_{i=1}^{N} e^i(a) \otimes T(e_i)(b), \quad a,b \in V,$$

where $\{e_1,\ldots,e_N\}$ is a basis of $\mathrm{End}(V)$, $\{e^1,\ldots,e^N\}$ is the corresponding dual basis with repsect to the trace form, and $T^{\vee}$ is the adjoint operator of T with respect to the trace form.

I.6, Rota-Baxter algebras on on matrix algebras

Theorem

The following data are equivalent:

(i) *A double Lie algebra structure* $\{\{-,-\}\}$ *on* V.

(ii) *A linear operator* $T : \mathrm{End}(V) \to \mathrm{End}(V)$ *constitute a Rota-Baxter operator on* $(\mathrm{End}(V), \circ)$*, with* T *cyclic, that is* $T^\vee = -T$.

(iii) *A skew-symmetric solution* $r \in \mathrm{End}(V) \otimes \mathrm{End}(V)$ *of the associative Yang-Baxter equation.*

M.E. Goncharov and P.S. Kolesnikov, Simple finite-dimensional double algebras, J. Algebra **500** (2018), 425–438.

I.7, Cyclic Rota-Baxter algebras

Definition

Let A be a finite dimensional algebra. We call that A is a symmetric algebra if there is a nondegenerate symmetric bilinear form

$$< -, - >: A \times A \to \mathbf{k}$$

satisfying

$$< a, bc > = < ab, c > = < ca, b > \quad \forall a, b, c \in A.$$

Definition (Q.-Wang 2025)

Let $(A, \cdot, < -, - >)$ be a symmetric algebra and T be a Rota-Baxter operator on A. We say that T is a **cyclic Rota-Baxter operator** *if $T^* = -T$, where T^* is the following compositions:*

$$T^* : A \xrightarrow{\rho} A^\vee \xrightarrow{T^\vee} A^\vee \xrightarrow{\rho^{-1}} A.$$

In this case, $(A, \cdot, T)$ is called a **cyclic Rota-Baxter algebra***.*

Y. Qin and K. Wang, From Cyclic Rota-Baxter algebras to Pre-Calabi-Yau algebras and double Poisson algebras, in preparation.

Definition

Let $A = \oplus_{n\in\mathbb{Z}} A_n$ be a graded vector space equipped a family of maps $\{m_n : A^{\otimes n} \to A\}_{n\geq 1}$, with $|m_n| = n-2$ satisfying the Stasheff identity: for all $n \geq 1$,

$$\sum_{\substack{i+j+k=n,\\ i,k\geqslant 0, j\geqslant 1}} (-1)^{i+jk} m_{i+1+k} \circ \left(\mathrm{id}^{\otimes i} \otimes m_j \otimes \mathrm{id}^{\otimes k} \right) = 0. \qquad (4)$$

Then $(A, \{m_n\}_{n\geq 1})$ is called an A_∞**-algebra (homotopy associative algebra)**.

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(iii) when $n = 3$
$$m_2 \circ (\mathrm{id} \otimes m_2 - m_2 \otimes \mathrm{id}) = m_1 \circ m_3 + m_3 \circ (m_1 \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes m_1 \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \otimes m_1),$$
that is, (A, m_1, m_2) is a differential graded associative algebra up to homotopy.

Definition

Let $(A, \{m_n\}_{n\geq 1})$ be an A_∞-algebra. An A_∞**-bimodule** over A is a graded space $M = \bigoplus_{n\in\mathbb{Z}} M_n$ equipped with a family of homogeneous maps $\{m_{p,q} : A^{\otimes p} \otimes M \otimes A^{\otimes q} \to M\}_{p,q\geq 0}$ with $|m_{p,q}| = p + q - 1$ satisfying: for all $p, q \geqslant 0$,

$$\sum_{\substack{i+r=p,s+k=q\\ i,r,s,k\geqslant 0}} (-1)^{i+(r+s-1)k+1} m_{i+1,k+1} \circ \left(\mathrm{id}^{\otimes i} \otimes m_{r,s} \otimes \mathrm{id}^{\otimes k} \right)$$

$$= \sum_{\substack{1\leqslant j\leqslant p\\ 0\leqslant i\leqslant p-j}} (-1)^{i+j(p-i-j+1+q)} m_{p-j+1,q} \circ \left(\mathrm{id}^{\otimes i} \otimes m_j \otimes \mathrm{id}^{\otimes p-i-j} \otimes \mathrm{id}_M \otimes \mathrm{id}^{\otimes q} \right)$$

$$+ \sum_{\substack{1\leqslant j\leqslant q\\ 0\leqslant i\leqslant q-j}} (-1)^{p+i+1+j(q-i-j)} m_{p,q-j+1} \circ \left(\mathrm{id}^{\otimes p} \otimes \mathrm{id}_M \otimes \mathrm{id}^{\otimes i} \otimes m_j \otimes \mathrm{id}^{\otimes q-i-j} \right).$$

II.2, Cyclic operators and pre-Calabi-Yau structures

Now, we mainly work with the locally finite graded space, i.e., each component A_i of space $A = \oplus_{i \in \mathbb{Z}} A_i$ is finite dimensional.

Definition

Let d be an integer. Let A be a graded space endowed with a graded symmetric bilinear form $\gamma : A^{\otimes 2} \to k$ of degree d. An operator $m_n : A^{\otimes n} \to A$ is called **d-cyclic** *if it satisfies*

$$\gamma\left(m_n\left(a_1, \ldots, a_n\right), a_0\right) = (-1)^{n+|a_0|\left(\sum_{i=1}^{n}|a_i|\right)} \gamma\left(m_n\left(a_0, \ldots, a_{n-1}\right), a_n\right),$$

for all homogeneous $a_0, \ldots, a_n \in A$.

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for all homogeneous $a_0, \ldots, a_n \in A$.

Let $d \in \mathbb{Z}$. Set $\partial_d A = A \oplus s^{-d} A^{\vee}$. There is a natural bilinear form, for all homogeneous $a, b \in A$ and $f, g \in A^{\vee}$

$$\zeta_A : \partial_d A \otimes \partial_d A \to \mathbf{k}$$

by

$$\zeta_A(s^{-d} f, a) = (-1)^{|a||s^{-d} f|} \zeta_A(a, s^{-d} f) = f(a),$$

and

$$\zeta_A(a, b) = \zeta_A(s^{-d} f, s^{-d} g) = 0.$$

Definition

Let $(A, \{m_n\}_{n \geqslant 1})$ *be an* A_∞*-algebra with a non-degenerate bilinear form* γ *of degree* d*. If each operator* m_n *is* d*-cyclic with respect to the bilinear form* γ*, we say that* $(A, \{m_n\}_{n \geq 1})$ *is a* **d-cyclic** A_∞**-algebra**.

Definition

Let $(A, \{m_n\}_{n\geqslant 1})$ be an A_∞-algebra with a non-degenerate bilinear form γ of degree $\mathbf{d}$. If each operator $\mathbf{m_n}$ is $\mathbf{d}$-cyclic with respect to the bilinear form γ, we say that $(A, \{m_n\}_{n\geq 1})$ is a **d-cyclic A_∞-algebra**.

Example

Any symmetric algebra is a 0-cyclic A_∞-algebra.

Definition (Kontsevich-Takeda-Vlassopoulos 2018)

*A d-**pre-Calabi-Yau** structure on a graded space $A = \oplus_{n \in \mathbb{Z}} A_n$ is:*

- *a $(d-1)$-cyclic A_∞ structure on $\partial_{d-1} A$ w.r.t. the natural bilinear form ζ_A,*
- *and such that A is an A_∞-subalgebra of $\partial_{d-1} A$.*

A 0-pre-Calabi-Yau algebra will be simply called a **pre-Calabi-Yau algebra**.

II.2, Cyclic operators and pre-Calabi-Yau structures

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- *and such that A is an A_∞-subalgebra of $\partial_{d-1}A$.*

*A 0-pre-Calabi-Yau algebra will be simply called a **pre-Calabi-Yau algebra**.*

Example (Kontsevich-Takeda-Vlassopoulos 2018)

Let M be a compact oriented manifold of dimension d with compact boundary ∂M then the cohomology $H^*(M)$ of M has the structure of a pre-Calabi-Yau algebra of dimension d.

M. Kontsevich, A. Takeda and Y. Vlassopoulos, Pre-Calabi-Yau algebras and topological quantum field theories, European Journal of Mathematics **11**, 15(2025).

Motivations

Theorem (Iyudu-Kontsevich-Vlassopoulos 2021)

The pre-Calabi-Yau structures of type B*, whose terms of order higher than three are trivial, are in one-to-one correspondence with the double Poisson brackets.*

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The good manageable special pre-Calabi-Yau structures are in one-to-one correspondence with the homotopy double Poisson algebras.

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The pre-Calabi-Yau structures are equivalent to the homotopy double Poisson gebras.

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Theorem (Leray-Vallette 2023)

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Problem

What is the relation between homotopy Rota-Baxter algebras and pre-Calabi-Yau algebras?

Definition (Das-Misha 2022, Wang-Zhou 2024)

Let $(A, \{m_i\}_{i\geq 1})$ *be an* A_∞*-algebra and* $(M, \{m'_{i,s}\}_{i\geq 1, 1\leq s\leq i})$ *an* A_∞*-bimodule over* A*. A* **homotopy relative Rota-Baxter** *operator* $\{T_i\}_{i\geq 1}$ *on* (A, M) *is a family of operators* $T_i : M^{\otimes i} \to A$, $i \geq 1$ *satisfying:*

$$\sum_{\substack{l_1+\cdots+l_k=n,\\ l_1,\ldots,l_k\geqslant 1}} (-1)^{\delta} m_k \circ \left(T_{l_1} \otimes \cdots \otimes T_{l_k}\right) = \sum_{1\leqslant j\leqslant p} \sum_{\substack{r_1+\cdots+r_p=n,\\ r_1,\ldots,r_p\geqslant 1}} \tag{5}$$

$$(-1)^{\eta} T_{r_1} \circ \left(\mathrm{id}^{\otimes i} \otimes m'_{j-1,p-j} \circ (T_{r_2} \otimes \cdots \otimes T_{r_j} \otimes \mathrm{id} \otimes T_{r_{j+1}} \otimes \cdots \otimes T_{r_p}) \otimes \mathrm{id}^{\otimes k}\right).$$

The triple $(A, M, \{T_i\}_{i\geq 1})$ *is called a homotopy relative Rota-Baxter algebra.*

Definition (Das-Misha 2022, Wang-Zhou 2024)

Let $(A, \{m_i\}_{i\geq 1})$ *be an* A_∞*-algebra and* $(M, \{m'_{i,s}\}_{i\geq 1, 1\leq s\leq i})$ *an* A_∞*-bimodule over* A*. A* **homotopy relative Rota-Baxter** *operator* $\{T_i\}_{i\geq 1}$ *on* (A, M) *is a family of operators* $T_i : M^{\otimes i} \to A$, $i \geq 1$ *satisfying:*

$$\sum_{\substack{l_1+\cdots+l_k=n,\\ l_1,\ldots,l_k\geqslant 1}} (-1)^\delta m_k \circ \left(T_{l_1} \otimes \cdots \otimes T_{l_k}\right) = \sum_{1\leqslant j\leqslant p} \sum_{\substack{r_1+\cdots+r_p=n,\\ r_1,\ldots,r_p\geqslant 1}} \tag{5}$$

$$(-1)^\eta T_{r_1} \circ \left(\mathrm{id}^{\otimes i} \otimes m'_{j-1,p-j} \circ (T_{r_2} \otimes \cdots \otimes T_{r_j} \otimes \mathrm{id} \otimes T_{r_{j+1}} \otimes \cdots \otimes T_{r_p}) \otimes \mathrm{id}^{\otimes k}\right).$$

The triple $(A, M, \{T_i\}_{i\geq 1})$ *is called a homotopy relative Rota-Baxter algebra.*
Moreover,

- *If* $T_n\sigma = (-1)^{\mathrm{sgn}(\sigma)} T_n$*, for all* $\sigma \in \mathbb{S}_n$*, we call it* **skew-symmetric homotopy relative Rota-Baxter algebra**.

Definition (Das-Misha 2022, Wang-Zhou 2024)

Let $(A, \{m_i\}_{i\geq 1})$ *be an* A_∞*-algebra and* $(M, \{m'_{i,s}\}_{i\geq 1, 1\leq s\leq i})$ *an* A_∞*-bimodule over* A*. A* **homotopy relative Rota-Baxter** *operator* $\{T_i\}_{i\geq 1}$ *on* (A, M) *is a family of operators* $T_i : M^{\otimes i} \to A$, $i \geq 1$ *satisfying:*

$$\sum_{\substack{l_1+\cdots+l_k=n,\\ l_1,\ldots,l_k\geqslant 1}} (-1)^{\delta} m_k \circ \left(T_{l_1} \otimes \cdots \otimes T_{l_k}\right) = \sum_{1\leqslant j\leqslant p} \sum_{\substack{r_1+\cdots+r_p=n,\\ r_1,\ldots,r_p\geqslant 1}} \tag{5}$$

$$(-1)^{\eta} T_{r_1} \circ \left(\mathrm{id}^{\otimes i} \otimes m'_{j-1,p-j} \circ (T_{r_2} \otimes \cdots \otimes T_{r_j} \otimes \mathrm{id} \otimes T_{r_{j+1}} \otimes \cdots \otimes T_{r_p}) \otimes \mathrm{id}^{\otimes k}\right).$$

The triple $(A, M, \{T_i\}_{i\geq 1})$ *is called a homotopy relative Rota-Baxter algebra.*
Moreover,

- *If* $T_n\sigma = (-1)^{\mathrm{sgn}(\sigma)} T_n$*, for all* $\sigma \in \mathbb{S}_n$*, we call it* **skew-symmetric homotopy relative Rota-Baxter algebra**.
- *If* $M = A$*, we call* $(A, \{T_i\}_{i\geq 1})$ **homotopy Rota-Baxter algebra**.

II.3, Homotopy (relative) Rota-Baxter algebras

In this paper, we mainly work with relative homotopy Rota-Baxter algebras on dual space over dg algebras:
Take A to be a dg algebra and $M = A^\vee$. In this case, the Rota-Baxter equation takes the following explicit form:

$$\begin{aligned} & d_A \circ T_n + \sum_{i+j=n} (-1)^{1+i} m \circ \left(T_i \otimes T_j\right) \\ = & \sum_{s+k+1=n} (-1)^{n-1} T_n \circ \left(\mathrm{id}^{\otimes s} \otimes d_{A^\vee} \otimes \mathrm{id}^{\otimes k}\right) \\ & + \sum_{s+k+j+1=n} (-1)^{s+(j-1)(i-s)} T_i \circ \left(\mathrm{id}^{\otimes s} \otimes m'^{l} \circ (T_j \otimes \mathrm{id}) \otimes \mathrm{id}^{\otimes k}\right) \\ & + \sum_{s+k+j+1=n} (-1)^{s+(j-1)(i-s+1)} T_i \circ \left(\mathrm{id}^{\otimes s} \otimes m'^{r} \circ (\mathrm{id} \otimes T_j) \otimes \mathrm{id}^{\otimes k}\right). \end{aligned}$$

II.4, Cyclic homotopy Rota-Baxter algebras and cyclic completion

Definition (Q.-Wang 2025)

Let A be a cyclic A_∞-algebra with respect to a nondegenerate bilinear form $\gamma : A \otimes A \to \mathbf{k}$. A homotopy Rota-Baxter operator $\{T_n\}_{n\geqslant 1}$ on A is said to be cyclic if each operator $T_n : A^{\otimes n} \to A$ is cyclic. Then $(A, \{T_n\}_{n\geqslant 1})$ is called a **cyclic homotopy Rota-Baxter algebra**.

II.4, Cyclic homotopy Rota-Baxter algebras and cyclic completion

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Let A be a cyclic A_∞-algebra with respect to a nondegenerate bilinear form $\gamma : A \otimes A \to \mathbf{k}$. A homotopy Rota-Baxter operator $\{T_n\}_{n\geqslant 1}$ on A is said to be cyclic if each operator $T_n : A^{\otimes n} \to A$ is cyclic. Then $(A, \{T_n\}_{n\geqslant 1})$ is called a **cyclic homotopy Rota-Baxter algebra**.

Moreover, if each operator T_n is cyclic and skew-symmetric, we call $\{T_n\}_{n\geqslant 1}$ an **ultracyclic homotopy Rota-Baxter operator** *and $(A, \{T_n\}_{n\geqslant 1})$ an* **ultracyclic homotopy Rota-Baxter algebra**.

Y. Qin and K. Wang, From Cyclic Rota-Baxter algebras to Pre-Calabi-Yau algebras and double Poisson algebras, in preparation.

II.4, Cyclic homotopy Rota-Baxter algebras and cyclic completion

We give a method to construct the cyclic homotopy Rota-Baxter algebras from homotopy Rota-Baxter algebras, called the **cyclic completion construction for homotopy Rota-Baxter algebras**.

Proposition (Q.-Wang 2025)

Let $(A, \{T_n\}_{n\geq 1})$ *be a homotopy Rota-Baxter algebra. Define a family of operators* $\{\widehat{T}_n\}_{n\geq 1}$ *on* $\partial_0 A$ *as follows: for homogeneous elements* $(a_1, f_1), \ldots, (a_n, f_n) \in \partial_0 A = A \oplus A^\vee$,

$$\widehat{T}_n : (\partial_0 A)^{\otimes n} \to \partial_0 A$$

$$((a_1, f_1), \ \ldots, (a_n, f_n)) \mapsto \tag{6}$$

$$\left(T_n(a_1, \ldots, a_n), \sum_{j=1}^{n} (-1)^{\sigma_n} f_j \circ T_n(a_{j+1}, \ldots, a_n, -, a_1, \ldots, a_{j-1}) \right).$$

Then $(\partial_0 A, \{\widehat{T}_n\}_{n\geqslant 1})$ *is a cyclic homotopy Rota-Baxter algebra. Moreover, if* $\{T_n\}_{n\geq 1}$ *is skew-symmetric, then* $\{\widehat{T}_n\}_{n\geq 1}$ *is an ultracyclic homotopy Rota-Baxter operator on* $\partial_0 A$.

II.5, Cyclic homotopy relative Rota-Baxter algebras

Definition (Q.-Wang 2025)

Let A be an A_∞-algebra. The operator $\{T_n : (A^\vee)^{\otimes n} \to A\}_{n\geqslant 1}$ is called a **cyclic homotopy relative Rota-Baxter operators**, *if we can define the operators*

$$\overline{T}_n : (\partial_0 A)^{\otimes n} \twoheadrightarrow (A^\vee)^{\otimes n} \xrightarrow{T_n} A \hookrightarrow \partial_0 A, \ \forall n \geq 1.$$

such that $(\partial_0 A, \{\overline{T}_n\}_{n\geqslant 1})$ is a cyclic homotopy Rota-Baxter algebra.

II.6, Left compatible pair (A, B) and B-derivative

Definition (Q.-Wang 2025)

A **left compatible pair** (A, B) *consists of the following data:*

(i) *A pair of dg algebras* $(A, d_A, \cdot)$ *and* $(B, d_B, *)$.

(ii) *A dg left* B*-module structure on the complex* (A, d_A) *and a left dg* A*-module structure on the complex* (B, d_B)*. To distinguish between them, the left action of* A *on* B *is denoted by* $\triangleright$*, while the left action of* B *on* A *is denoted by* $\blacktriangleright$.

(iii) *A compatibility condition ensuring that for all* $a \in A, b_1, b_2 \in B$*, the following identity holds:*

$$(b_1 \blacktriangleright a) \triangleright b_2 = b_1 * (a \triangleright b_2).$$

Example

(1) *Let A be a dg algebra. Then (A, A) is a left compatible pair.*

(2) *Let $(B, \cdot)$ be a finite dimensional dg algebra. The graded space $\mathrm{End}(B)$ is a dg algebra with multiplication being composition and B is a left dg $\mathrm{End}(B)$-module in the canonical way. Given an element $b \in B$, one has a map $l_b \in \mathrm{End}(B)$, which takes $x \in B$ to bx. Then we have a left action of B on $\mathrm{End}(B)$ given as*

$$b \triangleright f := l_b \circ f,$$

which makes $\mathrm{End}(B)$ into a left dg B-module. Moreover, we have: for all $b_1, b_2 \in B$, $f \in \mathrm{End}(B)$,

$$(l_{b_1} \circ f)(b_2) = b_1 \cdot (f(b_2)).$$

Thus, $(\mathrm{End}(B), B)$ is a left compatible pair.

II.6, Left compatible pair (A, B) and B-derivative

Definition (Q.-Wang 2025)

Let (A, B) be a left compatible pair. An operator $T_n : (A^\vee)^{\otimes n} \to A$ is called

(i) *an* **n-derivation** *on left compatible pair (A, B), if for all $b_1, b_2 \in B$, and $f_1 \cdots, f_n \in A^\vee$:*

$$T_n(f_1 \otimes \cdots \otimes f_n) \triangleright (b_1 * b_2) = T_n(f_1 \otimes \cdots \otimes f_n \blacktriangleleft b_1) \triangleright b_2 + (T_n(f_1 \otimes \cdots \otimes f_n) \triangleright b_1) * b_2;$$

(ii) *an* **$(n, 1)$-derivation** *on left compatible pair (A, B), if for all $b_1, b_2 \in B$, $f_1 \in B^\vee$, and $f_2, \cdots, f_n \in A^\vee$:*

$$T_n\big(\kappa(b_1 * b_2 \otimes f_1) \otimes f_2 \otimes \cdots \otimes f_n\big) = (-1)^{|T_n||b_1|}\Big(b_1 \blacktriangleright \big(T_n\big(\kappa(b_2 \otimes f_1) \otimes f_2 \otimes \cdots \otimes f_n\big)\big)\Big) + T_n\big(\kappa\big(b_1 \otimes b_2 \blacktriangleright f_1\big) \otimes f_2 \otimes \cdots \otimes f_n\big);$$

(iii) *an* **(n, l)-derivation** *on left compatible pair (A, B) with $1 < l \leq n$, if for all $b_1, b_2 \in B$, $f_l \in B^\vee$, and $f_1, \cdots, f_{l-1}, f_{l+1}, \cdots, f_n \in A^\vee$:*

$$\begin{aligned} &T_n\big(f_1 \otimes \cdots \otimes f_{l-1} \otimes \kappa(b_1 * b_2 \otimes f_l) \otimes f_{l+1} \otimes \cdots \otimes f_n\big) \\ &= T_n(f_1 \otimes \cdots \otimes f_{l-1} \blacktriangleleft b_1 \otimes \kappa(b_2 \otimes f_l) \otimes f_{l+1} \otimes \cdots \otimes f_n) \\ &+ T_n(f_1 \otimes \cdots \otimes f_{l-1} \otimes \kappa(b_1 \otimes b_2 \blacktriangleright f_l) \otimes f_{l+1} \otimes \cdots \otimes f_n), \end{aligned}$$

where $\kappa : B \otimes B^\vee \to A^\vee$ as $\kappa(b \otimes f)(a) = (-1)^{|b|(|f|+|a|)} f(a \triangleright b)$, for any $b \in B$, $f \in B^\vee$ and $a \in A$.

II.6, Left compatible pair (A, B) and B-derivative

Definition

Let (A, B) be a left compatible pair. A family of relative operators $\{T_n : (A^\vee)^{\otimes n} \to A\}_{n \geq 1}$ is called

(i) *a* ***B*-derivative** *if each T_n is an n-derivation.*

(ii) *a* **strong** ***B*-derivative** *if each T_n is an n-derivation and (n, l)-derivation for each $1 \leq l \leq n$.*

Proposition

Let (A, B) be a left compatible pair with B being locally finite dimensional. Then a cyclic B-derivative operator $\{T_n : (A^\vee)^{\otimes n} \to A\}_{n \geq 1}$ is a strong B-derivative operator.

Lemma (Q.-Wang 2025)

Let (A, B) be a left compatible pair and $(A, A^\vee, \{T_n\}_{n\geq 1})$ a strong B-derivative homotopy relative Rota-Baxter algebra. Define a family of operations $\{m_n\}_{n\geqslant 1}$ on $\partial_{-1}B$ as

(i) $m_1 = -d_{\partial_{-1}B}$,

(ii) *the operation m_2 is constrained to coincide with the associative product on B,*

(ii) *for all $n \geqslant 1$, $b_i \in B, f_i \in B^\vee$,*

$$m_{2n+1}(b_1, s^{-1}f_1, b_2, \dots, s^{-1}f_n, b_{n+1}) = (-1)^{\alpha n} T_n\left(\kappa(b_1 \otimes f_1), \dots, \kappa(b_n \otimes f_n)\right) \triangleright b_{n+1},$$

(iii) *for all $n \geqslant 1$, $b_i \in B, f_i \in B^\vee$,*

$$m_{2n+1}(s^{-1}f_0, b_1, s^{-1}f_1, \dots, b_n, s^{-1}f_n) = (-1)^{\beta n} s^{-1}f_0 \triangleleft T_n\left(\kappa(b_1 \otimes f_1), \dots, \kappa(b_n \otimes f_n)\right),$$

(iv) *m_n vanishes in all other cases.*

Then $(\partial_{-1}B, \{m_n\}_{n\geqslant 1})$ is an A_∞-algebra.

III.1, Homotopy relative Rota-Baxter algebras and A_∞ algebras

Lemma (Q.-Wang 2025)

Let (A, B) be a left compatible pair and $(A, A^\vee, \{T_n\}_{n\geq 1})$ a strong B-derivative homotopy relative Rota-Baxter algebra. Define a family of operations $\{m_n\}_{n\geqslant 1}$ on $\partial_{-1}B$ as

(i) $m_1 = -d_{\partial_{-1}B}$,

(ii) *the operation m_2 is constrained to coincide with the associative product on B,*

(ii) *for all $n \geqslant 1$, $b_i \in B, f_i \in B^\vee$,*

$$m_{2n+1}(b_1, s^{-1}f_1, b_2, \dots, s^{-1}f_n, b_{n+1}) = (-1)^{\alpha n} T_n\left(\kappa(b_1 \otimes f_1), \dots, \kappa(b_n \otimes f_n)\right) \rhd b_{n+1},$$

(iii) *for all $n \geqslant 1$, $b_i \in B, f_i \in B^\vee$,*

$$m_{2n+1}(s^{-1}f_0, b_1, s^{-1}f_1, \dots, b_n, s^{-1}f_n) = (-1)^{\beta n} s^{-1}f_0 \lhd T_n\left(\kappa(b_1 \otimes f_1), \dots, \kappa(b_n \otimes f_n)\right),$$

(iv) *m_n vanishes in all other cases.*

Then $(\partial_{-1}B, \{m_n\}_{n\geqslant 1})$ is an A_∞-algebra.

Remark

In particular, if $(A, A^\vee, \{T_n\}_{n\geq 1})$ is a homotopy relative Rota-Baxter algebra and B is a left dg A-module, then $(\partial_{-1}B, \{m_n\}_{n\geqslant 1})$ is an A_∞-algebra with trivial m_2.

III.2, Cyclic homotopy relative Rota-Baxter algebras and pre-Calabi-Yau structures

Theorem (Q.-Wang 2025)

Let $(A, A^{\vee}, \{T_n\}_{n\geq 1})$ *be a homotopy relative Rota-Baxter algebra and* B *a left dg* A*-module.*

- *If the operator* $\{T_n\}_{n\geq 1}$ *is a cyclic, then* $(B, \{m_n\}_{n\geq 1})$ *is a good pre-Calabi-Yau algebra with trivial* m_2*;*

III.2, Cyclic homotopy relative Rota-Baxter algebras and pre-Calabi-Yau structures

Theorem (Q.-Wang 2025)

Let $(A, A^\vee, \{T_n\}_{n\geq 1})$ *be a homotopy relative Rota-Baxter algebra and* B *a left dg* A*-module.*

- *If the operator* $\{T_n\}_{n\geq 1}$ *is a cyclic, then* $(B, \{m_n\}_{n\geq 1})$ *is a good pre-Calabi-Yau algebra with trivial* m_2*;*
- *moreover, if* (A, B) *is a left compatible pair and* $\{T_i\}_{i\geq 1}$ *is also* B*-derivative, then* B *is a good manageable pre-Calabi-Yau algebra;*

III.2, Cyclic homotopy relative Rota-Baxter algebras and pre-Calabi-Yau structures

Theorem (Q.-Wang 2025)

Let $(A, A^\vee, \{T_n\}_{n\geq 1})$ *be a homotopy relative Rota-Baxter algebra and* B *a left dg* A*-module.*

- *If the operator* $\{T_n\}_{n\geq 1}$ *is a cyclic, then* $(B, \{m_n\}_{n\geq 1})$ *is a good pre-Calabi-Yau algebra with trivial* m_2*;*
- *moreover, if* (A, B) *is a left compatible pair and* $\{T_i\}_{i\geq 1}$ *is also* B*-derivative, then* B *is a good manageable pre-Calabi-Yau algebra;*
- *furthermore, if the* $\{T_n\}_{n\geq 1}$ *is also ultracyclic, then* B *is a good manageable special pre-Calabi-Yau algebra.*

III.2, Cyclic homotopy relative Rota-Baxter algebras and pre-Calabi-Yau structures

Corollary

In special case where B is finite dimensional, the above constructions, when restricted to $\mathrm{End}(B)$, give rise to the following three bijections:

$$\mathfrak{F}_1 : \left\{ \begin{array}{c} \text{differentials } \boldsymbol{d} \text{ on } B \text{ and cyclic relative homotopy} \\ \text{Rota-Baxter operators on } \mathrm{End}(B)^\vee \end{array} \right\}$$

$$\to \left\{ \begin{array}{c} \text{good pre-Calabi-Yau algebras} \\ \{\boldsymbol{m}_n\}_{n\geqslant 1} \text{ on } B \text{ with trivial } \boldsymbol{m}_2 \end{array} \right\},$$

$$\mathfrak{F}_2 : \left\{ \begin{array}{c} \text{differentials } \boldsymbol{d} \text{ on } B \text{ and ultracyclic relative} \\ \text{homotopy Rota-Baxter operators on } \mathrm{End}(B)^\vee \end{array} \right\}$$

$$\to \left\{ \begin{array}{c} \text{good special pre-Calabi-Yau algebras} \\ \{\boldsymbol{m}_n\}_{n\geqslant 1} \text{ on } B \text{ with trivial } \boldsymbol{m}_2 \end{array} \right\},$$

$$\mathfrak{F}_3 : \left\{ \begin{array}{c} \text{dg algebra structures } (\boldsymbol{d}, \cdot) \text{ on } B \text{ and ultracyclic } B\text{-derivative} \\ \text{relative homotopy Rota-Baxter operators on } \mathrm{End}(B)^\vee \end{array} \right\}$$

$$\to \left\{ \begin{array}{c} \text{good manageable special pre-Calabi-Yau} \\ \text{algebras } \{\boldsymbol{m}_n\}_{n\geqslant 1} \text{ on } B \end{array} \right\}.$$

Definition (Schedler 2009)

- *A* DL_∞*-algebra (homotopy double Lie algebra) is a graded space* $V = \oplus_{n\in\mathbb{Z}} V^n$ *endowed with maps* $\{\{-,\dots,-\}\}_{n+1} : V^{\otimes n+1} \to V^{\otimes n+1}$*, for all* $n \geq 0$*, where* $\{\{-,\dots,-\}\}_{n+1}$ *has degree* $n-1$*, satisfying that:*
  - (i) *Double skew-symmetry:*

$$\sigma\{\{-,\dots,-\}\}_{n+1}\sigma^{-1} = \mathrm{sgn}(\sigma)\{\{-,\dots,-\}\}_{n+1}\text{, for all } \sigma \in \mathbb{S}_{n+1};$$

  - (ii) *Double Jacobi*$_\infty$*:*

$$\sum_{i+j=n}(-1)^{j(i+1)}\sum_{\sigma\in C_{n+1}}\mathrm{sgn}(\sigma)\sigma\left(\{\{-,\dots,-,\{\{-,\dots,-\}\}_{i+1}\}\}_{L,j+1}\right)\sigma^{-1} = 0.$$

- *A* DP_∞*-algebra (homotopy double Poisson algebra) is a graded algebra* A*, equipped with a* DL_∞*-algebra structure* A *satisfying the double Leibniz*$_\infty$*-rule: for* $n \geq 0$*, and* $a_1,\dots a_n, a'_{n+1}, a''_{n+1} \in A$*,*

$$\{\{a_1,\dots,a_n,a'_{n+1}a''_{n+1}\}\}_{n+1} = \{\{a_1,\dots,a_n,a'_{n+1}\}\}_{n+1}a''_{n+1}$$
$$+(-1)^{a'_{n+1}(n-1+\sum_{k=1}^{n}a_k)}a'_{n+1}\{\{a_1,\dots,a_n,a''_{n+1}\}\}_{n+1}.$$

Theorem (Q.-Wang 2025)

Let $(V, \{\{\{-, \ldots, -\}\}_n\}_{n\geq 1})$ *be a homotopy double Lie algebra. Denote a family of* $\{l_n\}_{n\geq 1}$ *on the graded symmetric space* $S(V)$*, which satisfies: for all* $u_1^1, \ldots, u_{k_1}^1, \ldots, u_1^n, \ldots, u_{k_n}^n \in V$

$$l_n(u_1^1 \cdots u_{k_1}^1 \otimes \cdots \otimes u_1^n \cdots u_{k_n}^n) := (n-1)! \sum_{1\leq q_1\leq k_1, \ldots, 1\leq q_n\leq k_n} (-1)^{\theta_n} \{\{u_{q_1}^1, \ldots, u_{q_n}^n\}\}_n^{[1]} \cdots \{\{u_{q_1}^1, \ldots, u_{q_n}^n\}\}_n^{[n]} \cdot u_1^1 \cdots \widehat{u_{q_1}^1} \cdots u_{k_1}^1 \cdots u_1^n \cdots \widehat{u_{q_n}^n} \cdots u_{k_n}^n.$$

Then $\left(S(V), \{l_n\}_{n\geq 1}\right)$ *is a homotopy Possion algebra. Thus* $V^\vee$ *can be considered as a* **derived Poisson manifold**.

Definition (Schedler 2009)

Let A be a unitary graded associative algebra. A solution of **associative Yang-Baxter-infinity equation** *is a family of elements* $\{r_n \in A^{\otimes n}\}_{n\geq 1}$ *with each* $|r_n| = n-2$*, satisfying: for* $n \geq 1$*,*

$$\sum_{i+j=n+1} (-1)^{(j+1)i} \sum_{\sigma\in C_n} \operatorname{sgn}(\sigma) r_i^{\sigma(1),\sigma(2),\ldots,\sigma(i)} r_j^{\sigma(i),\sigma(i+1),\sigma(i+2),\ldots,\sigma(n)} = 0.$$

Theorem (Schedler 2009)

Let V be a graded space. There is a bijection between the set of homotopy double Lie algebra structures on V and the set of skew-symmetric solutions of the associative Yang-Baxter-infinity equation on $\mathrm{End}(V)$.

T. Schedler, Poisson algebras and Yang-Baxter equations, Advances in quantum computation, Contemp. Math., Contemp. Math., Amer. Math. Soc., Providence, RI, **482** (2009), 91-106.

IV.3, Homotopy double Lie algebras and pre-Calabi-Yau structures

Theorem (Fernández-Herscovich 2021)

Let $A = \oplus_{n\in\mathbb{Z}} A^n$ be a graded space. For a good manageable special pre-Calabi-Yau structure $\{m_n\}_{n\geq 1}$ on A, one can define a family of maps $\{\{\{-,\dots,-\}\}_n : A^{\otimes n} \to A^{\otimes n}\}_{n\geq 1}$ by

$$(f_1 \otimes \cdots \otimes f_n)\left(\{\{a_1,\dots,a_n\}\}_n\right) = s^{a_1,\dots,a_n}_{f_1,\dots,f_n} \zeta_A \left(m_{2n-1}\left(a_n, s^{-1}f_n, \dots, a_2, s^{-1}f_2, a_1\right), s^{-1}f_1\right).$$

Then the map determines a bijection

$$\left\{ \begin{array}{c} \text{good manageable special pre-Calabi-Yau} \\ \text{structures } \{m_n\}_{n\geq 1} \text{ on } A \end{array} \right\} \to \left\{ \begin{array}{c} \text{homotopy double Poisson algebra} \\ \text{structures } \{\{\{-,\dots,-\}\}_n\}_{n\geq 1} \text{on } A \end{array} \right\}.$$

Theorem (Q.-Wang 2025)

Let $(A, A^\vee, \{T_i\}_{i\geq 1})$ *be a finite dimensional ultracyclic relative homotopy Rota-Baxter algebra and* B *a dg left* A*-module. We define a family of maps* $\{\{\{-,\ldots,-\}\}_n\}_{n\geq 1}$ *as:* $\{\{-\}\}_1 = d_B : B \to B$ *and for all* $n \geq 1$*,*

$$\{\{-,\ldots,-\}\}_{n+1} = \Psi^n\left(\mathrm{id}_{A^{\otimes n}}\right), \tag{7}$$

where

$$\Psi^n : \mathrm{End}(A^{\otimes n}) \cong A^{\otimes n} \otimes (A^\vee)^{\otimes n} \stackrel{\mathrm{id}^{\otimes n} \otimes T_n}{\longrightarrow} A^{\otimes n+1} \stackrel{\Phi^{\otimes n+1}}{\longrightarrow} \mathrm{End}(B)^{\otimes n+1} \to \mathrm{End}(B^{\otimes n+1}).$$

Then $\{\{\{-,\ldots,-\}\}_n\}_{n\geq 1}$ *defines a homotopy double Lie algebra structure on* B*.*

Theorem (Q.-Wang 2025)

Let $(A, A^\vee, \{T_i\}_{i\geq 1})$ *be a finite dimensional ultracyclic relative homotopy Rota-Baxter algebra and* B *a dg left* A*-module. We define a family of maps* $\{\{\{-,\dots,-\}\}_n\}_{n\geq 1}$ *as:* $\{\{-\}\}_1 = d_B : B \to B$ *and for all* $n \geq 1$,

$$\{\{-,\dots,-\}\}_{n+1} = \Psi^n\left(\mathrm{id}_{A^{\otimes n}}\right), \tag{7}$$

where

$$\Psi^n : \mathrm{End}(A^{\otimes n}) \cong A^{\otimes n} \otimes (A^\vee)^{\otimes n} \xrightarrow{\mathrm{id}^{\otimes n} \otimes T_n} A^{\otimes n+1} \xrightarrow{\Phi^{\otimes n+1}} \mathrm{End}(B)^{\otimes n+1} \to \mathrm{End}(B^{\otimes n+1}).$$

Then $\{\{\{-,\dots,-\}\}_n\}_{n\geq 1}$ *defines a homotopy double Lie algebra structure on* B*. Moreover, if* (A, B) *is a left compatible pair and* $\{T_i\}_{i\geq 1}$ *is* B*-derivative, then* $\{\{\{-,\dots,-\}\}_n\}_{n\geq 1}$ *defines a homotopy double Poisson algebra structure on* B*.*

V, Conclusions

Algebra	**Module**
Cyclic homotopy Rota-Baxter algebras	Good pre-Calabi-Yau without product
Ultracyclic homotopy Rota-Baxter algebras	Homotopy double Lie algebras/ $AYBE_\infty$
Ultracyclic homotopy Rota-Baxter algebras $+$ $\{T_i\}_{i\geq 1}$ is B-derivative	Good manageable special pre-Calabi-Yau / Homotopy double Poisson algebras

Thank you!