Efficient algorithm to decompose circuit varieties

Emiliano Liwski
(KU Leuven)
(joint with Fatemeh Mohammadi and Rémi Prébet)

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Motivation

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- A matroid formalizes this structure: a ground set $[d]$, together with a consistent labeling of its subsets as dependent or independent.
- This labeling is not arbitrary, it must satisfy axioms that mirror linear dependence relations in vector spaces.

Matroids

Matroid

A matroid M consists of a ground set $[d]$ together with a collection $\mathcal{I}$ of subsets of $[d]$, called **independent sets**, that satisfy the following three axioms:

- $\emptyset \in \mathcal{I}$.
- If $I \in \mathcal{I}$ and $I' \subset I$, then $I' \in \mathcal{I}$.
- If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, then there exists an element $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

Matroids

- A subset of $[d]$ that is not independent is called **dependent**.
- A minimally dependent set is called a **circuit**. The set of **circuits** of M is denoted by $\mathcal{C}(M)$.
- A **basis** of M is a maximal independent subset.
- The **rank** of M is the size of any basis.

Example

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- A determines a matroid on [6].
- $\{1, 2, 3\}$ is dependent, while $\{2, 4, 5\}$ is independent.
- $\mathcal{C}(A) = \{12, 134, 234, 156, 256, 3456\}$.
- $\mathrm{rk}(A) = 3$.

Realizable matroids

More generally, consider any matrix

$$A = [v_1 \mid \cdots \mid v_d].$$

- A determines a matroid on $[d]$, where

$$F \subset [d] \text{ is dependent} \longleftrightarrow \{v_i : i \in F\} \text{ is linearly dependent.}$$

- Matroids arising in this way are called **realizable.**

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- Are all matroids realizable?
  - Some matroids are not realizable (Whitney, 1935).
  - Almost all matroids are non-linear (Nelson, 2018).
  - There are at most $2^{n^3/4}$ representable matroids with ground set $[n]$.

Main questions

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- **Combinatorics:** Is there a matrix A with $\mathcal{C}(A) = \mathcal{C}$?

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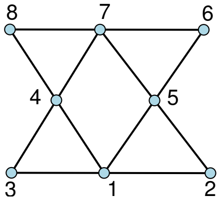
- **Statistics:** is there such a matrix with all real non-negative entries?
- **Total positivity:** is there such a matrix with non-negative minors?
- **Rigidity theory:** How to expand $\mathcal{C}$ to be the circuits of a matrix.

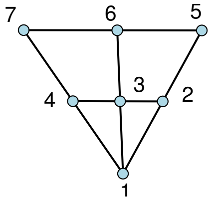
Point-line configurations

Definition

Point-line configurations: matroids of rank 3 whose circuits have size 3 or 4.

- *Points:* Elements of the ground set.
- *Line:* maximal dependent subset where every subset of three elements is dependent.
- The *degree* of a point is the number of lines it belongs.





Algebraic geometry

Definition

Affine algebraic variety: common zero set of a collection $\{F_i\}_{i \in I}$ of complex polynomials on $\mathbb{C}^n$. Notation: $\mathbb{V}(\{F_i\}_{i \in I})$.

Example:

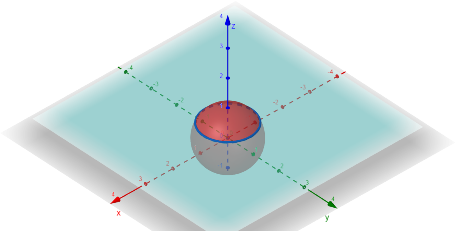


Figure: $\mathbb{V}(x^2 + y^2 + z^2 - 1, z - \frac{1}{2})$

Algebraic geometry

- Given an affine algebraic variety $V \subseteq \mathbb{C}^n$, we define

$$\mathbb{I}(V) = \{f \in \mathbb{C}[x_1, \cdots, x_n] : f(x) = 0 \text{ for all } x \in V\}.$$

- For an ideal I of a ring R, the **radical** $\sqrt{I}$ is

$$\{r \in R : r^n \in I \text{ for some } n \in \mathbb{N}\}.$$

- If Y is an algebraic variety and J is an ideal of $\mathbb{C}[X]$, we have the following equalities :

$$Y = \mathbb{V}(\mathbb{I}(Y)), \quad \text{and} \quad \sqrt{J} = \mathbb{I}(\mathbb{V}(J)).$$

Zariski topology

Definition

The **Zariski topology** on $\mathbb{C}^n$ is the topology where the closed sets are the affine algebraic varieties.

Note that if A is closed for the Zariski topology, then A is closed for the Euclidean topology.

Matroid varieties

Definition

A **realization** of a matroid M of rank n on $[d]$ is a collection of vectors $\gamma = \{\gamma_1, \dots, \gamma_d\} \subset \mathbb{C}^n$ such that

$$\{i_1, \dots, i_p\} \text{ is a dependent set of } M \iff \{\gamma_{i_1}, \dots, \gamma_{i_p}\} \text{ is l. d.}$$

The **realization space** of M is defined as

$$\Gamma_M := \{\gamma \subset \mathbb{C}^n : \gamma \text{ is a realization of } M\} \subset \mathbb{C}^{nd}.$$

Note: $\gamma = \{\gamma_1, \cdots, \gamma_d\} \subseteq \mathbb{C}^n$ can be considered as $[\gamma_1 \cdots \gamma_d] \in \mathbb{C}^{nd}$.

Definition

The **matroid variety** V_M is the Zariski closure of Γ_M in $\mathbb{C}^{nd}$. The associated ideal $I_M = \mathbb{I}(V_M)$ is called the **matroid ideal**.

Matroid ideal

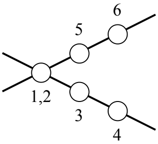
Since $V_M \subseteq \mathbb{C}^{nd}$, $I_M \subseteq \mathbb{C}[X]$, with

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1d} \\ x_{21} & x_{22} & \cdots & x_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nd} \end{pmatrix}.$$

Example

Example. A realization of the matroid M:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \in V_M$$



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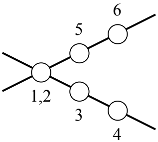

- Consider a matrix of variables

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$[134] = 0 \iff x_1 y_3 z_4 - x_3 y_1 z_4 + \cdots + x_4 y_3 z_1 = 0$$

$$[12] = 0 \iff x_1 y_2 - x_2 y_1 = x_1 z_2 - x_2 z_1 = y_1 z_2 - y_2 z_1 = 0$$

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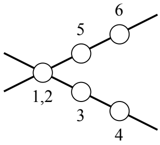

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- $[12], [134], [234], [156], [256]$ are the defining polynomials of V_M.

Main goal

Main goal

Find a generating set for I_M, up to radical.

- I_M can be computed as a saturation of ideals.
- This saturation is usually not possible to compute with current computer algebra systems.

Circuit variety

Definition

A collection of vectors $\gamma = \{\gamma_1, \dots, \gamma_d\} \subset \mathbb{C}^n$ **includes the dependencies** of M if it satisfies:

$$\{i_1, \dots, i_k\} \text{ is dependent in } M \Longrightarrow \{\gamma_{i_1}, \dots, \gamma_{i_k}\} \text{ is l.d.}$$

The **circuit variety** of M is

$$V_{\mathcal{C}(M)} = \{\gamma : \gamma \text{ includes the dependencies of } M\} \subset \mathbb{C}^{nd}.$$

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- $\mathcal{C}(A) = \{12, 134, 234, 156, 256, 3456\}$
- A collection $\gamma = (\gamma_1, \ldots, \gamma_6) \subset \mathbb{C}^3$ belongs to $V_{\mathcal{C}(A)}$ if and only if

$$\{\gamma_1, \gamma_2\}, \{\gamma_1, \gamma_3, \gamma_4\}, \ldots\ldots$$

  are linearly dependent.

Main goal

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Let M be a point-line configuration. Develop an algorithm to determine the irreducible decomposition of $V_{\mathcal{C}(M)}$.

Plan

Develop an algorithm for identifying the set of minimal matroids of M.

Decomposing $V_{\mathcal{C}(M)}$: Minimal matroids

Let N_1 and N_2 be matroids on $[d]$. We say that $N_1 \leq N_2$ if $\mathcal{D}(N_1) \subseteq \mathcal{D}(N_2)$. We define the set of minimal matroids of M as:

$$\min(M) = \min\{N : N > M\}.$$

Theorem (Clarke et al., 2022)

Let M be a point-line configuration. Then

$$V_{\mathcal{C}(M)} = V_M \bigcup_{N \in \min(M)} V_{\mathcal{C}(N)} .$$

Oliver Clarke, Kevin Grace, Fatemeh Mohammadi, and Harshit J. Motwani. Matroid stratifications of hypergraph varieties, their realization spaces, and discrete conditional independence models. arXiv: 2103.16550.

Decomposing $V_{\mathcal{C}(M)}$: step-by-step plan

Step-by-step plan [1]:

1. Identify the minimal matroids over M (algorithm).

[1] Emiliano Liwski, Fatemeh Mohammadi and Rémi Prébet. Efficient algorithms for minimal matroid extensions and irreducible decompositions of circuit varieties arXiv: 2502.00799. arXiv: 2504.16632.

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3. In the recursive decomposition step, this process is applied to each circuit variety that appears in the decomposition, until we obtain a decomposition of $V_{\mathcal{C}(M)}$ as a union of matroid varieties.

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5. Remove redundant components.

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Decomposing $V_{\mathcal{C}(M)}$: Minimal matroids

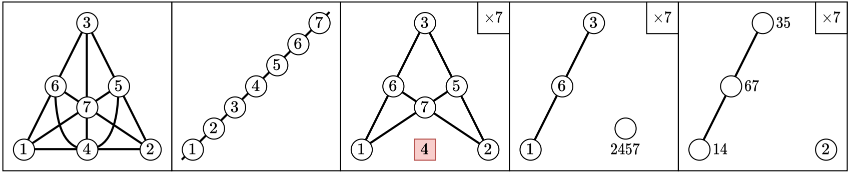
Theorem (Liwksi-Mohammadi, 2024)

We present a combinatorial algorithm to determine the minimal matroids of a given matroid M.

Emiliano Liwski, Fatemeh Mohammadi and Rémi Prébet. Efficient algorithms for minimal matroid extensions and irreducible decompositions of circuit varieties arXiv: 2504.16632.

Example

Applying the algorithm, we find that the Fano plane has exactly 22 minimal matroids, depicted in the figure below.

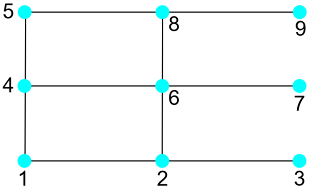


Decomposing $V_{\mathcal{C}(M)}$: nilpotent and solvable point-line configuration

- $S_M := \{p \in [d] : \deg(p) \geq 2\}, Q_M := \{p \in [d] : \deg(p) \geq 3\}$.
- Define the chains $M_0 = M$, $M_1 = S_M$, $M_{j+1} = S_{M_j}$ and $M^0 = M$, $M^1 = Q_M$, $M^{j+1} = Q_{M_j}$ for all $j \geq 1$.

Definition

If $M_j = \emptyset$ for some j, then M is **nilpotent**. If $M^j = \emptyset$ for some j, then M is **solvable**.



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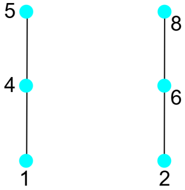

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If $M_j = \emptyset$ for some j, then M is **nilpotent**. If $M^j = \emptyset$ for some j, then M is **solvable**.

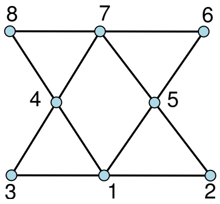
Example

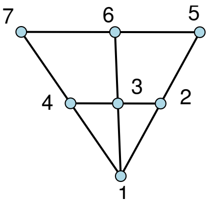
- Let M and N be the point-line configurations depicted below.
- M is solvable, since we have

$$M^1 = \{1, 7\}, \quad \text{and} \quad M^2 = \emptyset.$$

- N is solvable, since we have

$$N^1 = \{1\}, \quad \text{and} \quad N^2 = \emptyset.$$





Decomposing $V_{\mathcal{C}(M)}$: Useful results

Theorem (Liwksi-Mohammadi, 2024)

1. If M is nilpotent without points of degree greater than two, then

$$V_{\mathcal{C}(M)} = V_M.$$

Emiliano Liwski, and Fatemeh Mohammadi. Solvable and Nilpotent Matroids: Realizability and Irreducible Decomposition of Their Associated Varieties. arXiv: 2408.12784.

Decomposing $V_{\mathcal{C}(M)}$: Useful results

Theorem (Liwksi-Mohammadi, 2024)

1. If M is nilpotent without points of degree greater than two, then
$$V_{\mathcal{C}(M)} = V_M.$$
2. If M is solvable, then V_M is irreducible.

Emiliano Liwski, and Fatemeh Mohammadi. Solvable and Nilpotent Matroids: Realizability and Irreducible Decomposition of Their Associated Varieties. arXiv: 2408.12784.

Decomposing $V_{\mathcal{C}(M)}$

Integrate this results in the step-by-step plan:

1. Identify the minimal matroids over M (algorithm).
2. Decompose the circuit variety using
$$V_{\mathcal{C}(M)} = \bigcup_{N \in \min(M)} V_{\mathcal{C}(N)} \ \cup \ V_M.$$
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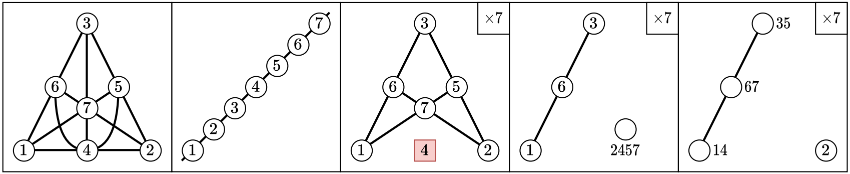
4. Establish the irreducibility of the matroid varieties involved, **using solvability.**
5. Remove redundant components.

Fano Plane

The irreducible decomposition of the circuit variety of the Fano plane is:

$$V_{\mathcal{C}(M_{\text{Fano}})} = V_{U_{2,7}} \bigcup_{i=1}^{7} V_{A_i} \bigcup_{j=1}^{7} V_{B_j} \bigcup_{k=1}^{7} V_{C_k}.$$



Examples

- The circuit variety of the **Fano plane** has 22 irreducible components.
- The circuit variety of the **Pappus configuration** has 23 irreducible components.
- The circuit variety of the **second configuration** 9_3 has 29 irreducible components.

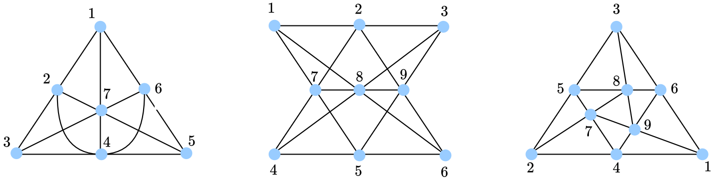


Figure: (Left) Fano plane; (Center) Pappus configuration; (Right) Second configuration 9_3

Examples

- The circuit variety of the **MacLane** configuration has 23 irreducible components.
- The circuit variety of the **Affine plane** has 78 irreducible components.

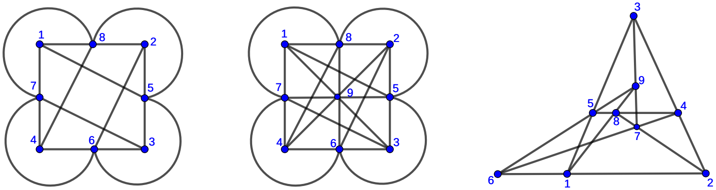


Figure: (Left) MacLane configuration; (Center) Affine plane; (Right) Third configuration 9_3

References

- Efficient algorithms for minimal matroid extensions and irreducible decompositions of circuit varieties.

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Thank you for your attention!