Some problems in Hopf–Galois theory.

VUB algebra seminar

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$n = pq$

- Two groups$N = C_{pq}$ or $C_p \rtimes C_q$.
- Let
$$N = C_{pq} = \langle \sigma, \tau \; od \sigma^p = \tau^q = 1, \sigma\tau = \tau\sigma \rangle.$$

$M \leq \mathrm{Hol}(C_{pq})$ transitive:

- $M = N \rtimes A,\ A \leq \mathrm{Aut}(N)$,
- Let $\langle \alpha \rangle := \mathrm{Syl}_q(\mathrm{Aut}(N))$

$$M = \underbrace{\langle \sigma, (\tau, \alpha^a) \rangle}_{=:J} \rtimes B,\ B \leq \mathrm{Norm}_{\mathrm{Hol}(C_{pq})}(J).$$

Note: J is *minimally transitive* with $|J| > pq$ (if a small enough) – can't "see a skew brace inside".

$n = pq$

Structure	#groups	$\lvert\mathrm{Aut}(M, M')\rvert$	#HGS
$(N \rtimes (C_p \rtimes C_{q^{c_0}})) \rtimes C_d,\ c_0 \neq 0$	1	$2p(p-1)$	2
$(C_p \rtimes C_q) \times C_p$	2	$p^2(p-1)$	$2p$
$\mathbb{F}_p^2 \rtimes_u C_{q^{c_0}},\ (c_0, u) \neq (0, u), (1, -1),$ $u \in (\mathbb{Z}/q^{c_0}\mathbb{Z})^\times \setminus \{\frac{1}{2}(q^{c_0} - q^{c_0-1})\}$	2	$p^2(p-1)$	$2p$
$\mathbb{F}_p^2 \rtimes_{\frac{1}{2}(q^{c_0}-q^{c_0-1})} C_{q^{c_0}},\ c_0 \neq 0$	1	$2p^2(p-1)$	$2p$
$(C_p \times (C_p \rtimes C_q)) \rtimes C_d,\ d > 1$	2	$p(p-1)$	2
$\mathbb{F}_p^2 \rtimes_u C_{dq^{c_0}},\ (c_0, u) \neq (0, u), (1, -1), d > 1$ $u \in (\mathbb{Z}/q^{c_0}\mathbb{Z})^\times \setminus \{\frac{1}{2}(q^{c_0} - q^{c_0-1})\}$	2	$p(p-1)$	2
$\mathbb{F}_p^2 \rtimes_{\frac{1}{2}(q^{c_0}-q^{c_0-1})} C_{dq^{c_0}},\ c_0 \neq 0, d > 1$	1	$2p(p-1)$	2
$C_p \rtimes C_{dq^{c_0}},\ (c_0, d) \neq (1, 1), (0, d)$	$2p\varphi(q^{c_0})$	$p-1$	$2\varphi(q^{c_0})$
$C_p \rtimes C_q$	$2p(q-1)+2$	$p(p-1)$	$2p(q-2)+2$
$(C_p \rtimes C_{dq^{c_0}}) \times C_q$	$2p$	$(p-1)(q-1)$	$2(q-1)$

Table 1: Transitive subgroups for $N = C_p \rtimes C_q$.

Computer results

Degree	Types	Total		Regular		AC		BC
		#HGS	#Sbracoids	#Gal	#Sbraces	#HGS	#Sbracoids	HGS
2	1	1	1	1	1	1	1	1
3	1	2	2	1	1	2	2	2
4	2	10	8	6	4	6	6	7
5	1	3	3	1	1	3	3	3
6	2	15	12	8	6	7	6	9
7	1	4	4	1	1	4	4	4
8	5	348	148	190	47	74	47	147
9	2	38	23	12	4	26	20	28
10	2	27	20	10	6	11	9	17
11	1	4	4	1	1	4	4	4
12	5	249	134	102	38	56	38	81
13	1	6	6	1	1	6	6	6
14	2	32	24	12	6	14	12	19
15	1	8	8	1	1	8	8	8
16	14	49913	9739	25168	1605	2636	815	8216
17	1	5	5	1	1	5	5	5
18	5	881	333	289	49	123	89	253
19	1	6	6	1	1	6	6	6
20	5	434	203	166	43	79	62	156
21	2	78	36	28	8	22	18	46
22	2	36	24	16	6	14	12	19
23	1	4	4	1	1	4	4	4
24	15	14908	4752	5618	855	844	504	2682
25	2	106	58	30	4	70	54	74
26	2	58	40	18	6	22	18	35
27	5	6699	739	4329	101	766	283	1100
28	4	388	202	128	29	84	72	143
29	1	6	6	1	1	6	6	6
30	4	479	304	80	36	99	72	197
31	1	8	8	1	1	8	8	8

Some results

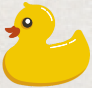
Degree	Types	Total		Regular		AC		BC
		#HGS	#Sbracoids	#Gal	#Sbraces	#HGS	#Sbracoids	HGS
33	1	10	10	1	1	10	10	10
34	2	59	36	22	6	19	15	33
35	1	16	16	1	1	16	16	16
36	14	16512	4159	5980	400	1099	753	2474
37	1	9	9	1	1	9	9	9
38	2	57	36	24	6	21	18	29
39	2	133	55	46	8	34	28	77
40	14	29534	8873	8556	944	1486	831	5931
41	1	8	8	8	1	8	8	8
42	6	1041	484	374	78	148	112	329
43	1	8	8	1	1	8	8	8
44	4	466	200	184	29	82	70	141
45	2	166	115	12	4	126	104	132
46	2	48	24	28	6	14	12	19
47	1	4	4	1	1	4	4	4
48	52	—	—	—	—	—	—	—
49	2	200	97	56	4	122	92	128
50	5	3430	978	969	51	339	235	865
51	1	14	14	1	1	14	14	14
52	5	1023	409	374	43	161	127	343
53	1	6	6	1	1	6	6	6
54	15	—	16017	—	1028	—	1953	—
55	2	192	54	88	12	32	24	94
56	13	32721	9227	10010	815	1620	968	5747
57	2	169	61	64	8	35	27	93
58	2	74	40	34	6	22	18	35
59	1	4	4	1	1	4	4	4
60	13	13457	4621	3128	418	947	668	2529
61	1	12	12	1	1	12	12	12
62	2	82	48	36	6	28	24	39
63	4	1875	501	504	47	335	207	749
64	267	—	—	—	—	—	—	—
65	1	30	30	1	1	30	30	30
66	4	608	352	128	36	118	90	211

Some results

Degree	Types	Total		Regular		AC		BC
		#HGS	#Sbracoids	#Gal	#Sbraces	#HGS	#Sbracoids	HGS
67	1	8	8	1	1	8	8	8
68	5	1162	391	478	43	145	108	352
69	1	10	10	1	1	10	10	10
70	4	1012	608	120	36	198	144	411
71	1	8	8	1	1	8	8	8
72	50	2004057	329821	646560	17790	—	13060	—
73	1	12	12	1	1	12	12	12
74	2	105	60	42	6	33	27	53
75	3	1795	357	597	6	290	230	330
76	4	763	304	296	14	127	109	220
77	1	20	20	1	29	20	20	20
78	6	1957	828	650	78	244	177	637
79	1	8	8	1	1	8	8	8
80	52	—	—	—	—	—	—	—
81	15	—	68549	—	8436	—	7470	—
82	2	106	56	46	6	30	24	61
83	1	4	4	1	1	4	4	4
84	15	—	6371	—	606	—	925	—
85	1	29	29	1	1	29	29	29
86	2	94	48	48	6	28	24	39
87	1	16	16	1	1	16	16	16
88	12	—	9120	—	800	—	934	—
89	1	8	8	1	1	8	8	8
90	10	30167	10256	2890	294	2165	1365	6611
91	1	48	48	1	1	48	48	48
92	4	706	200	352	29	82	70	141
93	2	246	72	100	8	44	36	130
94	2	72	24	52	6	14	12	19
95	1	24	24	1	1	24	24	24
96	231	—	—	—	—	—	—	—
97	1	12	12	1	1	12	12	12
98	5	—	1541	—	53	—	413	—
99	2	202	136	12	4	150	122	158

Thank You!

Questions?



References i