An overview on pretorsion theories

Federico Campanini

Joint work with F. Borceux, M. Gran and W. Tholen (first part), F. Fedele and E. Yıldırım (second part)

Definition: Let $\mathbb{C}$ be an abelian category.
A pair $(\mathcal{T}, \mathcal{F})$ of full replete subcategories of $\mathbb{C}$ is a torsion theory if

- $\mathrm{Hom}(T, F) = 0$ for all $T \in \mathcal{T}$, $F \in \mathcal{F}$;
- for every $X \in \mathbb{C}$ there exists a short exact sequence

$$0 \to T_X \to X \to F_X \to 0 \qquad \text{with } T \in \mathcal{T}, F \in \mathcal{F}.$$

Example:
$(\mathcal{T}, \mathcal{F})$ in the category Ab of abelian groups, where

- $\mathcal{T}$ = torsion groups;
- $\mathcal{F}$ = torsionfree groups

$$0 \longrightarrow t(G) \longrightarrow G \longrightarrow G/t(G) \longrightarrow 0 \qquad s.e.s$$

with $t(G)$ = torsion subgroup of G.

Definition: Let $\mathbb{C}$ be any pointed category.
A pair $(\mathcal{T}, \mathcal{F})$ of full replete subcategories of $\mathbb{C}$ is a torsion theory if

- $\mathrm{Hom}(T, F) = 0$ for all $T \in \mathcal{T}$, $F \in \mathcal{F}$;
- for every $X \in \mathbb{C}$ there exists a short exact sequence

$$0 \to T_X \to X \to F_X \to 0 \qquad \text{with } T \in \mathcal{T}, F \in \mathcal{F}.$$

Example:
$(\mathcal{T}, \mathcal{F})$ in the category Ab of abelian groups, where

- $\mathcal{T}$ = torsion groups;
- $\mathcal{F}$ = torsionfree groups

$$0 \longrightarrow t(G) \longrightarrow G \longrightarrow G/t(G) \longrightarrow 0 \qquad s.e.s$$

with $t(G)$ = torsion subgroup of G.

Torsion theories in pointed categories

- (Div, Red) in the category Ab of abelian groups, where
  - Div = divisible groups;
  - Red = reduced groups

$$0 \longrightarrow D(G) \longrightarrow G \longrightarrow G/D(G) \longrightarrow 0 \qquad s.e.s$$

- (NilCRng, RedCRng) in the category CRngs of commutative rings, where
  - NilCRng = nilpotent rings
  - RedCRng = reduced rings

$$0 \longrightarrow Nil(R) \longrightarrow R \longrightarrow R/Nil(R) \longrightarrow 0 \qquad s.e.s.$$

- (GrpInd, GrpHaus) in the category GrpTop of topological groups, where
  - GrpInd = groups with the indiscrete topology
  - GrpHauss = Hausdorff groups

$$0 \longrightarrow \overline{\{1\}} \longrightarrow G \longrightarrow G \,/\, \overline{\{1\}} \longrightarrow 0 \qquad s.e.s.$$

- ($\mathsf{PrimHopf}_K$, $\mathsf{GrpHopf}_K$) in the category $\mathsf{Hopf}_{K,coc}$ of cocommutative Hopf algebras, where
  - $\mathsf{PrimHopf}_K$ = primitive Hopf algebras
  - $\mathsf{GrpHopf}_K$ = group Hopf algebras

$$0 \longrightarrow \mathcal{U}(L_H) \longrightarrow H \cong \mathcal{U}(L_H) \rtimes K[G_H] \longrightarrow K[G_H] \longrightarrow 0 \qquad s.e.s.$$

Objects: sets endowed with a preorder (A, ρ) (reflexive + transitive relation)

Morphisms: monotone maps between (preordered) sets.

There is a pretorsion theory $(\mathcal{T}, \mathcal{F})$ in PreOrd given by:

- $\mathcal{T}$ = equivalence relations (symmetric preorders)
- $\mathcal{F}$ = partial orders (antysimmetric preorders)
- $\mathcal{Z} := \mathcal{T} \cap \mathcal{F}$ = discrete relations (the "equality" relations).

The short $\mathcal{Z}$-exact sequence of an object (A, ρ) is of the form

$$(A, \equiv) \xrightarrow{Id_A} (A, \rho) \xrightarrow{\pi} (A/\equiv, \leq)$$

where
· $a \equiv b$ if and only if $a\rho b$ and $b\rho a$; · $[a] \leq [b]$ if and only if $a\rho b$.

Replace the zero object and the zero morphisms

Let $\mathbb{C}$ be any category.

- Consider two full replete subcategories $\mathcal{T}$ and $\mathcal{F}$ of $\mathbb{C}$.
- Define $\mathcal{Z} := \mathcal{T} \cap \mathcal{F}$, the class of trivial objects.
- We say that a morphism $X \xrightarrow{f} Y$ in $\mathbb{C}$ is $\mathcal{Z}$-trivial if it factors through an object in $\mathcal{Z}$:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \\ & Z \arrow[ur] & \end{tikzcd} \qquad Z \in \mathcal{Z}$$

- The class of trivial morphisms forms an ideal of morphisms (denoted by Triv) in $\mathbb{C}$:
  if $f \in \mathrm{Triv}(A, B)$ or $g \in \mathrm{Triv}(B, C)$, then $g \cdot f \in \mathrm{Triv}(A, C)$.

Remark: if $\mathbb{C}$ is pointed and $\mathcal{Z} = \mathcal{T} \cap \mathcal{F} = 0$, then the ideal of trivial morphisms is the ideal of zero morphisms of $\mathbb{C}$.

Kernels and cokernels with respect to an ideal of morphisms

A morphism $k\colon K \to X$ is a $\mathcal{Z}$-kernel of $f\colon X \to Y$ if

(i) $K \xrightarrow{k} X \xrightarrow{f} Y$ is $\mathcal{Z}$-trivial;

(ii) for any $g\colon L \to X$ such that $f \cdot g$ is trivial, there is a unique $h\colon L \to K$ such that $k \cdot h = g$

$$\begin{tikzcd}
K \arrow[r, "k"] & X \arrow[r, "f"] & Y \\
& L \arrow[ul, dotted, "\exists! h"] \arrow[u, "\forall g"'] \arrow[ur, dashed, "f \cdot g \in \mathrm{Triv}"'] &
\end{tikzcd}$$

The notion of $\mathcal{Z}$-cokernel is defined dually. A sequence

$$W \xrightarrow{f} X \xrightarrow{g} Y$$

is a short $\mathcal{Z}$-exact sequence if f is the $\mathcal{Z}$-kernel of g and g is the $\mathcal{Z}$-cokernel of f.

Definition: Let $\mathbb{C}$ be any pointed category.
A pair $(\mathcal{T}, \mathcal{F})$ of full replete subcategories of $\mathbb{C}$ is a torsion theory if

- $\mathrm{Hom}(T, F) = 0$ for all $T \in \mathcal{T}$, $F \in \mathcal{F}$;
- for every $X \in \mathbb{C}$ there exists a short exact sequence

$$0 \to T_X \to X \to F_X \to 0 \qquad \text{with } T \in \mathcal{T}, F \in \mathcal{F}.$$

Definition: Let $\mathbb{C}$ be any category.
A pair $(\mathcal{T}, \mathcal{F})$ of full replete subcategories of $\mathbb{C}$ is a pretorsion theory if

- $\mathrm{Hom}(T, F) = \mathrm{Triv}(T, F)$ for all $T \in \mathcal{T}$, $F \in \mathcal{F}$;
- for every $X \in \mathbb{C}$ there exists a short $\mathcal{Z}$-exact sequence

$$T_X \to X \to F_X \qquad \text{with } T \in \mathcal{T}, F \in \mathcal{F}.$$

Basic properties of pretorsion theories

Given a pretorsion theory $(\mathcal{T}, \mathcal{F})$ in a category $\mathbb{C}$, there are two functors:

- a "torsion functor" $T\colon \mathbb{C} \to \mathcal{T}$ which is a left-inverse right-adjoint of the full embedding $E_T\colon \mathcal{T} \hookrightarrow \mathbb{C}$;
- a "torsion-free functor" $F\colon \mathbb{C} \to \mathcal{F}$ which is a left-inverse left-adjoint of the full embedding $E_F\colon \mathcal{F} \hookrightarrow \mathbb{C}$.

For every object $X \in \mathbb{C}$ there is a short $\mathcal{Z}$-exact sequence

$$T(X) \xrightarrow{\varepsilon_X} X \xrightarrow{\eta_X} F(X)$$

where the monomorphism ε_X is the X-component of the counit ε of the adjunction

$$\begin{tikzcd} \mathcal{T} \arrow[r, bend left, hook, "E_T"] \arrow[r, phantom, "\perp"] & \mathbb{C} \arrow[l, bend left, "T"] \end{tikzcd}$$

while the epimorphism η_X is the X-component of the unit η of the adjunction

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{F} \arrow[l, bend left, hook, "E_F"] \end{tikzcd}$$

Basic properties of pretorsion theories

- $X \in \mathcal{T} \iff T(X) \cong X$ and $Y \in \mathcal{F} \iff F(Y) \cong Y$.
- Two classes determine the third one, in the sense that:
  - if $\mathrm{Hom}(X, \mathcal{F}) = \mathrm{Triv}(X, \mathcal{F})$ then $X \in \mathcal{T}$ and
  - if $\mathrm{Hom}(\mathcal{T}, Y) = \mathrm{Triv}(\mathcal{T}, Y)$ then $Y \in \mathcal{F}$.
- $\mathcal{T}$ is closed under extremal quotients and $\mathcal{F}$ is closed under extremal monomorphisms.
- The three classes $\mathcal{T}, \mathcal{F}$ and $\mathcal{Z}$ are all closed under retracts.
- The initial object 0 is in $\mathcal{T}$, while the terminal object 1 is in $\mathcal{F}$ (if they exist).
  - In particular, if $\mathbb{C}$ is pointed, the zero object is in $\mathcal{Z}$.

Some examples

PreOrd = category of preordered sets [Facchini, Finocchiaro]

Objects: sets endowed with a preorder (A, ρ) (reflexive + transitive relation)

Morphisms: monotone maps between (preordered) sets.

There is a pretorsion theory $(\mathcal{T}, \mathcal{F})$ in PreOrd given by:

- $\mathcal{T}$ = equivalence relations (symmetric preorders)
- $\mathcal{F}$ = partial orders (antysimmetric preorders)
- $\mathcal{Z}$ = discrete relations (the "equality" relations).

The short $\mathcal{Z}$-exact sequence of an object (A, ρ) is of the form

$$(A, \equiv) \xrightarrow{Id_A} (A, \rho) \xrightarrow{\pi} (A/\equiv, \leq)$$

where
· $a \equiv b$ if and only if $a\rho b$ and $b\rho a$; · $[a] \leq [b]$ if and only if $a\rho b$.

There is an isomorphism of categories:

$$\begin{array}{ccc} \mathsf{PreOrd} & \xrightarrow{\sim} & \mathsf{AlexTop} \\ (A,\rho) & \longmapsto & (A,\tau_\rho) \end{array}$$

where

- AlexTop is the category of Alexandrov-discrete spaces (arbitrary intersections of open sets is open).
- $\mathcal{O} \in \tau_\rho$ if and only if $[\ x \in \mathcal{O}$ and $a\rho x \Rightarrow a \in \mathcal{O}\ ]$.

  The corresponging pretorsion theory in AlexTop is (PartAlex, T_0)
  - $\mathcal{T}$ = PartAlex = partition spaces (there exists a partition of the set which is a basis)
  - $\mathcal{F} = T_0$ - spaces
  - $\mathcal{Z}$ = discrete topological spaces

Two generalizations

1. A pretorsion theory in the category Cat of all small categories [Xarez]:
   - $\mathcal{T}$ = "symmetric categories" $\quad \mathrm{Hom}(X,Y) \neq \emptyset \Rightarrow \mathrm{Hom}(Y,X) \neq \emptyset$
   - $\mathcal{F}$ = "antisymmetric categories" $\quad \mathrm{Hom}(X,Y) \neq \emptyset,\ \mathrm{Hom}(Y,X) \neq \emptyset \Rightarrow X = Y$
   - $\mathcal{Z}$ = classes of monoids $\quad$ (no morphisms between distinct objects)

2. A pretorsion theory in the category $\mathrm{PreOrd}(\mathbb{C})$ of internal preorders in an <u>exact</u> category [Facchini, Finocchiaro, Gran]:
   - $\mathcal{T} = \mathrm{Eq}(\mathbb{C})$ = equivalence relations in $\mathbb{C}$
   - $\mathcal{F} = \mathrm{ParOrd}(\mathbb{C})$ = partial orders in $\mathbb{C}$
   - $\mathcal{Z} = \mathrm{Dis}(\mathbb{C})$ = discrete relations in $\mathbb{C}$.

There is another pretorsion theory in Cat:

- $\mathcal{T}$ = groupoids (every morphism is an isomorphism)
- $\mathcal{F}$ = skeletal categories (every isomorphism is an automorphism)
- $\mathcal{Z}$ = classes of groups (every morphism is an automorphism)

The short $\mathcal{Z}$-exact sequence of a category $\mathbb{C}$ is of the form

$$\mathsf{Iso}(\mathbb{C}) \longrightarrow \mathbb{C} \xrightarrow{Q} \mathbb{Q}$$

where the second functor is the following coequalizer in Cat

$$\coprod_{iso} 1 \overset{d}{\underset{c}{\rightrightarrows}} \mathbb{C} \xrightarrow{Q} \mathbb{Q}$$

Question: is it possible to associate a torsion theory (in an universal way) to a given pretorsion theory $(\mathcal{T}, \mathcal{F})$ in a category $\mathbb{C}$?

The idea is to consider a congruence $\mathcal{R}$ on $\mathbb{C}$ and a quotient <u>pointed</u> category

$$\Sigma\colon \mathbb{C} \longrightarrow \mathbb{C}/\mathcal{R} =: \mathsf{Stab}(\mathbb{C})$$

$$\text{“trivials”} \rightsquigarrow \text{“zeros”}$$

$$\text{Pretorsion theory} \rightsquigarrow \text{Torsion theory}$$

Torsion theory functor

Let $(\mathbb{A}, \mathcal{T}, \mathcal{F})_{pret}$ be a category $\mathbb{A}$ with a given pretorsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathbb{A}$. If $(\mathbb{B}, \mathcal{T}', \mathcal{F}')_t$ is a pointed category $\mathbb{B}$ with a given torsion theory $(\mathcal{T}', \mathcal{F}')$ in it, we say that a torsion theory functor is a functor $G \colon \mathbb{A} \to \mathbb{B}$ satisfying the following two properties:

- $G(\mathcal{T}) \subseteq \mathcal{T}'$ and $G(\mathcal{F}) \subseteq \mathcal{F}'$;
- if $T_A \to A \to F_A$ is the canonical short $\mathcal{Z}$-exact sequence associated with $A \in \mathbb{A}$ in the pretorsion theory $(\mathcal{T}, \mathcal{F})$, then
$$0 \to G(T_A) \to G(A) \to G(F_A) \to 0$$
is a short exact sequence in $\mathbb{B}$.

Theorem [F. Borceux, —, M. Gran]

Let $(\mathcal{T}, \mathcal{F})$ be a pretorsion theory in a lextensive category $\mathbb{L}$ and assume that $\mathcal{T}$ is closed under complemented subobjects. Then, there exists a "stable category" $\mathsf{Stab}(\mathbb{L})$ and a torsion theory functor $\Sigma\colon \mathbb{L} \to \mathsf{Stab}(\mathbb{L})$ which is universal among all finite coproduct preserving torsion theory functors $G\colon \mathbb{C} \to \mathbb{X}$.

$$\begin{tikzcd}
\mathbb{L} \arrow[rr, "\Sigma"] \arrow[dr, "\forall G"'] & & \mathsf{Stab}(\mathbb{L}) \arrow[dl, dotted, "\exists ! H"] \\
& \mathbb{X}. &
\end{tikzcd}$$

Examples of lextensive categories: Set, Top, $\mathsf{CRings}^{\mathsf{op}}$, Cat, PreOrd ...

Any (pre)topos is lextensive.

If $\mathbb{L}$ is lextensive, than $\mathsf{PreOrd}(\mathbb{L})$ and $\mathsf{Cat}(\mathbb{L})$ are lextensive.

Building pretorsion theories from torsion theories

Comparable torsion theories [—, Fedele]:

Let $\mathbb{C}$ be a pointed category and consider two torsion theories $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_2, \mathcal{F}_2)$ in it.

The following conditions are equivalent:

(i) $\mathcal{T}_2 \subseteq \mathcal{T}_1$ $(\mathcal{F}_1 \subseteq \mathcal{F}_2)$

(ii) $(\mathcal{T}_1, \mathcal{F}_2)$ is a pretorsion theory.

If these conditions hold, the $\mathcal{Z}$-short exact sequence of an object $X \in \mathbb{C}$ is given by

$$T_1X \longrightarrow X \longrightarrow F_2X$$

Notice: no hypothesis are required for $\mathbb{C}$ *or the torsion theories.*

Comparable torsion theories: example 1

Let R be a unital commutative ring and $S \subseteq R$ a multiplicatively closed subset ($1 \in S$ and $r, s \in S \Rightarrow r \cdot s \in S$).

There is a torsion theory $(\mathcal{T}_S, \mathcal{F}_S)$ in $\mathrm{Mod}(R)$ where $M \in \mathcal{T}_S$ iff $M \otimes_R S^{-1}R = 0$.

Explicitly, $M \in \mathcal{T}_S$ if, for every $m \in M$, there exists $s \in S$ such that $sm = 0$, while $M \in \mathcal{F}_S$ if there are no non-zero elements of M annihilated by elements of S.

Any inclusion $S \subseteq T$ of multiplicatively closed subsets of R induces a pretorsion theory $(\mathcal{T}_T, \mathcal{F}_S)$ where the class $\mathcal{Z}$ of trivial objects consists of those modules M with the following property: for every non-zero $m \in M$ we have $\mathrm{Ann}_R(m) \cap T \neq \emptyset$ and $\mathrm{Ann}_R(m) \cap S = \emptyset$.

As a particular case of what we have just seen, any inclusion of prime ideals induces a pretorsion theory, since the complement of a prime ideal is a multiplicatively closed set.

Comparable torsion theories: example 1

Let R be a domain of infinite Krull dimension and consider an infinite chain of prime ideals

$$0 = P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \dots$$

This chain induces a chain of torsion theories $(\mathcal{T}_i, \mathcal{F}_i)$, with $\mathcal{T}_0 \supsetneq \mathcal{T}_1 \supsetneq \mathcal{T}_2 \supsetneq \dots$

Thus we have pretorsion theories $(\mathcal{T}_0, \mathcal{F}_i)$, where $\mathcal{T}_0$ is the subcategory of "classical" torsion modules, while $N \in \mathcal{F}_i$ iff for every $n \in N$, $\mathrm{Ann}_R(n) \subseteq P_i$.

Conclusion:

A subcategory $\mathcal{T}$ of a given category $\mathbb{C}$ can be the torsion class of (possibly infinitely) many different pretorsion theories.

Comparable torsion theories: example 2 (suggested by S. Mantovani)

Let $\mathbb{C}$ be an homological category and consider $\mathsf{Grpd}(\mathbb{C})$. There are two (comparable) torsion theories:

$$(\ \mathsf{Ab}(\mathbb{C})\ ,\ \mathsf{Eq}(\mathbb{C})\ ) \qquad \text{and} \qquad (\ \text{connected groupoids}\ ,\ \mathbb{C}\ )$$

which then gives us a pretorsion theory

$$(\ \text{connected groupoids}\ ,\ \mathsf{Eq}(\mathbb{C})\ )$$

One last remark:

Not all pretorsion theories arise in this way.

Extension with a Serre subcategory [— , Fedele]:

- Let $\mathbb{C}$ be a pointed category where every morphism admits an (epi, mono)-factorization, and assume that $\mathbb{C}$ has pullbacks and pushouts which preserve normal epimorphisms and normal monomorphisms respectively.
- Let $\mathcal{S}$ be a Serre epireflective and monocoreflective subcategory of $\mathbb{C}$.
- Let $(\mathcal{U}, \mathcal{V})$ be a torsion theory in $\mathbb{C}$.

Then

the pair $(\mathcal{T}, \mathcal{F}) = (\mathcal{U} * \mathcal{S}, \mathcal{S} * \mathcal{V})$ is a pretorsion theory with class of trivial objects $\mathcal{S}$.

The short $\mathcal{S}$-exact sequence is given by

$$\begin{tikzcd}
0 \arrow[r] & U_X \arrow[r] \arrow[d, equal] & T_X \arrow[r] \arrow[d, "\varepsilon_X"] & S_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & U_X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "\eta_X"] & V_X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & S'_X \arrow[r] & F_X \arrow[r] & V_X \arrow[r] & 0
\end{tikzcd}$$

Recollements of abelian categories

Up to equivalence, any recollement of abelian categories is of the form:

$$\begin{tikzcd}
\mathcal{S} \arrow[r, "i_*"] & \mathcal{C} \arrow[l, bend right=40, "i^*"'] \arrow[l, bend left=40, "i^!"] \arrow[r, "j^*"] & {[\mathcal{C}/\mathcal{S}]} \arrow[l, bend right=40, "j_!"'] \arrow[l, bend left=40, "j_*"]
\end{tikzcd}$$

where $\mathcal{S}$ is a bilocalising Serre subcategory.

Theorem [— , F.Fedele]

Let $(\mathcal{T}, \mathcal{F})$ be a pretorsion theory in an additive category $\mathbb{C}$ with class of trivial objects $\mathbb{Z}$. Then, there exists a "stable category" $\mathrm{Stab}(\mathbb{C})$ and a torsion theory functor $\Sigma\colon \mathbb{C} \to \mathrm{Stab}(\mathbb{C})$ which is universal among all additive torsion theory functors $G\colon \mathbb{C} \to \mathbb{X}$.

$$\begin{tikzcd}
\mathbb{C} \arrow[rr, "\Sigma"] \arrow[dr, "\forall G"'] & & \mathrm{Stab}(\mathbb{C}) \arrow[dl, dotted, "\exists! H"] \\
& \mathbb{X}. &
\end{tikzcd}$$

Lattices of pretorsion classes

Some remarks:

- If $\mathcal{T}$ is a torsion class, then $\mathcal{F}$ is uniquely determined
$$\mathcal{F} = \mathcal{T}^{\perp} := \{X \in \mathbb{C} \mid \hom(T, X) = 0 \text{ for all } T \in \mathcal{T}\}$$
- The same is not true for pretorsion classes. A class $\mathcal{T}$ can be the torsion part of infinitely many pretorsion theories.
- Pretorsion classes in $\mathbb{C}$ are precisely the monocoreflective subcategories of $\mathbb{C}$ (strongly covering subcategories where all strong covers are monomorphisms).

A nice setting: $\mathbb{C} = \text{mod}kQ$

Ingredients:

- Q is a quiver and k is an algebraically closed field;
- kQ is the path algebra;
- $\text{mod}kQ$ is the category of finitely generated (right) modules over kQ.

Fact:

Any finite dimensional associative k-algebra is Morita equivalent to the path algebra of some bound quiver.

Gabriel classification theorem:

The category $\text{mod}kQ$ has only finitely many isomorphism classes of indecomposable modules if and only if its underlying graph (when the directions of the arrows are ignored) is one of the ADE Dynkin diagrams: A_n, D_n, E_6, E_7, E_8.

A nice setting: $\mathbb{C} = \mathrm{mod}kQ$

$\mathrm{mod}kQ$ is the category of finitely generated (right) modules over a path algebra kQ.

Why is $\mathrm{mod}kQ$ nice?

- $\mathrm{mod}kQ$ is a Krull-Schmidt Noetherian abelian category.
- $\mathcal{T} \subseteq \mathbb{C}$ is a pretorsion class if and only if $\mathcal{T}$ is closed under quotients and finite direct-sums.
- All the important information can be encoded into its Aulander-Reiten quiver.
- Torsion and pretorsion classes are quite easy to detect.

Example

Example: $Q = \mathbb{A}_2 : 1 \to 2$

$$\begin{tikzcd} & {\begin{smallmatrix}1\\2\end{smallmatrix}} \arrow[dr, two heads] & \\ 2 \arrow[ur, tail] \arrow[rr, dash, dashed] & & 1 \end{tikzcd},$$

Let me try to draw a picture...

Classification of distributive lattices for finite representation type (here $Q = A_n, D_n, E_6, E_7, E_8$)

The poset of pretorsion classes is a complete lattice, with meet and join given, for every $\mathcal{T}_1$ and $\mathcal{T}_2$, by

$$\mathcal{T}_1 \wedge \mathcal{T}_2 = \mathcal{T}_1 \cap \mathcal{T}_2 \qquad \text{and} \qquad \mathcal{T}_1 \vee \mathcal{T}_2 = \langle \mathcal{T}_1 \cup \mathcal{T}_2 \rangle_t .$$

Result 1 [— , Fedele, Yıldırım]

The lattice of pretorsion classes is distributive if and only if $\mathrm{add}\{\mathcal{T}_1 \cup \mathcal{T}_2\} = \langle \mathcal{T}_1 \cup \mathcal{T}_2 \rangle_t$ for every pair of pretorsion classes $\mathcal{T}_1$ and $\mathcal{T}_2$ in $\mathrm{mod}\, kQ$.

Result 2 [— , Fedele, Yıldırım]

The lattice of pretorsion classes is distributive if and only if Q does not contain subquivers of the form

$$\circ \longrightarrow \circ \longleftarrow \circ$$

$$\begin{tikzcd}
\circ & \\
\circ & \circ \arrow[l] \arrow[ul] \arrow[dl] \\
\circ &
\end{tikzcd}$$

$$\begin{tikzcd}
\circ & & \\
& \circ \arrow[ul] \arrow[dl] & \circ \arrow[l] \\
\circ & &
\end{tikzcd}$$

Classification of distributive lattices for finite representation type (here $Q = A_n, D_n, E_6, E_7, E_8$)

Result 3 [— , Fedele, Yıldırım]

There is a bijection between the isomorphism classes of indecomposable modules and the join-irreducible elements of the lattice of pretorsion classes, given by $M \mapsto \langle M \rangle_t$. Moreover, the join-irreducible elements are torsion classes.

Result 4 [— , Fedele, Yıldırım]

If the lattice of pretorsion classes is distributive, than it is the distributive completion of the lattice of torsion classes.

Thank you