Killing Forms on Finite Groups

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Introduction

Let $\mathfrak{g}$ be a Lie algebra with multiplication/bracket $[x, y]$.
Then for all $x \in \mathfrak{g}$, there exists an adjoint endomorphism

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The *Killing form* on $\mathfrak{g}$ is the bilinear form:

$$K(x, y) = \mathrm{Tr}(\mathrm{ad}(x) \circ \mathrm{ad}(y)) \qquad \forall\, x, y \in \mathfrak{g}.$$

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Cartan's Criterion

A finite-dimensional Lie algebra over a field of characteristic 0 is semisimple i.f.f. its Killing form is non-degenerate.

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  which is degenerate. $M_2(\mathbb{C})$ is semisimple as an algebra, but by Cartan's Criterion, it is not semisimple as a Lie algebra.
- For G a group and k a field, $k[G]$ is a Lie algebra with $[g, h] = gh - hg$.

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For G a finite group, $\mathcal{C} \subset G$, the vector space $\mathbb{C}[\mathcal{C}]$ becomes a Lie algebra with

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The *Killing form on groups* is defined by

$$\begin{aligned} K_{\mathcal{C}}(a, b) &= \mathrm{Tr}_{\mathbb{C}[\mathcal{C}]}(ab \, . \, b^{-1}a^{-1}) \\ &= \chi(ab) \\ &= |\{\text{fixed points under conjugation with } ab \text{ in } \mathcal{C}\}| \\ &= |C_G(ab) \cap \mathcal{C}|, \end{aligned}$$

where $a, b \in \mathcal{C}$ and χ is the character of

$$\rho : G \longrightarrow GL(\mathbb{C}[\mathcal{C}]) : g \mapsto (\rho_g : x \mapsto gxg^{-1}).$$

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- G a finite group, $\mathcal{C} = Z(G)$. For all $a, b \in Z(G)$:

$$K_{Z(G)}(a, b) = |C_G(ab) \cap Z(G)| = |Z(G)|.$$

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- $D_8 = \langle a, b \mid a^4 = b^2 = abab = e \rangle$, $\mathcal{C} = \langle a \rangle \cong \mathbb{Z}_4$. For $i, j = 0, \ldots, 3$:

$$K_{\mathcal{C}}(a^i, a^j) = |C_{D_8}(a^{i+j}) \cap \mathcal{C}| = |\mathcal{C}| = 4.$$

  $K_{\mathcal{C}}$ is a 4×4-matrix with all entries $= 4$.
  $\implies K_{\mathcal{C}}$ is degenerate.

Types of Non-degeneracy

G is …	if $K_{\mathcal{C}}$ is non-degenerate for …
non-degenerate	for $\mathcal{C} = G \setminus \{e\}$
class non-degenerate	for every $\mathcal{C} \subset G \setminus \{e\}$ a real, generating conjugacy class
strongly non-degenerate	for every $\mathcal{C} \subset G \setminus \{e\}$ a real, generating, G-stable set
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non-degeneracy class non-degeneracy

General Results

Proposition

Let G be a finite group and $\mathcal{C} \subset G \setminus \{e\}$ a G-stable set. If there exists $c \neq e \in Z(G)$ such that $\mathcal{C} \cap \mathcal{C}c \neq \emptyset$, then $K_{\mathcal{C}}$ is degenerate.

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Corollaries:

- If $|Z(G) \cap \mathcal{C}| > 1$, then $K_{\mathcal{C}}$ is degenerate.
- If $|G| > 2$ and $|Z(G)| > 1$, then G is degenerate.

Example: Dihedral Groups

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$$K_{\mathcal{C}} = \begin{pmatrix} (n-1)\mathbb{1}_{n-1,n-1} + n\bar{I}_{n-1} & \mathbb{1}_{n-1,n} \\ \mathbb{1}_{n,n-1} & (n-1)\mathbb{1}_{n,n} + nI_n \end{pmatrix},$$

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$$K_{\mathcal{C}}^{-1} = \frac{1}{mn}\begin{pmatrix} -(n^2-n-1)\mathbb{1}_{n-1,n-1} + m\bar{I}_{n-1} & -\mathbb{1}_{n-1,n} \\ -\mathbb{1}_{n,n-1} & -(n-1)^2\mathbb{1}_{n,n} + mI_n \end{pmatrix},$$

with $m = 1 - n^2 + n^3$.

$\implies D_{2n}$ is non-degenerate for n odd.

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Roth property $\Longrightarrow$ G non-degenerate

But G non-degenerate $\not\Longrightarrow$ Roth property!
Counterexample: $\mathbb{Z}_2$, $(((\mathbb{Z}_5 \times \mathbb{Z}_5) \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2$.

Example: S_3

S_3 has 3 irreducible characters: $\chi_{\text{triv}}, \chi_{\text{sign}}, \chi_{\text{stnd}}$.
For χ the character of the conjugation representation:

$$\chi((12)) = 2,\ \chi((123)) = 3,\ \chi(\text{id}) = 6.$$
$$\langle \chi, \chi_{\text{triv}} \rangle = 3,$$
$$\langle \chi, \chi_{\text{sign}} \rangle = 1,$$
$$\langle \chi, \chi_{\text{stnd}} \rangle = 1.$$

$\implies$ S_3 has the Roth property

$\implies$ S_3 is non-degenerate.

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Computationally verified for finite simple groups up to order 75.000:

- Most have the Roth property $\rightsquigarrow$ are non-degenerate.
  Exceptions: $PSU(3,3)$ is degenerate, $PSU(3,4)$ is out of reach.

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Computationally verified for finite simple groups up to order 75.000:

- Most have the Roth property $\rightsquigarrow$ are non-degenerate.
  Exceptions: $PSU(3,3)$ is degenerate, $PSU(3,4)$ is out of reach.
- For $\mathcal{C}$ a conjugacy class, $K_{\mathcal{C}}$ in most cases is non-degenerate.
  Exceptions appear for some non-real conjugacy classes.

Conjecture *(Lopez Peña, Majid, Rietsch. 2017)*

Let G be a simple group. If $\mathcal{C}$ is a non-trivial, real conjugacy class in G, $K_{\mathcal{C}}$ is non-degenerate.

Irreducibility

For $\mathcal{C}$ a G-stable subset, can we choose an ordering of $\mathcal{C}$ such that $K_{\mathcal{C}}$ is a diagonal block matrix?

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$\Gamma_{\mathcal{C}}$ graph with $V = \mathcal{C}$ and $E = \{(a, b) \in \mathcal{C}^2 \mid K_{\mathcal{C}}(a, b) \neq 0\}$.
$K_{\mathcal{C}}$ is *irreducible* if $\Gamma_{\mathcal{C}}$ is connected.

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Proposition

For $\mathcal{C}$ a conjugacy class, $K_{\mathcal{C}}$ has a positive, maximal eigenvalue given by

$$\lambda_{\max} = \text{ sum of any column or row of } K_{\mathcal{C}}.$$

The number of connected components of Γ_C is the dimension of the eigenspace of $\lambda_{\max}$.

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A *strongly embedded subgroup* is a subgroup H such that $|H|$ is even and for every $g \notin H$, $|H \cap gHg^{-1}|$ is odd.

For simple groups, these have been classified:

$$PSL(2, 2^n), PSU(3, 2^n), Suz(2^{2n-1}) \text{ where } n \geq 2.$$

Example: Unipotent Elements in $PSL(2, q)$

$q = p^a$, p a prime.

$$G := PSL(2, q) = \{2 \times 2\text{-matrices over } \mathbb{F}_q \text{ with } \det = 1\}/\{I_2, -I_2\}.$$

$$|G| = \frac{q(q^2 - 1)}{d}, \qquad d = \begin{cases} 1 & \text{if } q \text{ is even,} \\ 2 & \text{if } q \text{ is odd.} \end{cases}$$

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- $x \in G$ is *unipotent* if $|x| = p$.
- If $S \in \mathrm{Syl}_p(G)$, then $|S| = q$ and S is elementary abelian.
- If $S' \in \mathrm{Syl}_p(G) \setminus \{S\}$, then $S \cap S' = \{e\}$.
- $|\mathrm{Syl}_p(G)| = q + 1$.
- If $x \in G$ is unipotent and contained in Sylow p-subgroup S, then $C_G(x) = S$.

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$$K_{\mathcal{C}} = \begin{pmatrix} A & 0 & \dots & 0 \\ 0 & A & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A \end{pmatrix},$$

with $0 =$ zero matrix and A the $(q-1) \times (q-1)$-matrix

$$A = \begin{pmatrix} q^2 - 1 & q-1 & \dots & q-1 \\ q-1 & q^2-1 & \dots & q-1 \\ \vdots & \vdots & \ddots & \vdots \\ q-1 & q-1 & \dots & q^2-1 \end{pmatrix}.$$

$\rightsquigarrow$ $K_{\mathcal{C}}$ is non-degenerate and reducible.

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Proposition

For $y_1 \in \mathcal{C}$, and $S \in \mathrm{Syl}_p(G)$ such that $y_1 \in S$, $S' \in \mathrm{Syl}_p(G) \setminus \{S\}$, there exists a unique $y_2 \in S' \cap \mathcal{C}$ such that $y_1 y_2$ is unipotent.

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Lemma

If xy is unipotent, then so is yx.

Example: Unipotent Elements in $PSL(2, q)$, q odd

$K_{\mathcal{C}}$ is a block matrix $(A_{i,j})$ with $1 \leq i, j \leq q+1$ and each $A_{i,j}$ of size $\frac{q-1}{2} \times \frac{q-1}{2}$.

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For blocks on the diagonal,

$$A_{i,i} = \begin{pmatrix} \frac{q-1}{2} & \cdots & \frac{q-1}{2} & \frac{q^2-1}{2} \\ \frac{q-1}{2} & \cdots & \frac{q^2-1}{2} & \frac{q-1}{2} \\ \vdots & \iddots & \vdots & \vdots \\ \frac{q^2-1}{2} & \cdots & \frac{q-1}{2} & \frac{q-1}{2} \end{pmatrix}.$$

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Open Problem

Can we prove that $K_{\mathcal{C}}$ is invertible?