Local smallness and global largeness: a quantitative approach

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VUB Algebra Seminar

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Local finiteness conditions

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- Finite-dimensional algebras, e.g. $F[x]/\langle p(x) \rangle$, $M_n(F), \dots$
- K/F an algebraic field extension
- Finitary matrices: $\begin{pmatrix} * & \cdots & 0 & \\ \vdots & \ddots & & \\ 0 & & 0 & \cdots \\ \vdots & & & \ddots \end{pmatrix}$

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All of these examples are **locally finite**.

Local finiteness: Burnside and Kurosh Problems

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Wikipedia
https://en.wikipedia.org › wiki › William_Burnside_(c...

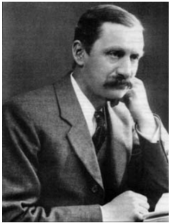
William Burnside (character)

William Burnside, PhD, also known as the Captain America of the 1950s, Commie Smasher or Bad Cap, : 50–51, 226–227 is a fictional character appearing in ...

Fandom
https://marvel.fandom.com › wiki › William_Burnside...

William Burnside (Earth-616) | Marvel Database - Fandom

Powers. Super-Soldier Serum: **Burnside** is a superb athlete, his physique having been



Burnside (1902):

Is every finitely generated periodic group finite?

Kurosh (1941):

Is every finitely generated algebraic algebra finite-dimensional?

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Affirmative for many important classes

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Example: strictly upper triangular matrices

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"Kurosh + Frobenius = Burnside"

 +  = 

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- Serre's conjecture: Arithmetic lattices in $SL_2(\mathbb{C})$ do not have the congruence subgroup property (True: Lubotzky)

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Golod-Shafarevich algebras and groups, Burnside groups are *huge*. Are there *small* counterexamples?

Growth of groups and algebras

$G = \langle S \rangle$ finitely generated group. The growth of G:

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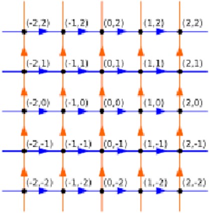
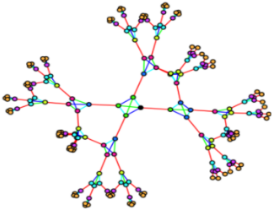
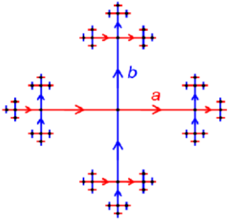


Polynomial growth



Intermediate growth



Exponential growth

Growth of groups and algebras

A - algebra generated by $x_1, \ldots, x_d$. The growth of A:

$$\gamma_A(n) = \dim_F \operatorname{Span}\{\text{Monomials in } x_1, \ldots, x_d \text{ of length} \leqslant n\}$$

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For group algebras $F[G]$, the growth coincides with the growth of G

Growth of algebras in real life

Geometric representation theory: Gel'fand-Kirillov conjecture
$U(\mathfrak{g}) \sim_{\text{bir}} \mathcal{A}_n(\mathbb{C}(x_1, \ldots, x_s))$ "parameter space" for representations of algebraic groups

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Symbolic dynamics: Infinite word, e.g.

$$0110100110010110\cdots$$

generates a symbolic dynamical system ('subshifts'). Complexity:

$$c_w(n) = |\{\text{Length-}n \text{ subwords of } w\}|$$

Complexity of words $\longleftrightarrow$ Growth of monomial algebras & convolution algebras

Fundamental Problem (e.g. Cassaigne, Ferenczi '90s)

Which functions occur as complexity functions of infinite words?

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An analogous problem: 'What are the growth functions of algebras?' (Bell-Zelmanov, '21).

All algebras are 'PBW deformations' of monomial algebras. Use them to 'pad' algebraic algebras and construct counterexamples to the Kurosh Problem of arbitrary growth rates.

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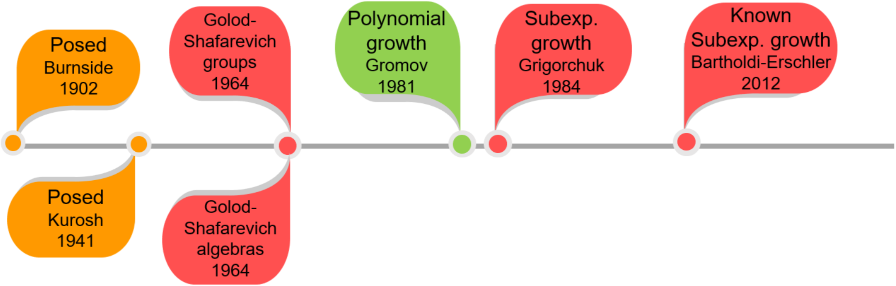

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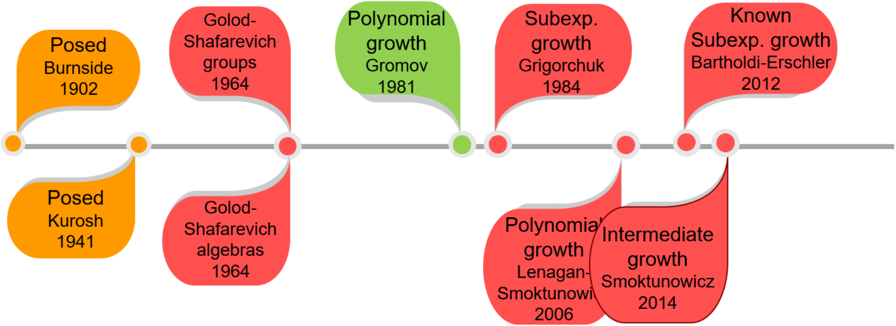

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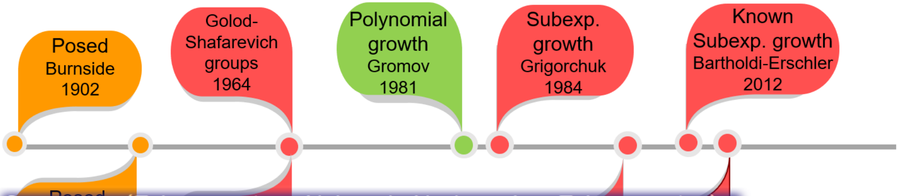

Conjecture (Zelmanov, '17; Alahmadi-Alsulami-Jain-Zelmanov, '17)

$$\left\{ \begin{array}{c} \textit{Growth functions*} \\ \textit{of algebras} \end{array} \right\} = \left\{ \begin{array}{c} \textit{Growth functions} \\ \textit{of nil algebras} \end{array} \right\}$$

**Except for algebras of linear growth*

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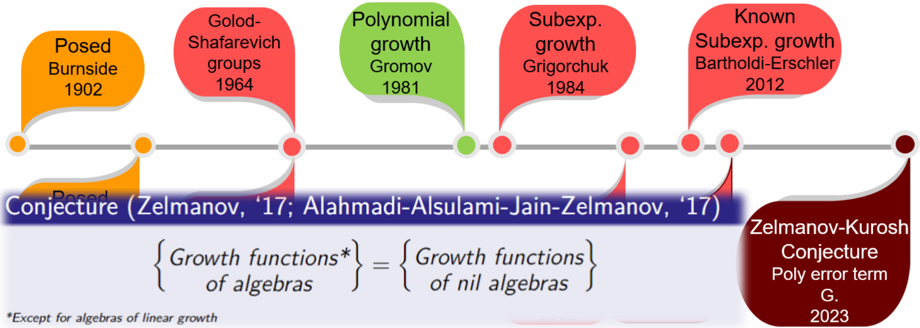

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Theorem (G., '23)

For any growth function f there exists a nil, graded algebra A such that:

$$f(n) \preceq \gamma_A(n) \preceq n^{4+\varepsilon} f(n) \qquad \text{countable fields}$$

$$f(n) \preceq \gamma_A(n) \preceq n^{\omega(n)} f(n) \quad \text{for any } \omega(n) \to \infty, \text{ any field}$$

own
p. growth
di-Erschler
2012

Corollary (G., '23)

First algebraic algebras and nil Lie algebras of known subexponential growth over arbitrary fields (e.g. $\exp(n^\alpha)$ for any $\alpha \in (0,1)$).

Zelmanov-Kurosh
Conjecture
Poly error term
G.
2023

Proposition (G., '23)

Nil, graded algebras (and nil Lie algebras) of any GKdim $\in [6,\infty)$.

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Local finiteness vs. Free substructures

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Question

Can a periodic group contain a free subgroup?
Can an algebraic algebra contain a free subalgebra?

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How close to periodic can a group with a free subgroup be?
How close to algebraic can an algebra with a free subalgebra be?

Nil algebras can contain free subalgebras after field extensions!
(Smoktunowicz, '09)

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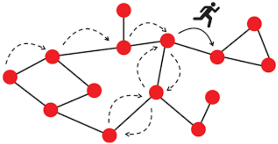
Finitely generated groups are geometric objects via Cayley graphs $\rightsquigarrow$ *measure* torsion by a sequence of probability measures $\{\mu_n\}_{n=1}^{\infty}$

- Uniform: $\mu_n = \mathcal{U}(B_{G,S}(n))$ uniform on the n-ball of the Cayley graph
- Random walks: $\mu_n = \nu^{\star n}$ for a non-degenerate distribution ν



Probabilistic identities in groups

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If a group law holds with positive probability, then the group virtually satisfies it.

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Another exercise for your students: if $\forall x \in G, x^2 = 1$ then G is abelian.

Theorem (Amir-Blachar-Gerasimova-Kozma, '23)

If $\lim_{n\to\infty} \Pr_{\mu_n}(x^2=1) > 0$ *then G is virtually abelian.*

Theorem (Amir–Blachar–Gerasimova-Kozma, '23)

There exists a group which is metabelian $[[w, x], [y, z]] = 1$ *with probability* $1 - \varepsilon$*, yet contains a free subgroup.*

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Question (Amir–Blachar–Gerasimova-Kozma, '23)

1. *If a group satisfies a law with probability* 1*, does it satisfy a law?*
2. *Does the limit probability always exist?*
3. *Is the limit probability sensitive to the generating set / random walk?*

Probabilistically Burnside groups

Theorem (Goffer–G., '23; Free subgroups)

There exists a finitely generated group G such that $\Pr_{\mu_n}(x^N = 1) \xrightarrow{n\to\infty} 1$ for every non-degenerate random walk and uniform measures; but $F_2 \hookrightarrow G$.

Theorem (Goffer–G.–Olshanskii, '24; Oscillating torsion probabilities)

There exists a group $G = \langle S \rangle$ such that every real number in $[0,1]$ is a partial limit of $\{\Pr_{\mu_n}(x^N = 1)\}_{n=1}^{\infty}$.

Probabilistically Burnside groups

Theorem (Goffer–G., '23; Free subgroups)

There exists a finitely generated group G such that $\Pr_{\mu_n}(x^N = 1) \xrightarrow{n\to\infty} 1$ for every non-degenerate random walk and uniform measures; but $F_2 \hookrightarrow G$.

Theorem (Goffer–G.–Olshanskii, '24; Oscillating torsion probabilities)

There exists a group $G = \langle S \rangle$ such that every real number in $[0,1]$ is a partial limit of $\{\Pr_{\mu_n}(x^N = 1)\}_{n=1}^{\infty}$.

Theorem (Goffer–G.–Olshanskii, '24; Sensitivity to generating sets)

There exists a finitely generated group G such that:

$$\lim_{n\to\infty} \Pr_{x\sim\mu_n}(x^N = 1) = 1, \quad \lim_{n\to\infty} \Pr_{x\sim\mu'_n}(x^N = 1) = 0$$

for different generating sets.

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Theorem (Goffer–G.–Olshanskii, '24; Oscillating torsion probabilities)

There exists a group $G = \langle S \rangle$ *such that every real number in* $[0,1]$ *is a partial limit of* $\{\Pr_{\mu_n}(x^N = 1)\}_{n=1}^{\infty}$.

Theorem (Goffer–G.–Olshanskii, '24; Sensitivity to generating sets)

There exists a finitely generated group G such that:

$$\lim_{n\to\infty} \Pr_{x\sim\mu_n}(x^{N_1} = 1) = 1, \quad \lim_{n\to\infty} \Pr_{x\sim\mu'_n}(x^{N_2} = 1) = 1$$

for co-prime numbers N_1, N_2 and for different generating sets.

Thank you!