The *Other* Yang–Baxter Equation

FROM SETS TO QUIVERS AND BACK AGAIN

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A quiver over Λ is the datum of a set Q (*arrows*), a set Λ (*vertices*), and two maps $\mathfrak{s}, \mathfrak{t} \colon Q \to \Lambda$ (*source* and *target*).

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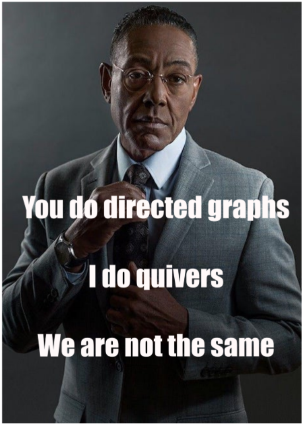

Let Q, R be quivers, over Λ and M respectively. A *weak morphism* of quivers $f\colon Q \to R$ is a pair $f = (f^1, f^0)$, where $f^1\colon Q \to R$ and $f^0\colon \Lambda \to \mathrm{M}$ are maps, such that the following diagram commutes:

$$\begin{array}{ccc} Q & \xrightarrow{f^1} & R \\ \mathfrak{s}_Q \Big\downarrow\Big\downarrow \mathfrak{t}_Q & & \mathfrak{s}_R \Big\downarrow\Big\downarrow \mathfrak{t}_R \\ \Lambda & \xrightarrow{f^0} & \mathrm{M} \end{array}$$

When Q and R have same set of vertices Λ, we say that a *morphism over* Λ is a weak morphism $f = (f^1, f^0)$ with $f^0 = \mathrm{id}_\Lambda$.

The category Quiv_Λ of quivers over Λ, with morphisms over Λ, is monoidal: the monoidal product $Q \otimes R$ is given by

$$\{(x, y) \in Q \times R \mid \mathfrak{t}_Q(x) = \mathfrak{s}_R(y)\},$$

$$\mathfrak{s}_{Q \otimes R}(x, y) := \mathfrak{s}_Q(x), \quad \mathfrak{t}_{Q \otimes R}(x, y) := \mathfrak{t}_R(y).$$

The monoidal unit $\mathbb{1}_\Lambda$ is the loop bundle with one loop on each object.

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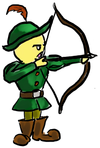

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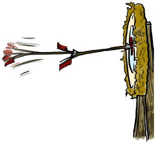


The Yang–Baxter Equation

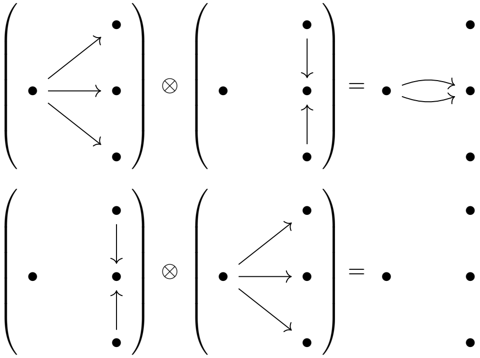
Let $(\mathscr{C}, \otimes, \mathbb{1})$ be a monoidal category. A solution to the Yang–Baxter equation in $\mathscr{C}$ is an object X with a morphism $\sigma\colon X^{\otimes 2} \to X^{\otimes 2}$ satisfying

$$(\sigma \otimes \mathrm{id}_X)(\mathrm{id}_X \otimes \sigma)(\sigma \otimes \mathrm{id}_X) = (\mathrm{id}_X \otimes \sigma)(\sigma \otimes \mathrm{id}_X)(\mathrm{id}_X \otimes \sigma).$$

The YBE is famous in the categories of vector spaces ($\mathsf{Vec}_\Bbbk$) and of sets (Set). These are also **symmetric** categories: the **flip** morphism $\tau_{X,Y}$ is a natural isomorphism between $X \otimes Y$ and $Y \otimes X$.

This means that **the YBE has always at least one solution** on each object in Set or $\mathsf{Vec}_\Bbbk$: namely, the flip.

This is not true for quivers: $Q \otimes R \not\cong R \otimes Q$ in general.



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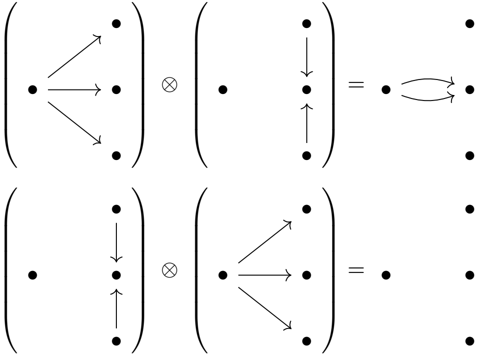

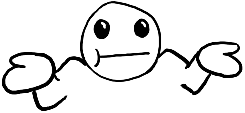

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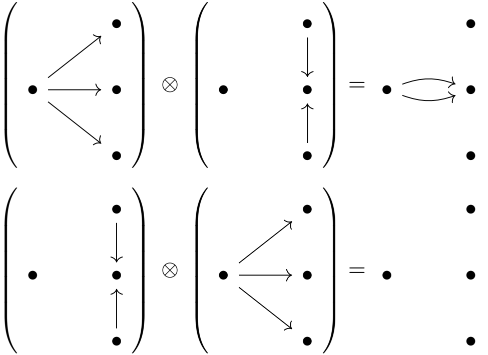

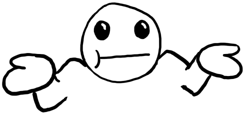

(also, the flip map just kinda
doesn't make any sense)

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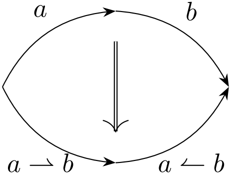
$\sigma\colon Q\otimes Q\to Q\otimes Q$



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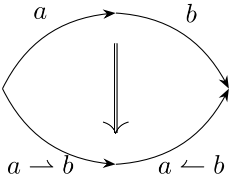

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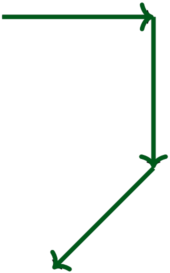

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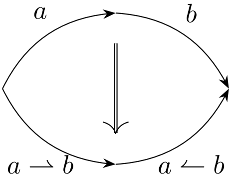

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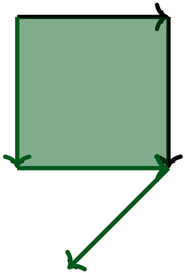

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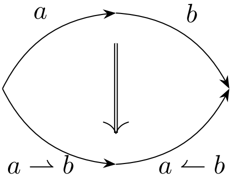

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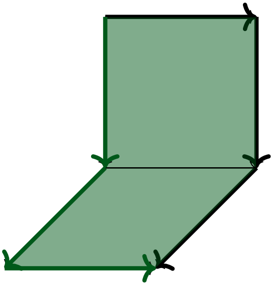

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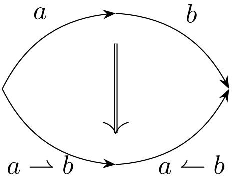

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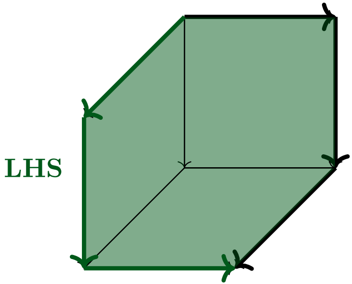

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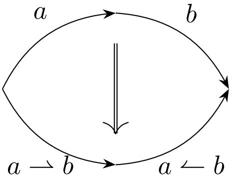

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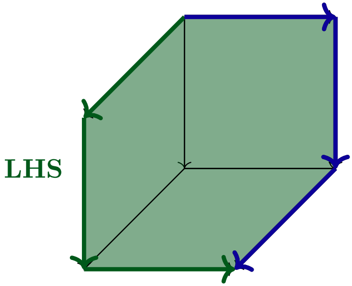

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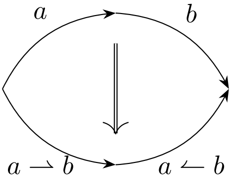

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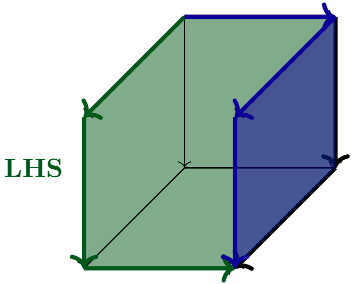

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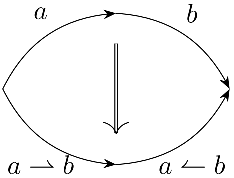

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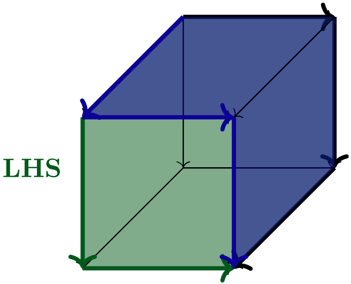

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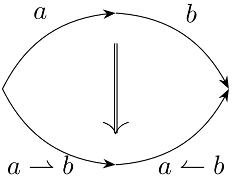

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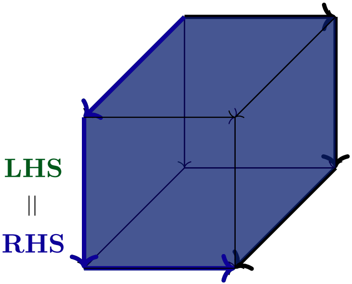
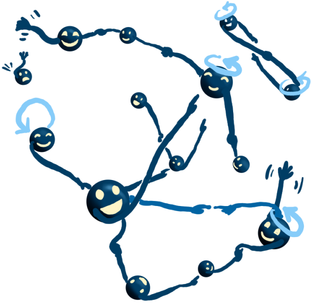


The quiver-theoretic YBE includes the set-theoretic YBE as a subcase.
Indeed, a set X can be seen as a quiver with a single object. Here, the **arrows** are in bijection with the elements of X, while the single object plays no role.



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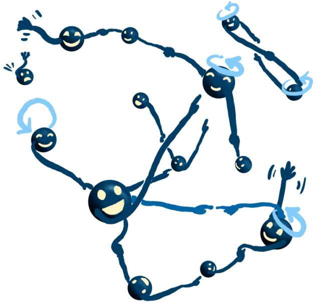


Yet *Another* Yang–Baxter Equation!. . .

A **dynamical set over** Λ (Shibukawa, 2005) is a set X with a function $\phi\colon \Lambda \times X \to \Lambda$, called the *transition map*.

The category of dynamical sets over Λ is monoidal: thus, a YBE on it makes sense. This is called the **set-theoretic Dynamical Yang–Baxter Equation** (DYBE).

...But not really.

Every dynamical set can be seen as a quiver, and this operation turns the DYBE into a quiver-theoretic YBE:

$$X, \ \ \phi\colon \Lambda \times X \to \Lambda \qquad \text{becomes} \qquad \Lambda \times X \underset{\phi}{\overset{\pi_1}{\rightrightarrows}} \Lambda$$

$$\lambda \xrightarrow{\;x\;} \phi(\lambda, x) \qquad\qquad =: [\lambda \| x]$$

Of course, the **essential image** of this functor is not the entire category of quivers. It is the subcategory of quivers Q such that all **sets** $Q(\lambda, \Lambda)$ **of outgoing arrows at** λ are in bijection with each other, for all vertices λ.

Overview:

Braided groupoids

$\Updownarrow$

Quiver-theoretic version of skew braces $\supseteq$ *Dynamical* version of skew braces

(turns out $\subseteq$, up to iso)

$\uparrow$
This one works nicely, has good-looking *ideals* and *quotients*, has geometric interpretations, and is very general: it comprises every braided groupoid.

$\uparrow$
This one, however, has better combinatorics, and makes you construct explicit stuff by hand.

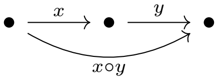
Groupoids

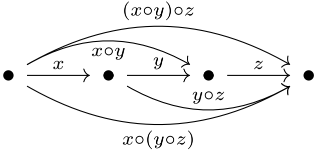
A groupoid is a quiver $\mathscr{G}$ over Λ, with a binary operation $\circ\colon \mathscr{G} \otimes \mathscr{G} \to \mathscr{G}$, and a family $\{1_\lambda\}_{\lambda\in\Lambda}$ (one loop on each vertex), such that

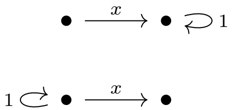
(1) $\circ$ is **associative**;

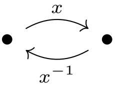
(2) the 1_λ's are **units**, i.e. $x \circ 1_{\mathfrak{t}(x)} = x$ and $1_{\mathfrak{s}(x)} \circ x = x$ for all $x \in \mathscr{G}$; and

(3) every arrow x from λ to μ has an **inverse** $x^{-1}\colon \mu \to \lambda$, such that $x \circ x^{-1} = 1_{\mathfrak{s}(x)}$ and $x^{-1} \circ x = 1_{\mathfrak{t}(x)}$.



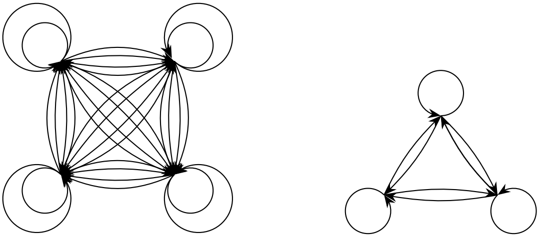






If you think about it for some minutes, you probably notice the following fact:

The underlying quiver of a **groupoid** is a union of connected components, where each connected component $\mathscr{K}$ is a **"complete quiver of degree $d_{\mathscr{K}}$"** for some number $d_{\mathscr{K}}$.



Braided groupoids

Let $\mathscr{G}$ be a groupoid over Λ. A **braiding** on $\mathscr{G}$ is a bijective morphism $\sigma\colon \mathscr{G}^{\otimes 2} \to \mathscr{G}^{\otimes 2}$, with $\sigma(x \otimes y) =: (x \rightharpoonup y) \otimes (x \leftharpoonup y)$, such that

(1) $\sigma(x \otimes 1) = 1 \otimes x$ and $\sigma(1 \otimes x) = x \otimes 1$;

(2) $(x \circ y) \rightharpoonup z = x \rightharpoonup (t \rightharpoonup z)$,

$x \leftharpoonup (y \circ z) = (x \leftharpoonup y) \leftharpoonup z$,

i.e., $\rightharpoonup$ is a left action and $\leftharpoonup$ is a right action; and

(3) $x \rightharpoonup (y \circ z) = (x \rightharpoonup y) \circ ((x \leftharpoonup y) \rightharpoonup z)$,

$(x \circ y) \leftharpoonup z = (x \leftharpoonup (y \rightharpoonup z)) \circ (y \leftharpoonup z)$.

A braiding is, in particular, a solution to the Yang–Baxter equation.
When $\Lambda = \{\bullet\}$, this is called a **braided group**, and is equivalent to a **skew brace**.

A **skew brace** is the datum of a set G with two group structures, $(G, +, 0)$ and $(G, \circ, 0)$, satisfying the compatibility

$$a \circ (b + c) = a \circ b - a + a \circ c.$$

Dynamical skew braces

Definition (Matsumoto)

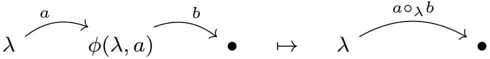
A **dynamical skew brace** is the datum of a dynamical set (A, ϕ) over Λ, with a group structure $(A, +, 0)$ and a family of left quasigroup structures $\{A, \circ_\lambda\}_{\lambda \in \Lambda}$ satisfying

$$a \circ_\lambda (b \circ_{\phi(\lambda,a)} c) = (a \circ_\lambda b) \circ_\lambda c \qquad \text{(dynamical associativity)}$$

$$a \circ_\lambda (b + c) = a \circ_\lambda b - a + a \circ_\lambda c \qquad \text{(brace compatibility)}$$

It is always true that $a \circ_\lambda 0 = a$.
A dynamical skew brace is called ZERO-SYMMETRIC if $0 \circ_\lambda a = a$ for all a, λ.

Let $Q := \mathcal{Q}(A, \phi)$ be the quiver over Λ associated with the dynamical set (A, ϕ). Then, we can describe a binary operation $\circ \colon Q \otimes Q \to Q$ by $[\lambda \| a] \circ [\phi(\lambda, a) \| b] := [\lambda \| a \circ_\lambda b]$.



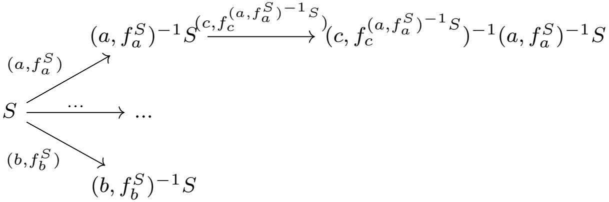
This is a left semiloopoid operation (forget about it), but **it is a GROUPOID operation, if and only if A is ZERO-SYMMETRIC**.

Dynamical skew braces admit the following, more handy description:

Let $(A, +)$ be a group. The *holomorph* is $\mathrm{Hol}(A) := A \rtimes \mathrm{Aut}(A)$.

A subset S of the holomorph is *regular* if $S = \{(a, f_a^S) \mid a \in A\}$ for suitable maps $f_a^S \in \mathrm{Aut}(A)$. (Basically, the projection on the first factor $S \to A$ is a bijection.)

A *dynamical subgroup* of $\mathrm{Hol}(A)$ is a family $\mathscr{S}$ of regular subsets $S \subseteq \mathrm{Hol}(A)$, such that for all $S \in \mathscr{S}$ and $(a, f_a^S) \in S$, the set $(a, f_a^S)^{-1}S$ is also in $\mathscr{S}$.



Take $\mathscr{S}$ as the set of vertices, and define $a \circ_S b := a + f_a^S(b)$. This is a dynamical skew brace. Every dynamical skew brace arises in this way.

It is ZERO-SYMMETRIC if and only if $f_0^S = \mathrm{id}$ for all S; i.e., if and only if every S contains the pair $(0, \mathrm{id})$.

Skew bracoids*

*A different structure with the same name already exists
(This is gonna make Isabel and Paul a bit upset, I am afraid)

Definition (Sheng–Tang–Zhu)

A **skew bracoid** $(\mathscr{G}, \{+_\lambda\}_{\lambda\in\Lambda}, \circ, \{1_\lambda\}_{\lambda\in\Lambda})$ over Λ is the datum of

(1) a groupoid structure $(\mathscr{G}, \circ, \{1_\lambda\}_{\lambda\in\Lambda})$; and

(2) a group structure $+_\lambda$ on each set $\mathscr{G}(\lambda, \Lambda)$;

satisfying the compatibility

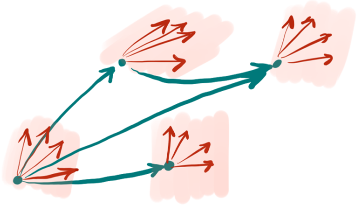
$$a \circ (b +_{\mathfrak{t}(a)} c) = a \circ b -_{\mathfrak{s}(a)} a +_{\mathfrak{s}(a)} a \circ c.$$

Theorem (Sheng–Tang–Zhu)

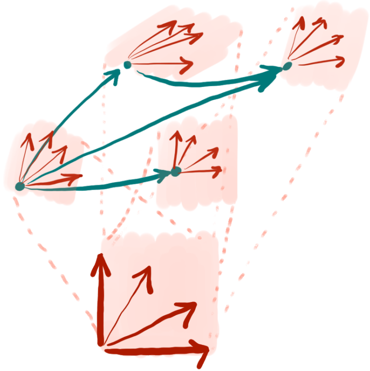
Skew bracoids are equivalent to braided groupoids.

Just not to lose the big picture:

Skew bracoid



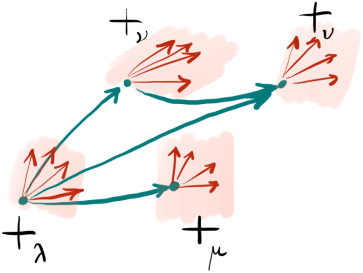
Dynamical skew brace



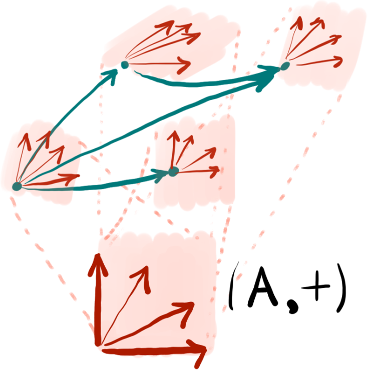
Turns out that **every dynamical skew brace yields a skew bracoid**. Conversely, every (connected) skew bracoid can be **“parallelised”**, making it **isomorphic to a dynamical skew brace**

Just not to lose the big picture:

Skew bracoid



Dynamical skew brace



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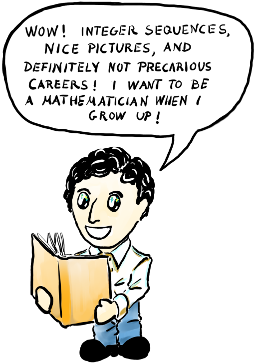
Enough with the boring stuff!

Now fun with Pictures!
and Weird Integers Sequences!



Me at age 14. (Yes, I was already wearing shirts.)

Recall that

a dynamical skew brace is the same thing as **a group** $(A, +)$, and a **dynamical subgroup** $\mathscr{S}$ of $\mathrm{Hol}(A)$.

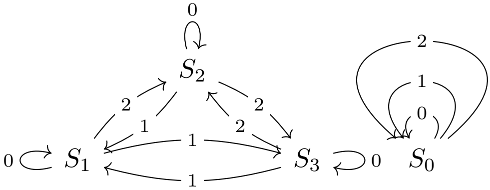
Of course, there is a **MAXIMAL dynamical skew brace** on A: choose $\mathscr{S}_A$ as the family of **all** the regular subsets of $\mathrm{Hol}(A)$. This is clearly a dynamical subgroup.
All dynamical skew braces on A are contained in $(A, +, \mathscr{S}_A)$.

The dynamical skew brace $(A, +, \mathscr{S})$ is **ZERO-SYMMETRIC** if and only if every element $S \in \mathscr{S}$ contains the pair $(0, \mathrm{id})$.

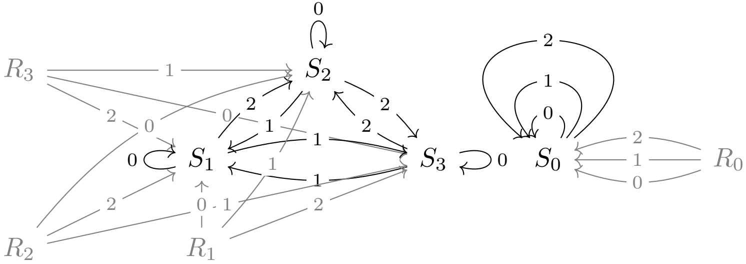
The family $\mathscr{S}_A^0$ of **all regular subsets containing** $(0, \mathrm{id})$ is also a dynamical subgroup (straightforward).
Thus, there is a **MAXIMAL ZERO-SYMMETRIC dynamical skew brace** on A, which is $(A, +, \mathscr{S}_A^0)$.

$A = \mathbb{Z}/3\mathbb{Z}$

$\mathscr{S}_A^0$

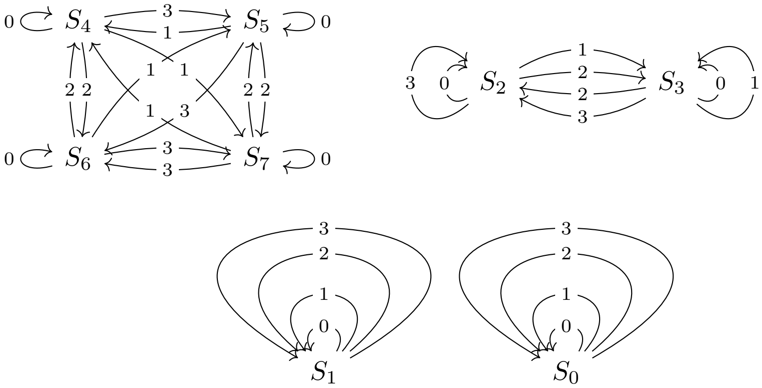


$\mathscr{S}_A$



$A = \mathbb{Z}/4\mathbb{Z}$

$\mathscr{S}_A^0$



For each connected component in $\mathscr{S}_A^0$,

$$(\#\text{vertices}) \cdot (\text{degree of the component}) = |A|.$$

Thus, the “shape” of the quiver $\mathscr{S}_A^0$ is determined by the string of numbers

$$N_s^A = \#\text{connected components with } s \text{ vertices},$$

where s divides $|A|$.

One clearly has

$$\sum_s N_s^A = |\operatorname{Aut}(A)|^{|A|-1}.$$

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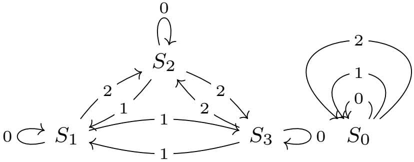
These numbers are an utter mystery.

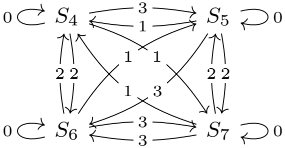
...and not an easy one: **observe that N_1^A is the number of skew braces on A.**

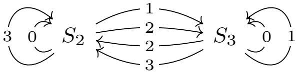
$A = \mathbb{Z}/3\mathbb{Z}$

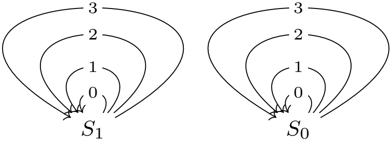


$$\implies N_1^A = 1,\ N_3^A = 1.$$

$A = \mathbb{Z}/4\mathbb{Z}$





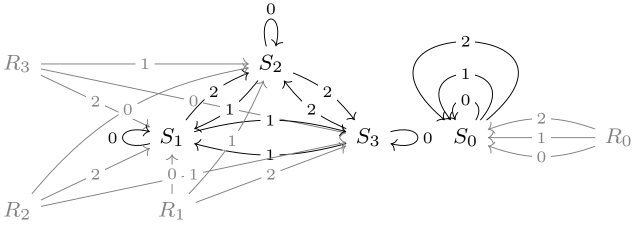


$$\implies N_4^A = 1,\ N_2^A = 1, N_1^A = 2.$$

…and for $A = (\mathbb{Z}/2\mathbb{Z})^2$ one can compute $N_4^A = 50$, $N_2^A = 6$, $N_1^A = 4$.

Recall that the quiver of $\mathscr{S}_A$ also has some **initial vertices**:

$(A = \mathbb{Z}/3\mathbb{Z})$



How many per connected component? Where and how many are the arrows? Fortunately, at least on this we can answer:

Let Q be the quiver of $\mathscr{S}_A$. Let $\mathscr{S}_{\text{in}}$ be the set of initial vertices, and $\mathscr{S}_{\text{un}}$ the set of vertices containing $(0, \text{id})$. Then:

(1) for all $S \in \mathscr{S}_{\text{in}}$, and for all $S', S'' \in \mathscr{S}_{\text{un}}$ in the same connected component, there is a bijection $Q(S, S') \cong Q(S', S'')$;

(2) for all $S \in \mathscr{S}_{\text{in}}$, and for all $S', S'' \in \mathscr{S}_{\text{un}}$ in the same connected component, there is a bijection $Q(S, S') \cong Q(S, S'')$;

(3) if $\mathscr{K}$ is a connected component in $\mathscr{S}_A^0$ with s vertices, then the number of initial vertices in $\mathscr{S}_A$ pointing at $\mathscr{K}$ is

$$\text{in}_{\mathscr{K}}^A = s(|\operatorname{Aut}(A)| - 1).$$

"While the other mathematicians climb the mountains, the combinatorist sticks around in the jungle, and looks for frogs."

—M. D'Adderio

We need to get done with the fun part, now, and climb some mountain.

Attempts at classifying Skew Bracoids

In the set-theoretic world, there are notions of **normal subgroup** and **ideal of a skew brace**.

Likewise, for quivers, there are notions of **normal sugbroupoid** and **ideal of a skew bracoid**.

Let $\mathscr{G}$ be a groupoid over Λ. A **subgroupoid** of $\mathscr{G}$ is a subquiver that is a groupoid with the restricted operation.
A subgroupoid $\mathscr{H}$ is **normal** if $g^{-1} \circ h \circ g \in \mathscr{H}$ for all $h \in \mathscr{H}$, $g \in \mathscr{G}$ (compatible).

The expression $g^{-1} \circ h \circ g$ is well-defined **only when h is a loop**, whence the following:

A subgroupoid $\mathscr{H} \subseteq \mathscr{G}$ is normal in $\mathscr{G}$ if and only if the **subgroupoid of loops** $\mathscr{H}^{\circlearrowright} \subseteq \mathscr{H}$ is normal in $\mathscr{G}$.

An **ideal** $\mathscr{H}$ of a skew bracoid $\mathscr{G}$ is a normal subgroupoid $\mathscr{H}$ of $\mathscr{G}$ such that every additive group $\mathscr{H}(\lambda, \Lambda)$ is normal in $\mathscr{G}(\lambda, \Lambda)$; and $\mathscr{H}$ is stable under the left action $\rightharpoonup$.

We can take **left and right quotients** of groupoids (resp. skew bracoids) by normal subgroupoids (resp. ideals), and obtain new groupoids (resp. skew bracoids):

Proposition

Let $\mathscr{G}$ be a connected groupoid, and hence complete of degree d, over a set of vertices Λ of cardinality n. Let $\mathscr{N}$ be a normal subgroupoid such that all the connected components of $\mathscr{N}$ have m vertices and are complete of degree k. Then, $\mathscr{N}\backslash\mathscr{G}$ is complete of degree d/k on a set of n/m vertices.

Examples: $\mathscr{G}^{\circlearrowleft}\backslash\mathscr{G}$ is a groupoid with same vertices as $\mathscr{G}$, but with degree 1.
If $\mathscr{H}$ is a **wide** normal subgroupoid of $\mathscr{G}$, then $\mathscr{H}\backslash\mathscr{G}$ is a **group**.

Groupoids of pairs

A **principal homogeneous groupoid of degree 1** is a very special object.

For every pair of vertices (a, b), there is **exactly one** arrow $a \to b$.
We denote it by $[a, b]$. Thus, such a groupoid is just the **groupoid of pairs** on the set of vertices Λ.
The composition $\circ$ is the only one possible: $[a, b] \circ [b, c] = [a, c]$.

We denote by $\hat{\Lambda}$ the groupoid of pairs on Λ.

Question

A braiding on $\hat{\Lambda}$ corresponds to what structure on Λ?

Answer

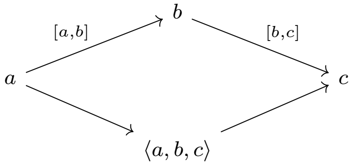
A group structure (kinda).
More precisely, a **heap** structure.

Heaps

Definition

A heap is a set Λ with a ternary operation $\langle \_\_, \_\_, \_\_ \rangle \colon \Lambda^3 \to \Lambda$, satisfying

$$\begin{aligned} \langle a, a, b\rangle &= b \\ \langle a, b, b\rangle &= a \\ \langle a, b, \langle b, c, d\rangle\rangle &= \langle a, c, d\rangle \\ \langle\langle a, b, c\rangle, c, d\rangle &= \langle a, b, d\rangle \end{aligned}$$



The map $[a, b] \otimes [b, c] \mapsto [a, \langle a, b, c\rangle] \otimes [\langle a, b, c\rangle, c]$ is a braiding on $\hat{\Lambda}$ if and only if $(\Lambda, \langle \_\_, \_\_, \_\_ \rangle)$ is a heap.

Heaps $\leftrightarrow$ groups

If $(G, \cdot)$ is a group, then it has a heap structure given by

$$\langle a, b, c \rangle := a - b + c.$$

If $(\Lambda, \langle \_, \_, \_ \rangle)$ is a heap, then every $b \in \Lambda$ provides a group structure with b as unit; given by

$$a \cdot_b c := \langle a, b, c \rangle.$$

Heaps are "affine" groups.
(Like affine spaces and vector spaces: you go from one to the other by fixing a neutral element.)

Dream...

For every skew bracoid $\mathscr{G}$, we would like to find a maximal **subgroupoid of pairs** $\mathscr{G}^P$ which is an **ideal**. The quotient $\mathscr{G}^P\backslash\mathscr{G}$ is thereby a **skew brace**, and we have squeezed $\mathscr{G}$ in between **(1) a braided groupoid of pairs (fundamentally a heap)**, and **(2) a skew brace**:

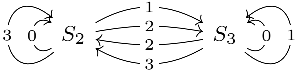
$$\mathbb{1} \to \mathscr{G}^P \to \mathscr{G} \to \mathscr{G}^P\backslash\mathscr{G} \to \mathbb{1}.$$

Thus every skew bracoid would be an extension, made using two objects that we already know.

. . . REALITY

Although it is always possible to find a maximal subgroupoid of pairs $\mathscr{G}^P$ (which is also normal), **it is not always possible to find it σ-invariant.**

E.g.



We try to do our best: find a bundle of normal subgroupoids which is **the widest** and **the "thinnest"** possible. But sometimes (as in the above example), even this "best we can do" is just taking the entire groupoid $\mathscr{G}$.

Thus, we have two "layers" that we understand: *(minimum degree)* heaps, and *(maximum degree)* skew braces; plus a number of **simple objects** in the middle, that can possibly be complicated

Possible classification programs

Idea 1:
Try to sandwich all skew bracoids between **(1)** a skew brace (below) and a simple object (above); or **(2)** a simple object (below) and a heap (above).

Try to classify the simple objects.

Assume that we have already "classified skew braces" (take them as a black box).

Idea 2:
Use the "elevated point" of skew bracoids, to attack the classification of skew braces from above.
In the "highest position" we have heaps, that are already fully understood: from there we can go down in a cascade, and try to classify (or at least count) skew braces.

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Bonkers

Bedankt voor het luisteren!
Hopelijk hebben jullie genoten :-)