Germs and Sylows for structure group of solutions to the Yang–Baxter equation

VUB Algebra Seminar

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Yang–Baxter Equation

Set-theoretical solution of the YBE *(Drinfeld '92)*

(X, r) where X is a set and $r : X \times X \to X \times X$ a bijection, such that

$$r_1 r_2 r_1 = r_2 r_1 r_2$$

where $r_i : X \times X \times X \to X \times X \times X$ acts on the coordinates i and $i+1$.

For any X, $r(x, y) = (y, x)$ defines a solution.

Definition *(Etingof-Schedler-Soloviev '99)*

Denote $r(x, y) = (\lambda(x, y), \rho(x, y))$. (X, r) is said to be:
- Involutive if $r^2 = \mathrm{id}_{X \times X}$
- Left non-degenerate (resp. right) if $\lambda(x, -)$ (resp. $\rho(-, y)$) is a bijection for any x (resp. y).

Cycle sets

Cycle set *(Rump '05)*

$(S, *)$ where S is a set and $*$ a binary operation such that for any s in S the map $\psi(s) : t \mapsto s * t$ is bijective, and for all s, t, u in S

$$(s * t) * (s * u) = (t * s) * (t * u).$$

Example: $S = \{s_1, \dots, s_n\}$ and $s * s_i = s_{\sigma(i)}$, with $\sigma = (12 \dots n)$ $(\psi(s_i) = \sigma)$.

Theorem *(Rump '05)*

There is a bijection correspondence

$$\text{involutive left non-degenerate solutions} \longleftrightarrow \text{Cycle sets}$$

Structure groups

Definition-Proposition *(Etingof-Schedler-Soloviev '99, Rump '05)*

Define the structure group G (resp. monoid M) by the presentation

$$\langle X \mid xy = x'y' \text{ if } r(x,y) = (x',y') \rangle \quad \longleftrightarrow \quad \langle S \mid s(s*t) = t(t*s) \rangle.$$

Example: $S = \{s_1, s_2\}$ with $\psi(s_i) = (12)$ yields $M = \langle s_1, s_2 \mid s_1^2 = s_2^2 \rangle^+$.

Suppose S finite and fix an enumeration $S = \{s_1, \ldots, s_n\}$.

Representation *(Dehornoy '15)*

We define the morphism $\Theta : M \to \mathrm{GL}_n(\mathbb{Q}[q, q^{-1}])$ induced by

$$s_i \mapsto D_i P_{s_i} = \mathrm{diag}(1, \ldots, q, \ldots, 1) \cdot P_{\psi(s_i)}.$$

Example: $\Theta(s_1) = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}$ et $\Theta(s_2) = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}$.

- A monomial matrix m decomposes uniquely as $m = D_m P_m$.

Dehornoy's calculus

Definition *(Dehornoy '15)*

Define inductively $\Omega_1(t) = t$ then

$$\Omega_k(s_{i_1}, \ldots, s_{i_k}) = \Omega_{k-1}(s_{i_1}, \ldots, s_{i_{k-1}}) * \Omega_{k-1}(s_{i_1}, \ldots, s_{i_{k-2}}, s_{i_k}) \in S$$

$$\Pi_k(s_{i_1}, \ldots, s_{i_k}) = \Omega_1(s_{i_1}) \ldots \Omega_k(s_{i_1}, \ldots, s_{i_k}) \in M.$$

$\Omega_3(s, t, u) = (s * t) * (s * u) = \Omega_3(t, s, u)$
$\Pi_2(s, t) = s(s * t) = \Pi_2(t, s)$

Lemma

Any word $s_{i_1} \ldots s_{i_k}$ in S^* can be written as a Π_k.

$$stu = s(s * t')u = s(s*t')((s*t')*u') = s(s*t')((s*t')*(s*u'')) = \Pi_3(s, t', u'').$$

Cayley graph

Fix s, t, u in S.

$$\begin{tikzcd}
 & \Pi_2(t,u) \arrow[rr, "\Omega_3(u,t,s)"] & & \Pi_3(s,t,u) \\
t \arrow[ur, "\Omega_2(t,u)"] \arrow[rr, "\Omega_2(t,s)"'] & & \Pi_2(s,t) \arrow[ur, "\Omega_3(s,t,u)"] & \\
 & u \arrow[uu, dashed, "\Omega_2(u,t)"] \arrow[rr, dashed, "\Omega_2(u,s)"'] & & \Pi_2(s,u) \arrow[uu, "\Omega_3(s,u,t)"'] \\
1 \arrow[uu, "\Omega_1(t)"] \arrow[ur, dashed, "\Omega_1(u)"'] \arrow[rr, "\Omega_1(s)"'] & & s \arrow[uu, "\Omega_2(s,t)"] \arrow[ur, "\Omega_2(s,u)"'] &
\end{tikzcd}$$

I-Structure

$\mathfrak{S}_n$ acts by conjugation on diagonal matrices: $P_\sigma D_s = D_{\sigma^{-1}(s)} P_\sigma$

Proposition

$$D_{\Theta(\Pi_k(s_{i_1},\ldots,s_{i_k}))} = D_{s_{i_1}} \ldots D_{s_{i_k}}$$

$$\Theta(\Pi_2(s,t)) = \Theta(s(s*t)) = D_s P_s D_{s*t} P_{s*t} = D_s D_t P_{\psi(s*t)\psi(s)} = \Theta(t(t*s)).$$

Theorem

G is the (left) fraction group of M.

Theorem (I-structure)

The only permutation matrix in $\Theta(G)$ is the identity.

- In other words, $\Theta(f)$ is uniquely determined by $D_{\Theta(f)}$ for f in G.
- $G < \mathbb{Z}^n \rtimes \mathfrak{S}_n$ such that projecting on the 1st coordinate is bijective.

Garside structure

Theorem

Θ is a faithful representation.

Theorem *(Chouraqui '10)*

M is a Garside monoid.

Example: $S = \{s_1, s_2, s_3\}$, $\psi(s_i) = (123)$, $f_1 = \begin{pmatrix} 0 & q^3 & 0 \\ 0 & 0 & 1 \\ q & 0 & 0 \end{pmatrix}$ and $f_2 = \begin{pmatrix} 0 & 0 & q^5 \\ q & 0 & 0 \\ 0 & q^2 & 0 \end{pmatrix}$.

- Length: sum of powers ($\lambda(f_1) = 3 + 0 + 1 = 4$, $\lambda(f_2) = 8$).
- Left divisibility: by the rows ($3 \leq 5, 0 \leq 1, 1 \leq 2$ thus $f_1 \preceq f_2$).
- Right divisibility: by the columns ($3 \not\leq 2$ thus $f_1 \not\preceq_r f_2$).
- GCD/LCM: min/max ($f_1 \wedge f_2 = \begin{pmatrix} 0 & q^3 & 0 \\ 0 & 0 & 1 \\ q & 0 & 0 \end{pmatrix}$, $f_1 \vee_r f_2 = \begin{pmatrix} q & 0 & 0 \\ 0 & q^3 & 0 \\ 0 & 0 & q^5 \end{pmatrix}$).
- $\Delta = \Pi_n(s_1, \ldots, s_n) = \begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q \end{pmatrix} \mathrm{P}$.

$\Longrightarrow$ Solution to the word problem, normal forms, torsion-free, etc.

A proof

- Dehornoy used a result of Rump to obtain the previous results.

Theorem *(Rump '05)*

(X, r) finite, Inv $+$ LND $\Rightarrow$ (X, r) RND $\longleftrightarrow T : \begin{array}{c} S \to S \\ s \mapsto s * s \end{array}$ is bijective.

$\mathfrak{S}_n$ acts by conjugation on diagonal matrices $\Longrightarrow P_s^{-1} D_t = D_{s*t} P_s^{-1}$

Proof.

Suppose $s * s = s' * s'$. Let $g = ss'^{-1}$.

$$g = D_s P_s P_{s'}^{-1} D_{s'}^{-1} = D_s P_s D_{s'*s'}^{-1} P_{s'}^{-1} = D_s P_s D_{s*s}^{-1} P_{s'}^{-1} = D_s D_s^{-1} P = P$$

By the I-structure, $P = 1$ thus $s = s'$. $\square$

Dehornoy's class and germ

$s_i^{[k]}$ the unique element of G with diagonal part D_i^k. $\quad s_2^{[3]} = \begin{pmatrix} 0 & 1 \\ q^3 & 0 \end{pmatrix}$

Proposition (Dehornoy's class)

There exists $d \in \mathbb{N}$ such that $s^{[d]}$ is diagonal for all $s \in S$.

Example: If $S = \{s_1, \dots, s_n\}$ et $\psi(s_i) = (12 \dots n)$, $d = n$.

- The quotient $\overline{G} = G/\langle s^{[d]} \rangle$ is finite.

Theorem (Germ)

(M, Δ^{d-1}) can be "recovered" from $\overline{G}$.

- Similar to the role of Coxeter groups for spherical type Artin groups.
- $G \twoheadrightarrow \overline{G}$ amounts to evaluating $q = \exp(\frac{2i\pi}{d})$.
- "Recovering" M amounts to forgetting $q^d = 1$ in the products in $\overline{G}$.

A conjecture on the class

Using Vendramin's enumeration :

n	$d_{\max}(n)$
3	3
4	4
5	6
6	8
7	12
8	15
9	24
10	30

Maximum of the classes of cycle sets with size n

Conjecture *(F.)*

$d_{\max}(n)$ is equal to "Maximum of products of distinct partitions of n".

- Example: $n = 9 = 2 + 3 + 4$, and $2 \cdot 3 \cdot 4 = 24$ is maximal.
- A034893 on the OEIS. And Došlić gave an explicit formula (with T_m).

More on Dehornoy's class

Denote $\mathcal{G} < \mathfrak{S}_n$ the group generated by the $\psi(s)$.

Proposition

If $T : s \mapsto s * s$, we have : $o(T) \mid d \mid \#\mathcal{G} \mid d^n$.
In particular, d and $\#\mathcal{G}$ have the same prime divisors.

Proposition *(F.)*

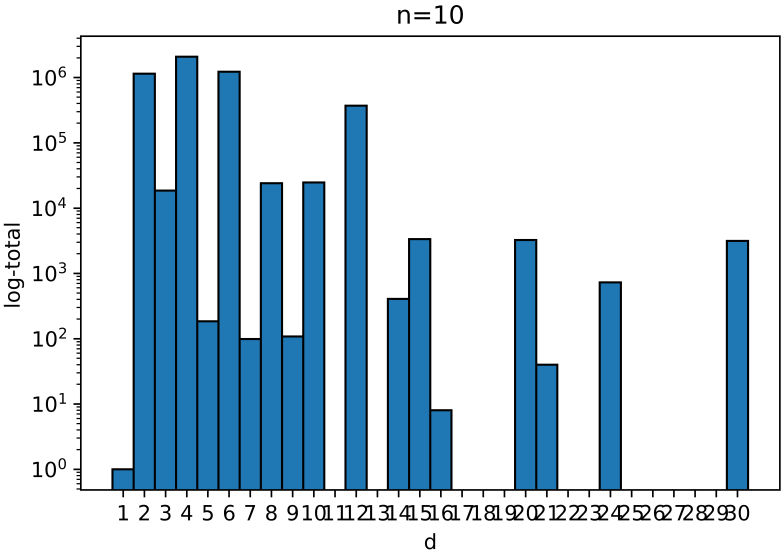
If $s * s = s$ for all s and $\mathcal{G}$ is abelian, the conjecture is verified.

Proof.

- $s * s = s \Rightarrow s^{[k]} = \Pi_k(s, \dots, s) = s^k$
- s^k diagonal $\Leftrightarrow o(\psi(s)) \mid k$.
- $\mathcal{G}$ abelian $\Rightarrow \exists f \in M,\ o(\psi(f)) = e(\mathcal{G})$.
- $o(\psi(s)) \mid o(\psi(f)) \leq \max\{\mathrm{lcm}(a_1, \dots, a_k) \mid a_1 + \dots + a_k = n\}$.

$\square$

Some histograms



Sylow for the germs

Theorem *(Lebed-Ramírez-Vendramin '22, F.)*

$G^{[k]} = \langle s^{[k]} \rangle$ induces a cycle set structure on $S^{[k]} = \{s^{[k]}\}_{s \in S}$.
Moreover, its class is $\frac{d}{d \wedge k}$ (if $k \leq d$).

- $G^{[k]}$ is the subgroup of G of matrices with coefficients powers that are multiples of k.

- Decompose $d = p_1^{a_1} \dots p_r^{a_r}$, and let $\alpha_i = p_i^{a_i}, \beta_i = \frac{d}{\alpha_i}$.

Lemma

The $\overline{G}^{[\beta_i]}$ are p_i-Sylow of $\overline{G}$, they commute two by two and their product is $\overline{G}$.

($H, K < G$ commute means $HK = KH$, i.e $\forall h, k, \exists h', k', hk = k'h'$.)

- We decomposed S in cycle sets with classes that are prime powers!

An example

Let $S = \{s_1, \ldots, s_6\}$ with $\psi(s_i) = (12\ldots6) = \sigma$. Then:

- $s_i * s_i = s_{\sigma(i)} \Rightarrow T = \sigma$
- $\psi(s_{i_1} \ldots s_{i_k}) = \sigma^k \Rightarrow d = 6$
- $\mathcal{G} = \langle\sigma\rangle \simeq \mathbb{Z}/6\mathbb{Z}$

- $G^{[3]}$ is generated by the $s_i^{[3]} = D_i^3 P_{\sigma^3}$, where $\sigma^3 = (14)(25)(36)$.
- $S^{[3]}$ is of class 2
- $\overline{G} = \overline{G}^{[2]}\overline{G}^{[3]}$, $s_i = (s_i^{[2]})^2 \cdot s^{[3]}_{\sigma^4(i)}$

Reconstructing

Denote Σ_n^d the group of monomial matrices with coefficient powers of ζ_d.
Define $\iota_d^{dk} : \Sigma_n^d \hookrightarrow \Sigma_n^{dk}$ sending ζ_d to ζ_{dk}^k.

- Let $(S, *_1)$ and $(S, *_2)$ be two cycle set structure on S.
- Suppose their classes d_1 and d_2 are coprime.
- Denote $\overline{G}_i < \Sigma_n^{d_i}$ their germs.
- Let $d = d_1 d_2$ and $\overline{G} = \iota_{d_1}^d(\overline{G}_1)\iota_{d_2}^d(\overline{G}_2)$.
- Bézout $\Rightarrow \exists u, v \in \mathbb{N},\ d_2 u + d_1 v = 1[d] \Rightarrow \forall s \in S, \exists g \in \overline{G}, D_g = D_s$.

Does $\overline{G}$ induces a cycle set structure on S?

- No in general. Yes if $\overline{G}_1$ and $\overline{G}_2$ commute in Σ_n^d !

Example

$S = \{s_1, \ldots, s_6\}$ with $(S', *_1)$ et $(S'', *_2)$ given by:

$$\psi_1(\{s'_1, \ldots, s'_6\}) = (14)(25)(36) \qquad d_1 = 2$$

$$\psi_2(\{s''_1, s''_3, s''_5\}) = (135) \qquad \psi_2(\{s''_2, s''_4, s''_6\}) = (246) \qquad d_2 = 3$$

- $3u + 2v = 1[6] \Rightarrow u = 1, v = 2$ $\qquad \overline{s_i} = \iota_2^6(\overline{s_i'^{[u]}})\iota_3^6(\overline{s''^{[v]}_{\psi(s_i'^{[u]})(i)}})$
- $\iota_2^6(\overline{s'_1}^{[1]}) = \iota_2^6\begin{pmatrix} 0&0&0&\zeta_2&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&0&\zeta_6^3&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0 \end{pmatrix}$
- $\iota_3^6(\overline{s''_4}^{[2]}) = \begin{pmatrix} 1&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0\\ 0&(\zeta_6^2)^2&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0 \end{pmatrix}$

$$\overline{s_1} = \begin{pmatrix} 0&\zeta_6&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0 \end{pmatrix} \Rightarrow \text{We take } \psi(s_1) = (125634)$$

- We find: $\psi(\{s_1, s_3, s_5\}) = (125634)$, $\psi(\{s_2, s_4, s_6\}) = (145236)$

Reconstructing

Theorem *(F.)*

If $(S, *_1)$ et $(S, *_2)$ satisfy a "mixed" cycle set condition:

$$\forall s, t, u \in S, (s *_1 t) *_2 (s *_1 u) = (t *_2 s) *_1 (t *_2 u)$$

Then $\overline{G}$ previously defined induces a cycle set structure on S of class (dividing) $d = d_1 d_2$.

We can restrict to cycle sets of class a prime-power to classify all cycle sets!

For $n = 10$, there is ~ 4.9m cycle sets, with ~ 3.3m of class a prime-power (67%)

Indecomposability

Definition

$(S, *)$ is said to be decomposable if there exists a partition $S = X \sqcup Y$ such that $(X, *_{|_X}), (Y, *_{|_Y})$ are cycle sets. Otherwise $(S, *)$ is called indecomposable.

- Up to a change of enumeration, S is decomposable iff the generators are diagonal by same blocks.

Proposition

If S is indecomposable and $d = p^k$, then $n = p^l$.

- We can "restrict" to cycle sets of size and class powers of the same prime.

Example: $(S, *)$ with $n = 16$ and $d = 30$ splits as cycle sets of class $2, 3$ and 5. Those with class 3 and 5 must be decomposable.

Future work

1. Better understand d
   Conjecture, relation with decomposability (indec. $\Rightarrow d \leq n$, relation between d_S and d_X, d_Y)

2. Study the irreducibility of the representation
   Relation with $|\mathcal{G}|$, induced by a subgroup character

3. Characterizing balanced elements and germs isomorphism.

4. Define the notion of Hecke algebra associated to a germ.

5. Generalize the approach to Weyl groups
   $G < T \rtimes W$, representation, class, etc.

Thank you for your attention!