The classification of Nichols algebras

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Let (V, c) be a braided vector space, i.e. V is a $\mathbb{K}$-vector space and $c \in \mathbf{GL}(V \otimes V)$ is a solution of the braid equation:

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c)$$

Examples:

- $V = \langle x_1, x_2, \ldots, x_\theta \rangle$, $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$, $q_{ij} \in \mathbb{K}^\times$;
- G a group, $V = \mathbb{K}G$, $c(g \otimes h) = ghg^{-1} \otimes g$.

A braided vector space (V, c) gives a special type of algebra called the Nichols algebra $\mathcal{B}(V, c)$.

The Nichols algebra of (V, c) is constructed as a quotient of the tensor algebra $T(V)$:

$$\mathcal{B}(V) = \mathbb{K} \oplus V \oplus \bigoplus_{n \geq 2} T^n(V) / \ker \mathfrak{S}_n,$$

where $\mathfrak{S}_n$ is the quantum symmetrizer. For example:

$$\mathfrak{S}_2 = \mathrm{id} + c,$$
$$\mathfrak{S}_3 = \mathrm{id} + c_{12} + c_{23} + c_{12}c_{23} + c_{23}c_{12} + c_{12}c_{23}c_{12}.$$

Let G be a group. A Yetter-Drinfeld module V over the group G is a G-graded $\mathbb{K}G$-module

$$V = \oplus_{g \in G} V_g$$

such that $g \cdot V_h \subseteq V_{ghg^{-1}}$ for all $g, h \in G$. A Yetter-Drinfeld module V is a braided vector space:

$$c(v \otimes w) = g \cdot w \otimes v,$$

where $v \in V_g$ and $w \in V$.

${}^G_G\mathcal{YD}$ denotes the category of Yetter-Drinfeld modules over G.

Some well-known examples of Nichols algebras:

- (V, flip) gives the symmetric algebra;
- $(V, -\text{flip})$ gives the exterior algebra.

Nichols algebras appear in:

- Quantum groups (Drinfeld; Jimbo; Lusztig; Rosso);
- Differential calculus in quantum groups (Woronowicz);
- Pointed Hopf algebras (Nichols; Andruskiewitsch, Schneider);
- Schubert calculus (Fomin, Kirillov; Postnikov; Bazlov);
- Mathematical-physics (Majid; Semikhatov; Lentner).

Problems

1. Classify finite-dimensional Nichols algebras.
2. Give a “nice” presentation of the Nichols algebras in the classification.

Definition:
A braided vector space V is of diagonal type if there exists a basis $\{v_1, \cdots, v_\theta\}$ of V such that

$$c(v_i \otimes v_j) = q_{ij} v_j \otimes v_i, \quad q_{ij} \in \mathbb{K}^\times.$$

Nichols algebras of braided vector spaces of diagonal type have many interesting properties and applications.

Over the complex numbers:

1. Heckenberger classified finite-dimensional Nichols algebras of diagonal type.
2. Andruskiewitsch and Schneider classified finite-dimensional pointed Hopf algebras with abelian coradical of order coprime to 210.
3. Angiono found a presentation for the Nichols algebras in Heckenberger's classification. With this, Angiono proved that finite-dimensional pointed Hopf algebras with abelian coradical are generated by grouplikes and skew-primitives (Andruskiewitsch–Schneider conjecture).
4. Andruskiewitsch, Angiono and García Iglesias found new infinite families of finite-dimensional pointed Hopf algebras.

What about non-diagonal Nichols algebras?

This is important to study combinatorial Schubert calculus and pointed Hopf algebras with non-abelian coradical.

Let (V, c) be a braided vector space not of diagonal type. There are two cases to consider:

1. V is *indecomposable*.
2. $V = V_1 \oplus \cdots \oplus V_\theta$ is *not indecomposable*, where $\theta \geq 2$.

The classification of Nichols algebras over indecomposable Yetter-Drinfeld modules is an open problem.

So far only few examples of finite-dimensional Nichols algebras over indecomposable braided vector spaces of group type are known!

Notation:
$(k)_t = 1 + t + \cdots + t^{k-1}$.

$\dim V$	$\dim \mathcal{B}(V)$	Hilbert series	characteristic
3	12	$(2)_t^2(3)_t$	
3	432	$(3)_t(4)_t(6)_t(6)_{t^2}$	2
4	36	$(2)_t^2(3)_t^2$	2
4	72	$(2)_t^2(3)_t(6)_t$	$\neq 2$
4	5184	$(6)_t^4(2)_{t^2}^2$	
6	576	$(2)_t^2(3)_t^2(4)_t^2$	
6	576	$(2)_t^2(3)_t^2(4)_t^2$	
6	576	$(2)_t^2(3)_t^2(4)_t^2$	
5	1280	$(4)_t^4(5)_t$	
5	1280	$(4)_t^4(5)_t$	
7	326592	$(6)_t^6(7)_t$	
7	326592	$(6)_t^6(7)_t$	
10	8294400	$(4)_t^4(5)_t^2(6)_t^4$	
10	8294400	$(4)_t^4(5)_t^2(6)_t^4$	

Let us study the decomposable case.

I will discuss my collaboration with István Heckenberger. The techniques used in these works heavily depend on the theory of Weyl groupoids developed by Andruskiewitsch, Cuntz, Heckenberger, Schneider, Yamane.

We first start with the case of two irreducible summands.

Remark (Graña)

If $c_{W,V}c_{V,W} = \mathrm{id}_{V\otimes W}$ then

$$\mathcal{B}(V \oplus W) \simeq \mathcal{B}(V) \otimes \mathcal{B}(W)$$

as graded vector spaces.

The remark implies that in order to be in the decomposable case one needs to assume that

$$c_{W,V}c_{V,W} \neq \mathrm{id}_{V\otimes W}.$$

Technical definition:
Let G be a group. The support of a Yetter-Drinfeld module $V = \oplus_{g\in G} V_g$ is the set

$$\mathrm{supp} V = \{g \in G : V_g \neq 0\}$$

Theorem (Heckenberger, V.)

Let G be a non-abelian group, and V and W be two absolutely irreducible finite-dimensional Yetter-Drinfeld modules over $\mathbb{K}G$. Assume that:

- G is generated by the support of $V \oplus W$,
- $c_{W,V}c_{V,W} \neq \mathrm{id}_{V\otimes W}$, and
- $\dim \mathcal{B}(V \oplus W) < \infty$.

Then G is an epimorphic image of a certain central extension T of the group $\mathbf{SL}(2,3)$, or an epimorphic image of

$$\Gamma_n = \langle g, h, \epsilon : hg = \epsilon gh,\ \epsilon g\epsilon = g,\ h\epsilon = \epsilon h,\ \epsilon^n = 1\rangle$$

for some $n \in \{2, 3, 4\}$.

The group T can be presented by generators z, x_1, x_2, x_3, x_4 and relations

$$zx_i = x_i z, \quad i \in \{1, 2, 3, 4\},$$

and

$$\begin{aligned} x_1x_2 &= x_4x_1 = x_2x_4, \\ x_1x_3 &= x_2x_1 = x_3x_2, \\ x_2x_3 &= x_4x_2 = x_3x_4, \\ x_1x_4 &= x_3x_1 = x_4x_3. \end{aligned}$$

The theorem has deep consequences. After some work one for example obtains:

- The structure of the braided vector spaces V and W.
- The dimension of $\mathcal{B}(V \oplus W)$.
- The Hilbert series of $\mathcal{B}(V \oplus W)$.

Theorem (Heckenberger, V.)

Let G be a non-abelian group, and V and W be two irreducible finite-dimensional Yetter-Drinfeld modules over $\mathbb{C}G$. Assume that:

- G is generated by the support of $V \oplus W$,
- $c_{W,V}c_{V,W} \neq \mathrm{id}_{V\otimes W}$, and
- $\dim \mathcal{B}(V \oplus W) < \infty$.

Then $\mathcal{B}(V \oplus W)$ is one of following Nichols algebras:

$\dim(V \oplus W)$	$\dim \mathcal{B}(V \oplus W)$
4	64
4	10368
5	10368
5	2304
5	80621568
6	262144

Let us show one of the algebras we found (over the complex numbers).

Let G be a non-abelian epimorphic image of the group T. We show the structure of the modules V and W.

Assume that $V \simeq M(z, \rho)$, where ρ is a character of the centralizer $G^z = G$. Let $v \in V_z \setminus \{0\}$. Then $\{v\}$ is basis of V and the action of G on V is given by

$$zv = \rho(z)v, \quad x_i v = \rho(x_1)v \quad \text{for all } i \in \{1, 2, 3, 4\}.$$

Let $W = M(x_1, \sigma)$, where σ is a character of $G^{x_1} = \langle x_1, x_2x_3, z \rangle$ with $\sigma(x_1) = -1$ and $\sigma(x_2x_3) = 1$. Let $w_1 \in W_{x_1}$ be such that $w_1 \neq 0$. Then the vectors

$$w_1,\ w_2 = -x_4w_1,\ w_3 = -x_2w_1,\ w_4 = -x_3w_1$$

form a basis of W. The degrees of these vectors are x_1, x_2, x_3 and x_4, respectively. The action of G on W is given by the following table:

W	w_1	w_2	w_3	w_4
x_1	$-w_1$	$-w_4$	$-w_2$	$-w_3$
x_2	$-w_3$	$-w_2$	$-w_4$	$-w_1$
x_3	$-w_4$	$-w_1$	$-w_3$	$-w_2$
x_4	$-w_2$	$-w_3$	$-w_1$	$-w_4$
z	$\sigma(z)w_1$	$\sigma(z)w_2$	$\sigma(z)w_3$	$\sigma(z)w_4$

Assume further that

$$(\rho(x_1)\sigma(z))^2 - \rho(x_1)\sigma(z) + 1 = 0, \quad \rho(x_1 z)\sigma(z) = 1.$$

Then $\mathcal{B}(V \oplus W)$ is finite-dimensional. Its Hilbert series is

$$H(t) = (6)_t(6)_{t^4}(6)_{t^5}(2)_t^2(3)_t(6)_t(2)_{t^4}(3)_{t^2}(6)_{t^2}(2)_{t^6}(3)_{t^3}(6)_{t^3}$$

and $\dim \mathcal{B}(V \oplus W) = 6^3\, 72^3 = 80621568$.

Open problem

Find a "nice" presentation for this Nichols algebra.

$\dim(V \oplus W)$	G	$\dim \mathcal{B}(V \oplus W)$	characteristic
4	Γ_2	64	
4	Γ_2	1296	3
4	Γ_3	10368	$\neq 2, 3$
4	Γ_3	5184	2
4	Γ_3	1152	3
4	Γ_3	2239488	2
5	Γ_3	10368	$\neq 2, 3$
5	Γ_3	5184	2
5	Γ_3	1152	3
5	Γ_3	2304	
5	Γ_3	2239488	2
5	T	80621568	$\neq 2$
5	T	1259712	2
6	Γ_4	262144	$\neq 2$
6	Γ_4	65536	2

We now study the case of at least three simple summands.

To be in the decomposable case, we need to assume that $M = (M_1, \dots, M_\theta)$ is connected, i.e. M admits no decomposition

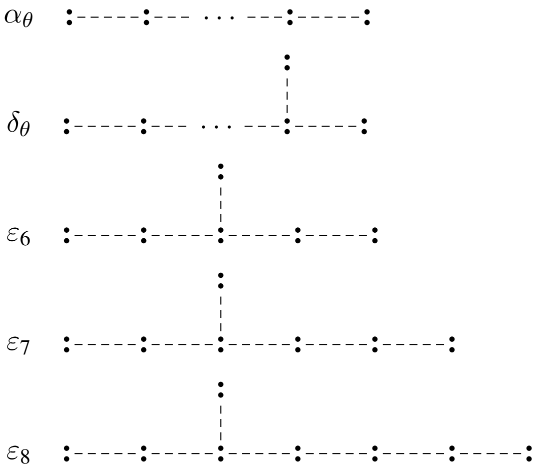
$$M_1 \oplus \cdots \oplus M_\theta = M' \oplus M''$$

as Yetter-Drinfeld modules over G with $M' \neq 0$, $M'' \neq 0$ and $c_{M'',M'} c_{M',M''} = \mathrm{id}$.

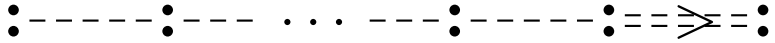
We need to introduce the following terminology.

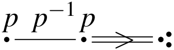
Skeletons (of finite type). A skeleton (of finite-type) is a decorated Dynkin diagram (of finite-type) that encodes the structure of the Yetter-Drinfeld module.

The following are the simply-laced skeleton of finite-type (i.e. Dynkin types ADE):

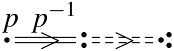


The other skeletons of finite type are:

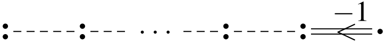
β_θ  $\mathrm{char}\mathbb{K} = 3$

β_3' 

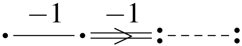
$(3)_{-p} = 0$

β_3'' 

$(3)_{-p} = 0$

γ_θ 

$\mathrm{char}\mathbb{K} \neq 2$

φ_4 

$\mathrm{char}\mathbb{K} \neq 2$

Theorem (Heckenberger, V.)

Let $\theta \geq 3$, G be a non-abelian group and

$$M = (M_1, \ldots, M_\theta)$$

be an connected tuple of finite-dimensional absolutely irreducible Yetter-Drinfeld modules over $\mathbb{K}G$. The following are equivalent:

1. $\mathcal{B}(M_1 \oplus \cdots \oplus M_\theta)$ is finite-dimensional.
2. M has a skeleton of finite-type.

The theorem gives the dimensions and the Hilbert series of

$$\mathcal{B}(M) = \mathcal{B}(M_1 \oplus \cdots \oplus M_\theta)$$

and the structure of the M_i can be read off from the skeletons of finite type.

Example:
In the case where M has a simply-laced skeleton of finite type (Dynkin type ADE), the dimensions of the Nichols algebras in the classification are:

$\dim \mathcal{B}(M)$	$4^{\theta(\theta+1)/2}$	$4^{\theta(\theta-1)}$	4^{36}	4^{63}	4^{120}
skeleton	α_θ	δ_θ	ε_6	ε_7	ε_8