Fomin and Kirillov algebras

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Nichols Algebras and Weyl Groupoids
Oberwolfach, October 2012

In 1995 Fomin & Kirillov introduced the quadratic algebras $\mathcal{E}_n$ to study the combinatorics of the cohomology of flag manifolds.

Definition:
Let $\mathcal{E}_n$ be the algebra (of type A_{n-1}) with generators $x_{(ij)}$, where $i,j \in \{1,\dots,n\}$, and relations

$$\begin{aligned}
&x_{(ij)} + x_{(ji)} = 0,\\
&x_{(ij)}^2 = 0,\\
&x_{(ij)}x_{(jk)} + x_{(jk)}x_{(ki)} + x_{(ki)}x_{(ij)} = 0,\\
&x_{(ij)}x_{(kl)} = x_{(kl)}x_{(ij)}
\end{aligned}$$

for any distinct i,j,k,l.

Remarks:

- $\mathcal{E}_n$ is quadratic,
- $\mathcal{E}_n$ is graded: $\deg(x_{(ij)}) = 1$,
- $\mathcal{E}_n$ provides a solution for the classical Yang-Baxter equation:

$$[x_{(ij)}, x_{(jk)}] = [x_{(jk)}, x_{(ik)}] + [x_{(ik)}, x_{(ij)}]$$

where $[u, v] = uv - vu$ is the usual commutator.

Example:
The algebra $\mathcal{E}_3$ can be presented with generators $x_{(12)}, x_{(23)}, x_{(13)}$ and relations

$$x_{(12)}^2 = x_{(23)}^2 = x_{(13)}^2 = 0$$

$$x_{(12)}x_{(23)} + x_{(23)}x_{(13)} = x_{(12)}x_{(13)},$$

$$x_{(23)}x_{(12)} + x_{(13)}x_{(23)} = x_{(13)}x_{(12)}.$$

It is a graded algebra of dimension 12.
The Hilbert series is

$$\mathcal{H}(t) = 1 + 3t + 4t^2 + 3t^3 + t^4 = (2)_t^2(3)_t,$$

where $(k)_t = 1 + t + \cdots + t^{k-1}$.
The degree of $\mathcal{H}(t)$ is four: $\operatorname{top}(\mathcal{E}_3) = 4$.

Problems (Fomin & Kirillov)

- Is $\mathcal{E}_n$ finite-dimensional?
- If $\mathcal{E}_n$ is finite-dimensional, compute $\dim \mathcal{E}_n$.
- Compute the Hilbert series of $\mathcal{E}_n$.

For example: $\mathcal{E}_3$, $\mathcal{E}_4$ and $\mathcal{E}_5$ are finite-dimensional:

	top	dimension	Hilbert series
$\mathcal{E}_3$	4	12	$(2)_t^2(3)_t$
$\mathcal{E}_4$	12	576	$(2)_t^2(3)_t^2(4)_t^2$
$\mathcal{E}_5$	40	8294400	$(4)_t^4(5)_t^2(6)_t^4$

Example:

The algebra $\mathcal{E}_6$ can be presented with 15 generators and 91 relations. The Hilbert series of $\mathcal{E}_6$ is

$$\mathcal{H}(t) = 1 + 15t + 125t^2 + 765t^3 + 3831t^4 + 16605t^5 + \cdots$$

Remark (Fomin & Kirillov):

- $\mathcal{H}(t)$ cannot expressed as a product of t-numbers.

Conjectures

- $\dim \mathcal{E}_n = \infty$ for $n \geq 6$.
- $\dim(\mathcal{E}_n)_k \sim \binom{\binom{n}{2}}{k}$.

Fomin & Kirillov introduced the algebras $\mathcal{E}_n$ to study the cohomology of flags varieties.

For example:
Let $\mathcal{A}$ be the subalgebra of $\mathcal{E}_3$ generated by the Dunkl elements:

$$\theta_1 = x_{(12)} + x_{(13)}, \quad \theta_2 = -x_{(12)} + x_{(23)}, \quad \theta_3 = -x_{(13)} - x_{(23)}.$$

Then $[\theta_i, \theta_j] = 0$ for all i, j and hence $\mathcal{A}$ is commutative.

Furthermore,

$$\theta_1 + \theta_2 + \theta_3 = \theta_1^2 + \theta_2^2 + \theta_3^2 = \theta_1\theta_2\theta_3 = 0,$$

and $\mathcal{A} \simeq H^*(\mathrm{Flags}(\mathbb{C}^3))$.

In general, the Dunkl elements are

$$\theta_j = \sum_{j \neq k} x_{(jk)}$$

for all $j \in \{1, 2, \ldots, n\}$.

Remarks:

- The Dunkl elements commute pairwise,
- The complete list of relations among the Dunkl elements is given by

  $$e_i(\theta_1, \ldots, \theta_n) = 0,$$

  for $i \in \{1, 2, \ldots, n\}$, where $e_1, e_2, \ldots, e_n$ are the elementary symmetric functions.
- $\mathcal{E}_n$ contains a commutative subalgebra isomorphic to the cohomology of flags manifolds.

What is the connection with Nichols algebras?

Recall that the algebras $\mathcal{E}_3$, $\mathcal{E}_4$ and $\mathcal{E}_5$ are finite-dimensional:

	top	dimension	Hilbert series
$\mathcal{E}_3$	4	12	$(2)_t^2(3)_t$
$\mathcal{E}_4$	12	576	$(2)_t^2(3)_t^2(4)_t^2$
$\mathcal{E}_5$	40	8294400	$(4)_t^4(5)_t^2(6)_t^4$

Let V_n be the vector space with basis $\{v_{(ij)} \mid 1 \leq i < j \leq n\}$ and the braiding $c \in \mathbf{GL}(V_n \otimes V_n)$ defined by

$$c(v_\sigma \otimes v_\tau) = \chi(\sigma,\tau) v_{\sigma\tau\sigma} \otimes v_\sigma,$$

where

$$\chi(\sigma,\tau) = \begin{cases} 1 & \text{if } \sigma(i) < \sigma(j), \\ -1 & \text{otherwise,} \end{cases}$$

where $\tau = (i\, j)$ with $i < j$.

It is well-known that

$$\mathfrak{B}(V_n) = \mathcal{E}_n$$

if $n \in \{3, 4, 5\}$.

Remarks:

- Bazlov proved that the Nichols algebra $\mathfrak{B}(V_n)$ contains a commutative subalgebra isomorphic to $H^*(\mathrm{Flags}(\mathbb{C}^n))$.
- This subalgebra is isomorphic to the subalgebra generated by the Dunkl elements.

Conjectures

- $\dim \mathfrak{B}(V_n) = \infty$ for $n \geq 6$.
- $\mathfrak{B}(V_n)$ is quadratic and hence $\mathcal{E}_n = \mathfrak{B}(V_n)$.

Bazlov's construction

Let Δ be a root system, and let V be the vector space spanned by the symbols $[\alpha]$, where $\alpha \in \Delta$, and $[-\alpha] = [\alpha]$.
The map $c \in \mathbf{GL}(V \otimes V)$ defined by

$$c([\alpha] \otimes [\beta]) = [s_\alpha \beta] \otimes [\alpha],$$

where

$$s_\alpha(x) = x - \frac{2(x, \alpha)}{(\alpha, \alpha)}\alpha,$$

is a solution of the braid equation. Hence (V, c) is a braided vector space.

Theorem (Bazlov):
$\mathfrak{B}(V)$ contains a commutative subalgebra isomorphic to the cohomology of flags manifolds.

Preprojective algebras

Let Q be an orientation of a Dynkin diagram of type A_{n-1}

and let $\overline{Q}$ be the double quiver: for each arrow $\alpha : i \to j$ add a new arrow $\overline{\alpha} : j \to i$.

Definition (Gelfand & Ponomarev):
The preprojective algebra of Q is

$$\Lambda = \mathbb{C}\overline{Q}/\mathcal{I},$$

where $\mathbb{C}\overline{Q}$ is the path algebra of $\overline{Q}$ and $\mathcal{I}$ is the two-sided ideal generated by $\sum_\alpha (\alpha\overline{\alpha} - \overline{\alpha}\alpha)$.

Remarks:

- Λ is finite-dimensional.
- Λ is of finite representation type if and only if $n \leq 5$.

In 2001 Majid and Marsh noticed that maybe there exists a relationship between the algebras $\mathcal{E}_n$ and the representation theory of preprojective algebras of type A_{n-1}.

Let d be the number of indecomposable modules of the preprojective algebra Λ. Then

n	d	$\mathrm{top}(\mathcal{E}_n)$
3	4	4
4	12	12
5	40	40
≥ 6	∞	?

Cluster algebras

Let $\mathbb{F} = \mathbb{Q}(x_1, \ldots, x_n)$ be a field.

A seed is a pair $(Q, y_1, \ldots, y_n)$, where Q is a quiver with n vertices, with no loops and no 2-cycles, and $\{y_1, \ldots, y_n\}$ is a free generating set of $\mathbb{F}$.

The k-mutation of the seed $(Q, y_1, \ldots, y_n)$ is the seed $(\mu_k(Q), \mu_k(y_1), \ldots, \mu_k(y_n))$, where

$$\mu_k(y_j) = \begin{cases} y_j & \text{if } j \neq k, \\ \frac{1}{y_k}\left(\prod_{i \to k} y_i + \prod_{k \to j} y_j\right) & \text{if } j = k, \end{cases}$$

and $\mu_k(Q)$ is obtained from Q by:

- adding a new arrow $i \to j$ for every $i \to k \to j$,
- erasing all the 2-cycles created,
- changing the orientation of every arrow incident to k.

The mutation class $\mu(\Sigma)$ of a seed $\Sigma = (Q, y_1, \ldots, y_n)$ is the set of all seeds obtained from a finite sequence of mutations.

If $(Q', y'_1, \ldots, y'_n)$ is a seed in $\mu(\Sigma)$, then

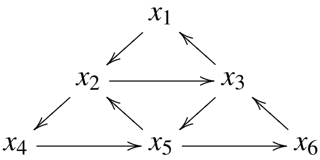
- the set $\{y'_1, \ldots, y'_n\}$ is called a cluster, and
- the elements of $\{y'_1, \ldots, y'_n\}$ are called cluster variables.

Definition (Fomin & Zelevinsky):
The cluster algebra A_Σ is the subring of $\mathbb{F}$ generated by all the cluster variables.

Theorem (Fomin & Zelevinsky):
A_Σ has a finite number of cluster variables if and only if the mutation class of Σ contains a seed whose quiver is an orientation of a Dynkin diagram of type A, D or E.

Example:

Let A be the cluster algebra with initial seed



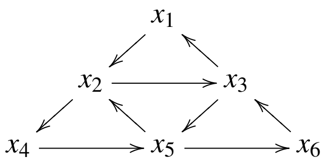
where the variables x_4, x_5 and x_6 are frozen (they belong to every cluster).

After removing the frozen variables, the cluster algebra A has finite type A_3. It has 12 cluster variables.

Let G be a simple algebraic group of type A, D or E, $N \subseteq G$ be a maximal unipotent subgroup, and $\mathbb{C}[N]$ be the coordinate ring of N.

Theorem (Berenstein, Fomin, Zelevinsky):
$\mathbb{C}[N]$ is a cluster algebra.

Example:
The case A_3. Let $G = \mathbf{SL}_4$ and N be the subgroup of upper triangular matrices with ones in the diagonal. The cluster algebra $\mathbb{C}[N]$ is the cluster algebra associated to the quiver



where x_4, x_5 and x_6 are frozen.

In general

Lie type of G	Cluster type of $\mathbb{C}[N]$	Clusters
A_2	A_1	4
A_3	A_3	12
A_4	D_6	40
others	∞	∞

Remarks:

- Berenstein and Zelevinsky proved that the cluster monomials coincide with the elements of Lusztig's dual canonical basis of $\mathbb{C}[N]$.
- Geiss, Leclerc and Schröer established a connection between the number of clusters of $\mathbb{C}[N]$ and the number of indecomposable modules over preprojective algebras.

Final remarks:

- Maybe there is a relationship between $\mathcal{E}_n$ and the number cluster of $\mathbb{C}[N]$.
- By the relationship between the representation theory of the preprojective algebras Λ and the cluster type of $\mathbb{C}[N]$ found by Geiss, Leclerc and Schröer, the existence of a relationship between $\mathcal{E}_n$ and $\mathbb{C}[N]$ turns out to be equivalent to the conjecture of Majid and Marsh.