Nichols algebras and root systems – Exercises

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Abstract. These are the exercises of Heckenberger's course **Nichols Algebras and Root Systems**, given at the Vrije Universiteit Brussel, Faculty of Sciences, Department of Mathematics and Data Sciences, as part of the Francqui VUB-Leerstoel 2025–2026.

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Introduction

This document contains the exercises for the 20-hour course Nichols Algebras and Root Systems, given in February and March 2026 at the Vrije Universiteit Brussel. The course is based on the book [1].

Several people attending this course helped write or improve exercises from the lectures, including Silvia Properzi, Lukas Simons, and Leandro Vendramin.

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§ 1. Braided structures

1.1. Exercise. Prove that the category of vector spaces is a monoidal category.

1.2. Exercise. Let $\mathcal{C}$ be a monoidal category and $\mathcal{C}^{\mathrm{op}}$ its dual category of $\mathcal{C}$. Prove that $\mathcal{C}^{\mathrm{op}}$ is a monoidal category.

1.3. Exercise. Let $\mathcal{C}$ be a monoidal category.

(a) Prove that the unit of an algebra A in $\mathcal{C}$ is unique.
(b) Prove that the counit of a coalgebra C in $\mathcal{C}$ is unique.

1.4. Exercise. Let $\mathcal{C}$ be a monoidal category, (C,Δ,ϵ) a coalgebra in $\mathcal{C}$, and (A,μ,η) an algebra in $\mathcal{C}$. Prove that the convolution product of $\mathrm{Hom}_{\mathcal{C}}(C,A)$ is associative with unit $\eta\epsilon$.

1.5. Exercise. Let $(\mathcal{C},\otimes,I,c)$ be a strict braided monoidal category. Prove that

$$c_{I,X}=c_{X,I}=\mathrm{id}$$

for all $X\in\mathcal{C}$.

1.6. Exercise. Let $(\mathcal{C},\otimes,I,c)$ be a braided strict monoidal category. Prove that $\mathcal{C}^{\mathrm{op}}$ is a braided strict monoidal category.

1.7. Exercise. Let $(\mathcal{C},\otimes,I,c)$ be a braided strict monoidal category. Prove that $\overline{\mathcal{C}}$ is a braided strict monoidal category.

1.8. Exercise. Let $\mathcal{C}$ be a strict monoidal category, A,B,C and D algebras in $\mathcal{C}$, and $\varphi\colon A\to C$ and $\psi\colon B\to D$ be algebra morphisms in $\mathcal{C}$. Let (V,λ_A) be a left A-module in $\mathcal{C}$ and (W,λ_W) be a left B-module in $\mathcal{C}$. Prove the following statements:

1) $A\otimes B$ is an algebra in $\mathcal{C}$ with multiplication
$$\mu_{A\otimes B}=(\mu_A\otimes\mu_B)(\mathrm{id}\otimes c_{B,A}\otimes\mathrm{id})$$
and unit $\eta_{A\otimes B}=\eta_A\otimes\eta_B$.
2) $\varphi\otimes\psi\colon A\otimes B\to C\otimes D$ is an algebra morphisms in $\mathcal{C}$.
3) $V\otimes W$ is an $A\otimes B$-module with $(\lambda_A\otimes\lambda_B)(\mathrm{id}\otimes c_{B,V}\otimes\mathrm{id})$.
4) The algebra structures of $(A\otimes B)\otimes C$ and $A\otimes(B\otimes C)$ coincide.

1.9. Exercise. Formulate and prove the dual version of Exercise 1.8.

1.10. Exercise. Let $\mathcal{C}$ be a strict braided monoidal category and H a bialgebra in $\mathcal{C}$.

(a) Prove that the category ${}_H\mathcal{C}$ of left H-modules in $\mathcal{C}$ is a strict monoidal category.
(b) Prove that the category ${}^H\mathcal{C}$ of left H-comodules in $\mathcal{C}$ is a strict monoidal category.

1.11. Exercise. Let K be a field and G be a group. Prove that a Yetter–Drinfeld module over KG is a G-graded KG-module $V=\bigoplus_{g\in G}V_g$ such that $g\cdot V_h\subseteq V_{ghg^{-1}}$ for all $g,h\in G$.

1.12. Exercise. Let $\mathcal{C}$ be a braided monoidal category. Define for every $(Y,\lambda)\in{}_H\mathcal{C}$ and $(X,\delta)\in{}^H\mathcal{C}$

$$c^{\mathcal{YD}(\mathcal{C})}_{X,Y}=(\lambda\otimes\mathrm{id})c_{X,Y}(\delta\otimes\mathrm{id}_Y).$$

Let $(V,\lambda)\in{}_H\mathcal{C}$ and $(V,\delta)\in{}^H\mathcal{C}$. Prove that the following statements are equivalent:

1) $(V,\lambda,\delta)\in{}^H_H\mathcal{YD}(\mathcal{C})$.
2) For all $X\in{}_H\mathcal{C}$, the map $c^{\mathcal{YD}(\mathcal{C})}_{V,X}$ is a morphism in ${}_H\mathcal{C}$.

3) $c^{\mathcal{YD}(\mathcal{C})}_{V,H}$ is a morphism in ${}_H\mathcal{C}$.

4) For all $X \in {}^H\mathcal{C}$, the map $c^{\mathcal{YD}(\mathcal{C})}_{X,V}$ is a morphism in ${}^H\mathcal{C}$.

1.13. Exercise. Let H be a Hopf algebra in $\mathcal{C}$ with invertible antipode. Prove that ${}^H_H\mathcal{YD}(\mathcal{C})$ is a strict braided monoidal category with monoidal structure as for modules and comodules and the braiding

$$c^{\mathcal{YD}(\mathcal{C})}_{X,Y} = (\lambda_Y \otimes \mathrm{id})c_{X,Y}(\delta_X \otimes \mathrm{id}_Y)$$

for $X, Y \in {}^H_H\mathcal{YD}(\mathcal{C})$.

§ 2. Nichols algebras

2.1. Exercise. Let $\mathcal{C}$ be a strict braided monoidal category. Prove the universal property of the tensor algebra in $\mathrm{gr}_{\mathbb{N}_0}\mathcal{C}$.

2.2. Exercise. Let $\mathcal{C}$ be a strict braided monoidal category and $V \in \mathcal{C}$. Prove that the tensor algebra $T(V)$ is a Hopf algebra with bijective antipode in $\mathrm{gr}_{\mathbb{N}_0}\mathcal{C}$.

2.3. Exercise. Let $\mathcal{C}$ be a strict braided monoidal category, $V \in \mathcal{C}$ and A an algebra in $\mathcal{C}$.

(a) Prove that for every morphism $f : V \to P_{\eta,\eta}(A)$ there is a unique bialgebra morphism $\varphi : T(V) \to A$ such that $\varphi|_V = f$.

(b) Show that, if A is in $\mathrm{gr}_{\mathbb{N}_0}\mathcal{C}$ and $f(V) \subseteq P_{\eta,\eta}(A(1))$, then φ is a morphism in $\mathrm{gr}_{\mathbb{N}_0}\mathcal{C}$.

2.4. Exercise. Let V be a complex braided vector space with basis $\{x_1, x_2\}$ and braiding

$$c(x_i \otimes x_j) = q_{ij}x_j \otimes x_i, \quad i, j \in \{1, 2\}.$$

Compute the Nichols algebra of (V, c) in the following cases:

1) $q_{11} = q_{22} = -1$ and $q_{12}q_{21} = 1$.

2) $q_{11} = q_{22} = -1$ and $q_{12}q_{21} = -1$.

3) $q_{11} = q_{22} = -1$ and $q_{12}q_{21} = \omega$, where ω is a primitive cube root of unity. This Nichols algebra is usually denoted by $\mathcal{U}_q(\mathfrak{sl}_3)^+$. Prove that in this algebra the elements $y^{n_1}(\mathrm{ad}_c\, x)(y)^{n_2}x^{n_3}$ span $\mathcal{U}_q(\mathfrak{sl}_3)^+$.

§ 3. Cartan graphs and Weyl groupoids

3.1. Exercise. Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a semi-Cartan graph. Let $i_1, \dots, i_k \in I$, $X, Y, Z \in \mathcal{X}$, $w \in \mathrm{Hom}(X, Y)$ and $w' \in \mathrm{Hom}(Y, Z)$. Prove the following statements:

1) $|l(w) - l(w')| \le l(w'w) \le l(w') + l(w)$ and $l(w^{-1}) = l(w)$.

2) $l(w'w) \equiv l(w') + l(w) \bmod 2$.

3) $l(s_iw) \in \{l(w)+1, l(w)-1\}$ and $l(ws_i) \in \{l(w)+1, l(w)-1\}$.

4) $k - l(\mathrm{id}_X\, s_{i_1} \cdots s_{i_k})$ is non-negative and even.

3.2. Exercise. Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a Cartan graph. Let $X \in \mathcal{X}$ and $i \in I$. Assume that both $\Delta^{X\mathsf{re}}$ and $\Delta^{r_i(X)\mathsf{re}}$ are contained in $\mathbb{N}_0^I \cup -\mathbb{N}_0^I$. Prove the following statements:

1) $s_i^X(\pm\alpha_i) = \mp\alpha_i$ and $s_i^X(\Delta_+^{X\mathsf{re}} \setminus \{\alpha_i\}) = \Delta_+^{r_i(X)\mathsf{re}} \setminus \{\alpha_i\}$.

2) $m_{ij}^X = m_{ij}^{r_i(X)}$.

3.3. Exercise. Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a Cartan graph, $X, Y \in \mathcal{X}$, $i \in I$ and $w \in \mathrm{Hom}(Y, X)$. Prove the following statements:

1) $N(w) = N(w^{-1})$.
2) Assume that both $\Delta^{Y\mathsf{re}}$ and $\Delta^{r_i(Y)\mathsf{re}}$ are contained in $\mathbb{N}_o^I \cup -\mathbb{N}_0^I$. If $w(\alpha_i) \in \mathbb{N}_0^I$, then $N(ws_i) = N(w) + 1$. If $w(\alpha_i) \in -\mathbb{N}_0^I$, then $N(ws_i) = N(w) - 1$.

§ 4. Classification of finite Cartan graphs of rank two

4.1. Exercise. Let $\mathcal{A}^+$ be the smallest set of all sequences of integers such that

a) $(0,0) \in \mathcal{A}^+$, and
b) $(c_1, \dots, c_n) \in \mathcal{A}^+ \implies V_i(c_1, \dots, c_n) \in \mathcal{A}^+$ for all $i \in \{2, \dots, n\}$, where

$$V_i \colon \mathbb{Z}^n \to \mathbb{Z}^n, \quad V_i(c_1, \dots, c_n) = (c_1, \dots, c_{i-2}, c_{i-1} + 1, 1, c_i + 1, c_{i+1}, \dots, c_n).$$

For $n \geq 3$, let $\mathcal{A}^+(n) = \mathcal{A}^+ \cap \mathbb{Z}^n$. Prove the following statements:

1) $c_i > 0$ for all $i \in \{1, \dots, n\}$.
2) There exists $1 \leq i < j \leq n$ such that $(i,j) \neq (1,n)$ and $c_i = c_j = 1$.
3) If $c_i = c_{i+1} = 1$, then $n = 3$ and $(c_1, c_2, c_3) = (1,1,1)$.

4.2. Exercise. For $n \geq 3$ let G be a convex n-gon. Label clockwise the vertices of G from 1 to n. Let $\mathcal{T}_n$ be the set of triangulations of G. For $T \in \mathcal{T}_n$ and $i \in \{1, \dots, n\}$, let $c_i(T)$ be the number of triangles of T incident to vertex i. Prove that the map $\mathcal{T}_n \to \mathcal{A}^+(n)$, $T \mapsto (c_1(T), \dots, c_n(T))$, is bijective.

4.3. Exercise. Let $n \geq 3$ and $(c_1, \dots, c_n) \in \mathcal{A}^+(n)$. Prove that if $c_i = 1$ for some $i \in \{2, \dots, n\}$, then

$$(c_1, \dots, c_{i-2}, c_{i-1} + 1, c_{i+1} - 1, c_{i+2}, \dots, c_n) \in \mathcal{A}^+(n-1).$$

4.4. Exercise. Consider the map $\eta : \mathbb{Z} \to SL_2(\mathbb{Z})$, $a \mapsto \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$. For $n \geq 2$ and $(c_1, \dots, c_n) \in \mathbb{Z}^n$, prove that the following are equivalent.

1) $(c_1, \dots, c_n) \in \mathcal{A}^+(n)$.
2) $\eta(c_1)\eta(c_2)\dots\eta(c_n) = -\mathrm{id}_{\mathbb{Z}^2}$

References

[1] I. Heckenberger and H.-J. Schneider. *Hopf algebras and root systems*, volume 247 of *Math. Surv. Monogr.* Providence, RI: American Mathematical Society (AMS), 2020.