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RESEARCH ON ALGEBRAIC STRUCTURES RELATED TO SOLUTIONS OF (QUANTUM) YANG-BAXTER EQUATION

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Abstract

The (quantum) Yang–Baxter equation is a fundamental equation in mathematical physics, originating in quantum and statistical mechanics. This thesis is divided into three parts and presents a concise study of several algebraic structures related to its solutions, with particular emphasis on Rota–Baxter structures on 3-Lie algebras and associative algebras, as well as on L-algebras.

The first part concerns 3-Lie algebras, which provide a natural framework for higher-order generalizations of Lie algebras and have important applications in mathematical physics. Within this setting, solutions to the classical 3-Lie Yang–Baxter equation are closely related to Rota–Baxter structures. We develop the representation theory, cohomology, and formal deformation theory of Rota–Baxter and modified Rota–Baxter 3-Lie algebras of arbitrary weight. We also construct two $L_\infty[1]$-algebras whose Maurer–Cartan elements correspond to relative and absolute modified Rota–Baxter 3-Lie algebra structures of non-zero weight, and compare this framework with the deformation-controlling $L_\infty[1]$-algebra introduced by Hou, Sheng, and Zhou.

The second part is devoted to associative algebras. Skew-symmetric solutions of the associative Yang–Baxter equation are closely related to double Lie algebras and cyclic Rota–Baxter algebras, especially for matrix algebras. We extend these correspondences to the homotopy setting by studying pre-Calabi–Yau algebras and homotopy double Poisson algebras arising from homotopy Rota–Baxter structures. We introduce cyclic homotopy Rota–Baxter algebras and construct them via cyclic completion. We further define interactive pairs of differential graded algebras and show that, under a suitable cyclic homotopy Rota–Baxter structure on the acting algebra, the base algebra inherits a pre-Calabi–Yau structure, hence a homotopy double Poisson structure. In particular, we show that any differential graded module over a differential graded algebra endowed with an ultracyclic (resp. cyclic) homotopy Rota–Baxter structure naturally carries a (resp. cyclic) homotopy double Lie algebra structure.

The third part studies L-algebras, motivated by set-theoretic solutions of the Yang–Baxter equation and classical logic. We characterize ideals in semidirect products of L-algebras and describe their prime spectra. We also construct a family of finite simple L-algebras, prove that every simple linear L-algebra belongs to this family, and apply these results to linear Hilbert algebras and their symmetric semidirect products.

Samenvatting

De (kwantum) Yang–Baxter-vergelijking is een fundamentele vergelijking in de mathematische fysica, met oorsprong in de kwantummechanica en de statistische mechanica. Dit proefschrift bestaat uit drie delen en behandelt verschillende algebraïsche structuren die verband houden met haar oplossingen, met bijzondere nadruk op Rota–Baxter-structuren op 3-Lie-algebra's en associatieve algebra's, evenals op L-algebra's.

Het eerste deel is gewijd aan 3-Lie-algebra's, die een natuurlijk kader vormen voor hogere-orde veralgemeningen van Lie-algebra's en belangrijke toepassingen hebben in de mathematische fysica. In deze context zijn oplossingen van de klassieke 3-Lie Yang–Baxter-vergelijking nauw verbonden met Rota–Baxter-structuren. Wij ontwikkelen de representatietheorie, cohomologietheorie en formele vervormingstheorie van Rota–Baxter- en gemodificeerde Rota–Baxter-3-Lie-algebra's van willekeurig gewicht. Daarnaast construeren wij twee $L_\infty[1]$-algebra's waarvan de Maurer–Cartan-elementen overeenkomen met relatieve en absolute gemodificeerde Rota–Baxter-3-Lie-structuren van niet-nulgewicht.

Het tweede deel behandelt associatieve algebra's. Scheefsymmetrische oplossingen van de associatieve Yang–Baxter-vergelijking zijn nauw verbonden met dubbele Lie-algebra's en cyclische Rota–Baxter-algebra's. Wij breiden deze verbanden uit naar de homotopiecontext via pre-Calabi–Yau-algebra's en homotopische dubbele Poisson-algebra's die voortkomen uit homotopische Rota–Baxter-structuren. Verder introduceren wij cyclische homotopische Rota–Baxter-algebra's, interactieve paren van differentiaalgegradeerde algebra's, en laten wij zien dat de basisalgebra onder geschikte voorwaarden op natuurlijke wijze een pre-Calabi–Yau-structuur, en dus ook een homotopische dubbele Poisson-structuur, erft. In het bijzonder verkrijgt elke differentiaalgegradeerde module over een geschikte algebra een homotopische dubbele Lie-algebrastructuur.

Het derde deel bestudeert L-algebra's, gemotiveerd door verzamelingstheoretische oplossingen van de Yang–Baxter vergelijking en de klassieke logica. Wij karakteriseren idealen in semidirecte producten van L-algebra's en beschrijven hun priemspectra. Bovendien construeren wij een familie van eindige eenvoudige L-algebra's, bewijzen wij dat elke eenvoudige lineaire L-algebra tot deze familie behoort, en passen wij deze resultaten toe op lineaire Hilbert-algebra's en hun symmetrische semidirecte producten.

Acknowledgement

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With my deepest appreciation, I dedicate these acknowledgements to all those who have helped, supported, and accompanied me throughout this journey.

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List of symbols

Below, the list of symbols that has been used throughout this thesis, can be found. This list is made per chapter to easily find the meaning for each symbol.

General

Symbol	**Description**
$\mathbb{N}$	The set of non-negative integers.
$\mathbb{N}_{\geq 1}$	The set of positive integers.
$\mathbb{Z}$	The set of integers.
$\mathrm{Hom}_{\mathbf{k}}(W,V)$	The space of linear maps from the linear space W to the space V over a field $\mathbf{k}$.
$\mathrm{End}_{\mathbf{k}}(V)$	The endomorphism algebra of the (graded) vector space V over a field $\mathbf{k}$.
$\mathfrak{gl}(V)$	The Lie algebra of the (graded) vector space V.
$\mathrm{End}(X)$	The set of maps from the set X to itself.
$\mathfrak{S}_n$	The cyclic group of order n.
$\mathrm{Sh}(i_1,i_2,\ldots,i_r)$	The set of $(i_1,\ldots,i_r)$-shuffles.

Part I

Symbol	**Description**
$\mathscr{C}(\mathfrak{g})$	The center of the 3-Lie algebra $\mathfrak{g}$.
$\mathrm{C}^{\bullet}_{\text{3-Lie}}(\mathfrak{g},M)$	The cochain complex of the 3-Lie algebra $\mathfrak{g}$ with coefficients in M.
$\mathrm{H}^{\bullet}_{\text{3-Lie}}(\mathfrak{g},M)$	The cohomology of the 3-Lie algebra $\mathfrak{g}$ with coefficients in M.
$(\mathfrak{g},[-,-,-],T)$	An absolute Rota–Baxter 3-Lie algebra of weight λ.
$((\mathfrak{h},\{-,-,-\}),(\mathfrak{g},[-,-,-]),\rho,\zeta)$	A relative 3-Lie algebra pair.
$((\mathfrak{h},\{-,-,-\}),(\mathfrak{g},[-,-,-]),\rho,\zeta,R)$	A relative modified Rota–Baxter 3-Lie algebra of weight λ.
$(\mathfrak{g},[-,-,-],R)$	An absolute modified Rota–Baxter 3-Lie algebra of weight λ.
$\mathrm{C}^{\bullet}_{\mathsf{RBO}^{\lambda}}(\mathfrak{g},M)$	The cochain complex of Rota–Baxter operators of weight λ on $\mathfrak{g}$ with coefficients in M.
$\mathrm{H}^{\bullet}_{\mathsf{RBO}^{\lambda}}(\mathfrak{g},M)$	The cohomology of Rota–Baxter operators of weight λ on $\mathfrak{g}$ with coefficients in M.
$C^{\bullet}_{\mathsf{RB3\text{-}Lie}^{\lambda}}(\mathfrak{g},M)$	The cochain complex of Rota–Baxter 3-Lie algebras of weight λ on $\mathfrak{g}$ with coefficients in M.
$\mathrm{H}^{\bullet}_{\mathsf{RB3\text{-}Lie}^{\lambda}}(\mathfrak{g},M)$	The cohomology of Rota–Baxter 3-Lie algebras of weight λ on $\mathfrak{g}$ with coefficients in M.
$\mathrm{C}^{\bullet}_{\mathsf{mRB3\text{-}Lie}^{\lambda}}(\mathfrak{g},M)$	The cochain complex of modified Rota–Baxter 3-Lie algebras of weight λ on $\mathfrak{g}$ with coefficients in M.
$\mathrm{H}^{\bullet}_{\mathsf{mRB3\text{-}Lie}^{\lambda}}(\mathfrak{g},M)$	The cohomology of modified Rota–Baxter 3-Lie algebras of weight λ on $\mathfrak{g}$ with coefficients in M.

Part II

Symbol	**Description**
$\mathrm{Hom}(W,V)$	The space of graded linear maps from the graded space W to V.
$\mathrm{End}_{\mathrm{gr}}(V)$	The endomorphism algebra of the graded vector space V.
$W\otimes V$	The tensor product of (graded) spaces W and V.
$\mathrm{T}(V)$	The tensor algebra generated by the (graded) space V.
$\mathrm{Sym}(V)$	The symmetric algebra generated by the (graded) space V.
$\{\{-,-\}\}$	A double Lie bracket.
$(A,\{m_n\}_{n\geqslant 1},\{T_n\}_{n\geqslant 1})$	A homotopy Rota–Baxter algebra.
$(A,M,\{T_i\}_{i\geqslant 1})$	A homotopy relative Rota–Baxter algebra.
$\rhd$	The left action of A on B in an interactive pair (A,B).
$\blacktriangleright$	The left action of B on A in an interactive pair (A,B).
$\blacktriangleleft$	The right action of B on $A^{\vee}$ induced by "$\blacktriangleright$".
$\big((A,\{T_n\}_{n\geqslant 1}),B\big)$	A homotopy Rota–Baxter interactive pair.

Part III

Symbol	**Description**
$\downarrow x$	The downset $\{y \in X \mid y \leq x\}$.
$\uparrow x$	The upset $\{y \in X \mid y \geq x\}$.
$\mathscr{I}(X)$	The set of ideals of the L-algebra X.
$\mathrm{Spec}(X)$	The prime ideals of $\mathscr{I}(X)$ form a topological space, denoted by $\mathrm{Spec}(X)$.
$X \rtimes_\rho Y$	The semidirect product of the L-algebras X and Y via ρ.
$X \bowtie_\rho Y$	The symmetric semidirect product of CKL-algebras X and Y via ρ.
$S(X)$	The self-similar closure of the L-algebra X.
$\rho\mathscr{I}(X)$	The set of ρ-ideals of X.
$\rho\,\mathrm{Spec}(X)$	The space of ρ-prime ideals.
$\mathbf{A}_n$	A simple linear L-algebra of size n.
LH_n	A linear Hilbert algebra of size n.

Contents

Chapter 1

Introduction

1.1 Introduction

1.1.1 Yang–Baxter equation

The (quantum) Yang–Baxter equation , introduced by Yang [96] and Baxter [12], is one of the fundamental equations in mathematical physics. It appears in various contexts such as integrable systems, statistical mechanics, quantum groups, and quantum information theory. Let V be a vector space over a field $\mathbf{k}$. A linear map $R\colon V\otimes V\to V\otimes V$ is called a *solution of the (quantum) Yang–Baxter equation* if the identity

$$R^{12}R^{13}R^{23}=R^{23}R^{13}R^{12} \tag{1.1}$$

holds in $\mathrm{End}_{\mathbf{k}}(V^{\otimes 3})$. Here R^{ij} denotes the operator on $V^{\otimes 3}$ acting as R on the (i,j)-th tensor factors and as the identity on the remaining factor.

The classical counterpart, the *classical Yang–Baxter equation* (CYBE), originated from inverse scattering theory in the work of Faddeev and his collaborators [34, 35]. Belavin and Drinfeld [14] later initiated a systematic study and partial classification of its solutions. For a Lie algebra $\mathfrak{g}$, an element $r=\sum_i a_i\otimes b_i\in\mathfrak{g}\otimes\mathfrak{g}$ is said to satisfy the CYBE if

$$\mathrm{CYBE}(r)=\left[r^{12},r^{13}\right]+\left[r^{12},r^{23}\right]+\left[r^{13},r^{23}\right]=0, \tag{1.2}$$

where $r^{12}=\sum_i a_i\otimes b_i\otimes 1$, $r^{13}=\sum_i a_i\otimes 1\otimes b_i$, and $r^{23}=\sum_i 1\otimes a_i\otimes b_i$. From the viewpoint of quantization, the matrix CYBE can be viewed as the classical limit of the quantum equation obtained from a quasi-classical asymptotic expansion, see [13, 32].

Motivated by the CYBE, Aguiar [1] introduced the *associative Yang–Baxter equation* (AYBE) as an associative analogue. For an associative algebra A, an element $r=\sum_i a_i\otimes b_i\in A\otimes A$ satisfies the AYBE if

$$\mathrm{AYBE}(r)=r^{13}r^{12}-r^{12}r^{23}+r^{23}r^{13}=0, \tag{1.3}$$

where $r^{12}=\sum_i a_i\otimes b_i\otimes 1$, $r^{13}=\sum_i a_i\otimes 1\otimes b_i$, and $r^{23}=\sum_i 1\otimes a_i\otimes b_i$. A solution r is said to be *skew-symmetric* if $r=-r^{21}$, where $r^{21}=\sum_i b_i\otimes a_i$. Aguiar [3] further clarified the relationship between AYBE and CYBE and provided conditions ensuring that an AYBE solution also satisfies the CYBE. In subsequent work, Schedler [88] extended this theory to the homotopical setting and introduced the notion of the associative Yang–Baxter-infinity equation (see Definition 10.10). More precisely, let A be a *graded* *associative algebra*. A solution of the associative Yang–Baxter-infinity equation is a family of elements $\{r_n \in A^{\otimes n}\}_{n\geqslant 1}$ satisfying, for all $n \geqslant 1$, the identity

$$\sum_{i+j=n+1} (-1)^{(j+1)i} \sum_{\sigma\in\mathfrak{C}_n} \operatorname{sgn}(\sigma)\, r_i^{\sigma(1),\sigma(2),\cdots,\sigma(i)} r_j^{\sigma(i),\sigma(i+1),\sigma(i+2),\cdots,\sigma(n)} = 0. \tag{1.4}$$

In a different direction, Drinfeld [31] proposed to study *set-theoretic solutions* of the Yang–Baxter equation. Here one considers a non-empty set X and a map $r\colon X\times X \to X\times X$ satisfying the braid relation in $\operatorname{End}(X^3)$, namely

$$r^{12}r^{23}r^{12} = r^{23}r^{12}r^{23},$$

where $r^{12} = r\times \operatorname{Id}_X$ and $r^{23} = \operatorname{Id}_X \times r$. For $n\geq 2$ and $1\leq i<j\leq n$, we denote by $r^{ij}\colon X^n\to X^n$ the map acting as r on the (i,j)-th components (in this order) and as the identity on the remaining components. Such an r induces a linear Yang–Baxter solution by extending r linearly on the vector space with basis X.

For a set-theoretic solution (X,r), write $r(x,y) = \big(\lambda_x(y),\rho_y(x)\big)$. It is *left* (resp. *right*) *non-degenerate* if each λ_x (resp. ρ_y) is bijective, and *non-degenerate* if it is both left and right non-degenerate. It is called *unitary* if $r^{21}r = \operatorname{Id}_{X\times X}$. The solution is *finite* if X is finite.

Finally, Rump [80] introduced *cycle sets*, that is, sets X endowed with bijections $\sigma_x : X\to X$ defined by $\sigma_x(y) = x\cdot y$ and satisfying the *cycloid equation*

$$(x\cdot y)\cdot(x\cdot z) = (y\cdot x)\cdot(y\cdot z). \tag{1.5}$$

Moreover, he proved that left non-degenerate unitary solutions are in bijection with cycle sets via $\lambda_x(y) := \sigma_y^{-1}(x)$ and $\rho_y(x) := \lambda_x(y)\cdot y$.

1.1.2 Rota–Baxter operators

Rota–Baxter operators on associative algebras originate in Baxter's 1960 work [11] on fluctuation theory in probability. They were subsequently investigated by Rota [75, 77] and Cartier [17]. Later, Guo and his collaborators [45–47] established a number of foundational results that substantially advanced the subject. Rota–Baxter algebras have since found applications in combinatorics [78], renormalization in quantum field theory [21], multiple zeta values [48], operad theory [6], Hopf algebras [21], and the Yang–Baxter equation [5].

A striking link with Yang–Baxter theory was established by Semenov-Tian-Shansky [89]. Assuming that $(\mathfrak{g},[\cdot,\cdot])$ admits a non-degenerate symmetric invariant bilinear form, Semenov-Tian-Shansky proved that solutions of the CYBE in $\mathfrak{g}$ give rise to Rota–Baxter operators of weight zero. Equivalently, one obtains a linear map $T\colon \mathfrak{g}\to\mathfrak{g}$ satisfying

$$[T(u),T(v)] = T\big([T(u),v]+[u,T(v)]\big), \qquad \forall\, u,v\in\mathfrak{g}.$$

Kupershmidt [64] later showed that skew-symmetric CYBE solutions yield relative Rota–Baxter operators. Moreover, Semenov-Tian-Shansky [89] introduced the *modified Yang–Baxter equation*, which takes the operator form

$$[R(u),R(v)] = R\big([R(u),v]+[u,R(v)]\big) - [u,v], \qquad \forall\, u,v\in\mathfrak{g}.$$

This equation has subsequently been applied to the study of non-commutative generalized Lax pairs, the affine geometry of Lie groups, and $\mathscr{O}$-operators, among other topics, see [7, 15]. Deformation and homotopy theories of Rota–Baxter structures on Lie algebras have also been studied by Tang, Bai, Guo, and Sheng [91], as well as by Lazarev, Sheng, and Tang [65].

The notion of an n-Lie algebra, introduced by Filippov [38], generalizes the classical concept of a Lie algebra and has connections to a broad range of problems in mathematics and mathematical physics. Bai, Guo, Li, and Wu [10] introduced Rota–Baxter operators of arbitrary weight on 3-Lie algebras and showed that such operators can be obtained from Rota–Baxter operators of arbitrary weight on Lie algebras. Furthermore, Bai, Guo, and Sheng [8] introduced the notion of a relative Rota–Baxter operator (also called an $\mathcal{O}$-operator) on a 3-Lie algebra with respect to a representation, in connection with the study of the 3-Lie classical Yang–Baxter equation.

Aguiar [2] obtained an analogue of the Semenov-Tian-Shansky correspondence in the associative setting. Specifically, for a solution $r = \sum_i a_i \otimes b_i$ in an associative algebra $(A, \cdot)$, the operator $T : A \to A$,

$$T(x) := \sum_i a_i \cdot x \cdot b_i$$

satisfies the weight-zero Rota–Baxter identity

$$T(u) \cdot T(v) = T\big(T(u) \cdot v + u \cdot T(v)\big), \qquad \forall u, v \in A.$$

In Chapter 8, we will simply refer to $(A, \cdot, T)$ as a Rota–Baxter algebra, see Definition 8.1 for details. For instance, on the polynomial algebra, the indefinite integral operator

$$T(f)(x) := \int_0^x f(t)\,dt$$

is a Rota–Baxter operator of weight zero. Bai [5] then systematically studied operator-form solutions of the Yang–Baxter equations. More recently, Gubarev [44] and Zhang, Gao, and Zheng [98] independently showed that solutions of the associative Yang–Baxter equation in matrix algebras correspond to Rota–Baxter algebra structures on those algebras. Das and Misha [23] studied deformations of relative Rota–Baxter associative algebras and introduced the notion of homotopy relative Rota–Baxter algebras. Building on operadic methods, Wang and Zhou [95] constructed a minimal model of the operad governing Rota–Baxter associative algebras of arbitrary weight, derived the corresponding L_∞-algebra controlling deformations, and introduced homotopy Rota–Baxter algebras of arbitrary weight. In Part II of this thesis, we focus on homotopy Rota–Baxter algebras of weight zero and homotopy relative Rota–Baxter algebras. More precisely, let $(A, \{m_n\}_{n\geqslant 1})$ be an A_∞-algebra and $(M, \{m_{p,q}\}_{p,q\geq 0})$ an A_∞-bimodule over A. A family of operators $\{T_n : M^{\otimes n} \to A\}_{n\geqslant 1}$ of degree $|T_n| = n-1$ is said to define a homotopy relative Rota–Baxter algebra structure if it satisfies the following identity (see Definition 8.9 for explicit definitions of the exponents δ resp. η):

$$\begin{aligned} &\sum_{\substack{l_1+\cdots+l_k=n,\\ l_1,\dots,l_k\geqslant 1}} (-1)^{\delta} m_k \circ \Big(T_{l_1} \otimes \cdots \otimes T_{l_k}\Big) \\ = &\sum_{1\leqslant j\leqslant p} \sum_{\substack{r_1+\cdots+r_p=n,\\ r_1,\dots,r_p\geqslant 1}} (-1)^{\eta} T_{r_1} \circ \Big(\mathrm{Id}^{\otimes i} \otimes m_{j-1,p-j} \circ (T_{r_2} \otimes \cdots \otimes T_{r_j} \otimes \mathrm{Id} \otimes T_{r_{j+1}} \otimes \cdots \otimes T_{r_p}) \otimes \mathrm{Id}^{\otimes k}\Big). \end{aligned}$$

In this case, $(A, M, \{T_n\}_{n\geqslant 1})$ is called a *homotopy relative Rota–Baxter algebra*. In particular, when $M = A$, the pair $(A, \{T_n\}_{n\geqslant 1})$ is called a *homotopy Rota–Baxter algebra*.

The concept of n-Lie algebras was introduced by Filippov [38] as a natural generalization of Lie algebras and has since found important applications in mathematics and mathematical physics. In particular, 3-Lie algebras play a significant role in the study of supersymmetry and gauge symmetry transformations in the world-volume theory of multiple coincident M2-branes [53, 73]. Bai, Guo, Li and Wu [10] introduced Rota–Baxter operators of arbitrary weight on n-Lie algebras, in particular on 3-Lie algebras, and showed that such operators can be induced from Rota–Baxter operators on Lie algebras, pre-Lie algebras, and associative algebras. Moreover, they established inheritance properties of Rota–Baxter 3-Lie algebras. Concretely, a Rota–Baxter operator of weight λ on a 3-Lie algebra $(\mathfrak{g}, [-,-,-])$ is a linear map

$T : \mathfrak{g} \to \mathfrak{g}$ satisfying

$$\begin{aligned}&[T(u),T(v),T(w)] \\ =&T\Big([T(u),T(v),w]+[u,T(v),T(w)]+[T(u),v,T(w)] \\ &+\lambda[T(u),v,w]+\lambda[u,T(v),w]+\lambda[u,v,T(w)]+\lambda^2[u,v,w]\Big),\end{aligned} \tag{1.6}$$

for all $u,v,w \in \mathfrak{g}$.

On the other hand, as a generalization of the classical Yang–Baxter equation to the 3-Lie setting, Bai, Guo and Sheng [8] introduced the 3-Lie classical Yang–Baxter equation together with the notion of relative Rota–Baxter operators (also called $\mathcal{O}$-operators) of weight zero, and established a correspondence between solutions of the 3-Lie classical Yang–Baxter equation and relative Rota–Baxter operators. Subsequently, Hou, Sheng and Zhou [55] introduced relative Rota–Baxter operators of arbitrary weight on 3-Lie algebras, developed their cohomology theory, and constructed an L_∞-algebra controlling their deformations. More precisely, a relative Rota–Baxter operator of weight λ is a linear map $R : \mathfrak{h} \to \mathfrak{g}$ satisfying

$$[R(u),R(v),R(w)] = R\Big(\rho(R(u),R(v))w+\rho(R(v),R(w))u+\rho(R(w),R(u))v+\lambda\{u,v,w\}\Big), \tag{1.7}$$

for all $u,v,w \in \mathfrak{h}$, where ρ denotes an action of $\mathfrak{g}$ on $\mathfrak{h}$ (see Definition 2.5).

It is worth noting that when $\mathfrak{g} = \mathfrak{h}$, the above two notions of Rota–Baxter operators of weight λ on 3-Lie algebras are not equivalent. Understanding the relationship between these two definitions is therefore a problem of independent interest. In this direction, for the first definition (1.6) (see also [10]), we developed a cohomology theory and showed that low-dimensional cohomology groups control formal deformations and abelian extensions [50]. For the second definition (1.7) (see [55]), we introduced modified Rota–Baxter operators of weight λ on 3-Lie algebras, studied their cohomology theory, and constructed the corresponding L_∞-algebra governing deformations [51].

From an operadic viewpoint, it would be desirable to show that the homotopy differential graded operad obtained in this way provides a minimal model of the original operad. However, whether the operad governing 3-Lie algebras is Koszul remains an open problem. Consequently, the operadic validity of Rota–Baxter structures on 3-Lie algebras, especially for non-zero weight, remains a challenging problem for future investigation.

1.1.3 Double Possion and pre-Calabi-Yau algebras

The notion of double Poisson algebras was introduced by Van den Bergh, who used this structure to establish a foundational framework for noncommutative Poisson geometry [92]. A *double Poisson algebra* is an associative algebra A equipped with a double bracket

$$\{\{-,-\}\} : A \otimes A \to A \otimes A$$

satisfying double skew-symmetry, the double Jacobi identity, and the double Leibniz rule (see Definition 8.2 for details). He showed that the representation schemes of such algebras naturally inherit a classical Poisson structure:

Proposition 1.1 ([92, Proposition 1.2]). *Let $(A,\{\{-,-\}\})$ be a double Poisson algebra. Then the coordinate ring*

$$\mathcal{O}(\mathrm{Rep}(A,n))$$

carries a Poisson algebra structure with bracket

$$\{a_{ij},b_{uv}\} = \{\{a,b\}\}'_{uj}\,\{\{a,b\}\}''_{iv}.$$

Here, by convention, any element $x \in A \otimes A$ is written as $x' \otimes x''$, omitting the summation sign.

From this perspective, double Poisson algebras provide a natural and robust framework for describing noncommutative Poisson geometry, in accordance with the Kontsevich–Rosenberg principle.

Along a different line of development, pre-Calabi–Yau algebras, which can be viewed as extensions of classical Calabi–Yau algebras to the setting of A_∞-algebras, also serve as a framework for noncommutative Poisson geometry. The notion of pre-Calabi–Yau algebras and, more generally, pre-Calabi–Yau categories was first introduced by Kontsevich and Vlassopoulos in unpublished notes [63], and later further developed in joint work with Takeda [62]. More precisely, if the graded space

$$\partial_{-1}A := A \oplus s^{-1}A^\vee$$

is endowed with a cyclic A_∞-algebra structure containing A as a A_∞-subalgebra, then A is called a pre-Calabi–Yau algebra (see Definition 7.18). Iyudu, Kontsevich, and Vlassopoulos [58] proved that a certain class of pre-Calabi–Yau algebras induces classical Poisson structures on their representation spaces. More generally, Yeung [97] showed that the derived moduli stack of a pre-Calabi–Yau algebra carries a shifted Poisson structure. Therefore, pre-Calabi–Yau algebras also provide a flexible and powerful framework for noncommutative Poisson geometry.

Since both double Poisson algebras and pre-Calabi–Yau algebras serve as frameworks for noncommutative Poisson geometry, there exists an intrinsic relationship between them. Iyudu and Kontsevich first established a bijection between double Poisson algebras and a certain class of special pre-Calabi–Yau algebras in the preprint [57] (later published in [58]). This correspondence was subsequently interpreted in [56] as a natural consequence of higher cyclic Hochschild cohomology. Fernández and Herscovich further generalized this bijection to the differential graded (dg) setting and to homotopy double Poisson algebras [36]. In particular, they proved a bijection between homotopy double Poisson algebras and a class of special pre-Calabi–Yau algebras, called good manageable special pre-Calabi–Yau algebras (see Theorem 10.4 for details):

Theorem 1.2. *[36, Theorem 6.3] Let $A = \oplus_{n\in\mathbb{Z}} A^n$ be a finite dimensional graded space. For a good manageable special pre-Calabi-Yau structure $\{m_n\}_{n\geqslant 1}$ on A, define a family of maps*

$$\{\{\{-,\cdots,-\}\}_n : A^{\otimes n} \to A^{\otimes n}\}_{n\geqslant 1}$$

by

$$(f_1\otimes\cdots\otimes f_n)\left(\{\{a_1,\cdots,a_n\}\}_n\right) = s^{a_1,\cdots,a_n}_{f_1,\cdots,f_n}\zeta_A\left(m_{2n-1}\left(a_n,s^{-1}f_n,\cdots,a_2,s^{-1}f_2,a_1\right),s^{-1}f_1\right) \tag{1.8}$$

for all homogeneous elements $a_1,\cdots,a_n\in A$ and $f_1,\cdots,f_n\in A^\vee$. The family of maps $\{\{\{-,\cdots,-\}\}_n\}_{n\geqslant 1}$, together with the dg algebra structure on A, defines a homotopy double Poisson algebra structure on the graded space A.

Moreover, the assignment

$$\left\{\begin{array}{l}\textit{good manageable special pre-Calabi-Yau}\\ \textit{algebra structures } \{m_n\}_{n\geqslant 1} \textit{ on } A\end{array}\right\} \to \left\{\begin{array}{l}\textit{homotopy double Poisson algebra}\\ \textit{structures } \{\{\{-,\cdots,-\}\}_n\}_{n\geqslant 1} \textit{ on } A\end{array}\right\}$$

defined by (1.8) *is a bijection.*

Moreover, they showed that double quasi-Poisson algebras can also be interpreted as pre-Calabi–Yau algebras [37]. More recently, Leray and Vallette established an equivalence between curved pre-Calabi–Yau algebras and curved double Poisson algebras by proving that the differential graded Lie algebras controlling their deformation theories are quasi-isomorphic [69].

The notions of homotopy double Lie algebras and homotopy double Poisson algebras were introduced by Schedler [88]. A *homotopy double Poisson algebra* is a graded associative algebra A equipped with a family of homogeneous maps

$$\left\{\{\{-,\cdots,-\}\}_n : A^{\otimes n} \to A^{\otimes n}\right\}_{n\geq 1},$$

which are skew-symmetric and satisfy the double Jacobi$_\infty$ identities together with the double Leibniz$_\infty$ rules (see Definition 10.3). In the same work, Schedler introduced the infinity associative Yang–Baxter equation (see Equation (1.4)) and investigated its relation to homotopy double Poisson structures. In particular, he showed that skew-symmetric solutions of this equation correspond bijectively to homotopy double Lie algebras, which can be viewed as homotopy double Poisson algebras without the associative multiplication (see Definition 10.1).

Leray introduced the notion of protoperads, generalizing operads, and proved that the properad governing double Lie algebras is Koszul [67, 68]. Building on this, he further showed that the properad governing double Poisson algebras is also Koszul, which leads to a minimal model for the properad of double Poisson algebras and provides a conceptual extension of Schedler's homotopy double Poisson structures.

1.1.4 L-algebras

Rump [81] introduced the notion of L-algebras as a unified algebraic framework arising from the study of the Yang–Baxter equation [33, 80, 85], lattice-ordered groups [4, 22], and algebraic logic [86]. In particular, L-algebras generalize various logical algebraic structures, including Brouwerian semilattices [60], MV-algebras [19, 20, 41], orthomodular lattices [71], Hilbert algebras [27, 28, 54], and Glivenko algebras [83]. These structures collectively capture the algebraic semantics of classical propositional logic, residuation theory, and major non-classical logics [43].

The notion of L-algebras provides a conceptual bridge between cycle sets, logical algebras, and group theory. An L-algebra is a set X equipped with a binary operation $(x,y) \mapsto x \cdot y$ and a logical unit 1, satisfying

$$1 \cdot x = x, \qquad x \cdot 1 = x \cdot x = 1, \quad \forall x \in X,$$

together with the cycloid equation (1.5) and the order condition

$$x \cdot y = y \cdot x = 1 \implies x = y.$$

If an L-algebra further satisfies the weak commutativity condition

$$x \cdot (y \cdot z) = y \cdot (x \cdot z), \quad \forall x, y, z \in X,$$

then it is called a CKL-algebra. If instead it satisfies

$$x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z), \quad \forall x, y, z \in X,$$

then X is called a Hilbert algebra (see Definition 11.1). If only the unit and the cycloid condition are required, then X is called a *unital cycloid* [81].

Similarly to the theory of set-theoretic solutions of the Yang–Baxter equation [31], L-algebras admit a structure group theory. The *structure group* $G(X)$ of an L-algebra X is defined as the quotient of its self-similar closure $S(X)$ (see Definition 11.17), where $S(X)$ itself is a quotient of the free monoid $M(X)$ generated by X (see Theorem 11.19). The group $G(X)$ naturally belongs to the class of *right ℓ-groups* [84], which are lattice-ordered groups invariant under right multiplication. This class includes, in particular, Artin braid groups [16, 26] and Garside groups [24, 25]. Moreover, it was shown in [84] that Noetherian right ℓ-groups with duality correspond precisely to non-degenerate unitary set-theoretic solutions of the Yang–Baxter equation.

In recent years, significant progress has been made in the study of ideals and structural properties of L-algebras. Rump [82] introduced the notions of operates and semidirect products of unital cycloids. An operate of Y on X is given by a map

$$\rho : Y \to \mathrm{End}(X)$$

satisfying $\rho_1 = \mathrm{Id}$ and

$$\rho_{u\cdot v}\circ\rho_u = \rho_{v\cdot u}\circ\rho_v, \quad \forall u,v\in Y.$$

The corresponding semidirect product is denoted by $X \rtimes_\rho Y$. Analogous constructions can be defined for CKL-algebras and Hilbert algebras, where they are referred to as symmetric semidirect products and denoted by $X \infty_\rho Y$ (see Section 11.2).

From the perspective of lattice theory and topology, Rump and Vendramin [87] proved that the lattice of ideals $\mathscr{I}(X)$ of an L-algebra X is distributive and described the prime spectrum of direct products:

Proposition 1.3 ([87, Proposition 7]). *Let X and Y be* L*-algebras. Then*

$$\mathrm{Spec}(X\times Y)\cong \mathrm{Spec}(X)\sqcup \mathrm{Spec}(Y).$$

In [29], Dietzel, Menchón, and Vendramin studied finite linear L-algebras and their isomorphism classes. They showed that for $n\geq 2$, the number of isomorphism classes of linear L-algebras with n elements coincides with the Bell number of order $n-1$. Here, linear L-algebras are those satisfying a linear order condition.

In [74], we systematically investigate semidirect products, ideals, spectra, and simplicity of L-algebras, finite linear L-algebras, and Hilbert algebras. In particular, we construct a family of simple CKL-algebras $\{\mathbf{A}_n\}_{n\geq 1}$ and prove that all finite simple linear L-algebras belong to this class. Moreover, we introduce a class of CKL-algebras and show that they are all linear. In our recent manuscript [30], we further prove that all finite simple CKL-algebras are linear, and hence are contained in the family $\{\mathbf{A}_n\}_{n\geq 1}$. Consequently, this provides a complete classification of finite simple CKL-algebras.

1.2 Outline

The dissertation is organized into three parts. The first part is devoted to deformation theory and cohomology of Rota–Baxter 3-Lie algebras and modified Rota–Baxter 3-Lie algebras. The second part concerns the relationships among homotopy Rota–Baxter algebras, pre-Calabi-Yau algebras, and homotopy double Poisson algebras. The third part addresses ideals of semidirect products of L-algebras and describes the structure of simple L-algebras.

1.2.1 Part I

The first part is organized as follows: In Chapter 2, we recall basic definitions and notation for 3-Lie algebras and briefly summarize the corresponding cohomology theory.

In Chapter 3, we introduce Rota–Baxter 3-Lie algebras and modified Rota–Baxter 3-Lie algebras, discuss representative examples, and present several constructions of modified Rota–Baxter operators of non-zero weight obtained from constructions of 3-Lie algebras. In Chapter 4, we define a cohomology theory for Rota–Baxter 3-Lie algebras of non-zero weight with coefficients in a suitable representation.

In Chapter 5, we introduce the corresponding cohomology theory for modified Rota–Baxter 3-Lie algebras of non-zero weight. As an application, we show that these cohomology theories control, respectively, the formal deformations of Rota–Baxter 3-Lie algebras and modified Rota–Baxter 3-Lie algebras of non-zero weight.

In Chapter 6, we construct an $L_\infty[1]$-algebra structure on the cochain complex associated with (relative and absolute) modified Rota–Baxter 3-Lie algebras. We prove that relative modified Rota–Baxter 3-Lie algebra structures of weight λ are in one-to-one correspondence with the Maurer–Cartan elements of the constructed $L_\infty[1]$-algebra (see also Theorem 6.2):

Theorem 1.4. *With all the above notations. Suppose there are maps*

$$\pi \in \mathrm{Hom}_{\mathbf{k}}(\wedge^3 \mathfrak{g}, \mathfrak{g}),$$

$$\mu \in \mathrm{Hom}_{\mathbf{k}}(\wedge^3 \mathfrak{h}, \mathfrak{h}),$$

$$\rho \in \mathrm{Hom}_{\mathbf{k}}(\wedge^2 \mathfrak{g} \otimes \mathfrak{h}, \mathfrak{h}),$$

$$\zeta \in \mathrm{Hom}_{\mathbf{k}}(\wedge^2 \mathfrak{h} \otimes \mathfrak{g}, \mathfrak{g}),$$

$$T \in \mathrm{Hom}_{\mathbf{k}}(\mathfrak{h}, \mathfrak{g})$$

and $\delta = \pi + \rho + \lambda(\mu + \zeta)$ with nonzero λ.

Then

$$(\delta[1], T) \in (V_{r\mathrm{Pair}}[1] \oplus \mathfrak{a})^0$$

is a Maurer-Cartan element in $L_\infty[1]$-algebra

$$(V_{r\mathrm{Pair}}[1] \oplus \mathfrak{a}, \{l_k^{m\mathrm{RB}}\}_{k\geq 1})$$

if and only

$$((\mathfrak{h}, \pi), (\mathfrak{g}, \mu), \rho, \zeta, T)$$

is a relative modified Rota–Baxter 3*-Lie algebra structure.*

This shows that the $L_\infty[1]$-algebra we constructed precisely governs the deformation theory of relative modified Rota–Baxter 3-Lie algebras. Under the twisting procedure, this subalgebra yields the $L_\infty[1]$-algebra constructed by Hou, Sheng, and Zhou, which controls the deformations of relative Rota–Baxter operators of weight λ on 3-Lie algebras.

1.2.2 Part II

In the second part. These results reveal a deep interplay among Rota–Baxter algebras, double Poisson algebras, and pre-Calabi-Yau algebras. In this part, we explore these connections within a more general homotopical framework. We introduce the notions of cyclic and ultracyclic homotopy relative Rota–Baxter algebras, in which the homotopy Rota–Baxter structures satisfy certain cyclic invariance conditions (see Section 8.3). These notions generalize the skew-symmetric Rota–Baxter operators studied by Goncharov and Kolesnikov in [42]. To investigate the pre-Calabi-Yau and double Poisson structures arising from homotopy Rota–Baxter algebras, we also define the concept of interactive pairs: pairs of differential graded algebras (A,B), referred to as the acting algebra A and the base algebra B, they are mutually left modules and act on each other in a compatible way (see Definition 9.1). In particular, we consider interactive pairs in which the acting algebra A is equipped with a suitable homotopy relative Rota–Baxter structure. Such pairs will be called homotopy Rota–Baxter interactive pairs. We then show that the base algebra B of a homotopy Rota–Baxter naturally acquires a pre-Calabi-Yau algebra structure. More precisely, we establish the following result (see Theorem 9.9):

Theorem 1.5. *Let $\big((A, \{T_n\}_{n\geqslant 1}), B\big)$ be a homotopy Rota–Baxter interactive pair, where the acting algebra A and the base algebra B are locally finite-dimensional.*

(i) *If each T_n is cyclic, then B admits a good manageable pre-Calabi–Yau algebra structure.*

(ii) *If each T_n is ultracyclic, then B admits a good manageable special pre-Calabi–Yau algebra structure.*

Then, using the correspondence between pre-Calabi-Yau algebras and homotopy double Poisson algebras established by Fernández and Herscovich in [36], we describe the induced homotopy double Poisson structures on base algebras in terms of homotopy Rota–Baxter algebra structures. This description is given in an explicit and streamlined form (see Theorem 10.7):

Theorem 1.6. *Let $\big((A,\{T_n\}_{n\geqslant 1}),B\big)$ be a homotopy Rota–Baxter interactive pair, where the acting algebra A is finite-dimensional and the base algebra B is locally finite-dimensional.*

Define a sequence of maps $\{\{\!\{-,\dots,-\}\!\}_n : B^{\otimes n}\to B^{\otimes n}\}_{n\geqslant 1}$ by setting $\{\!\{-\}\!\}_1 = d_B$, and for all $n\geqslant 1$,

$$\{\!\{-,\dots,-\}\!\}_{n+1} := \Psi^n(\mathrm{Id}_{A^{\otimes n}}),$$

where the map Ψ^n is the composition:

$$\Psi^n : \mathrm{End}(A^{\otimes n}) \cong A^{\otimes n}\otimes (A^\vee)^{\otimes n} \xrightarrow{\mathrm{Id}^{\otimes n}\otimes T_n} A^{\otimes(n+1)} \xrightarrow{\Phi^{\otimes(n+1)}} \mathrm{End}(B)^{\otimes(n+1)} \to \mathrm{End}(B^{\otimes(n+1)}),$$

and $\Phi : A\to \mathrm{End}(B)$ denotes the left A-action on B, i.e., $\Phi(a)(b) := a\rhd b$.

Then,

(i) *If each T_n is cyclic, the collection $\{\{\!\{-,\dots,-\}\!\}_n\}_{n\geqslant 1}$ defines a cyclic homotopy double Poisson algebra structure on B.*

(ii) *If each T_n is ultracyclic, the collection $\{\{\!\{-,\dots,-\}\!\}_n\}_{n\geqslant 1}$ defines a homotopy double Poisson algebra structure on B.*

The second part is organized as follows:

In Chapter 7, we begin by reviewing homotopy Possion algebras, A_∞-algebras and pre-Calabi–Yau structures.

In Chapter 8, we recall the definitions of Rota–Baxter algebras and double Lie algebras, along with their known connections. Building on homotopy framework, we introduce cyclic Rota–Baxter algebras, as well as cyclic and ultracyclic homotopy relative Rota–Baxter algebras. We also present a cyclic completion construction for homotopy Rota–Baxter algebras.

In Chapter 9, we introduce the notion of interactive pairs. We study homotopy Rota–Baxter structures on the acting algebra of such pairs under certain compatibility conditions, leading to the construction of pre-Calabi–Yau structures on the base algebra. This leads to the proof of Theorem 1.5 (see Theorem 9.9). In particular, we prove that a dg module over a dg algebra equipped with a cyclic (resp. ultracyclic) homotopy relative Rota–Baxter structure naturally carries a pre-Calabi—Yau (resp. an ultracyclic pre-Calabi–Yau) algebra structure .

In Chapter 10, we recall the definitions of homotopy double Lie algebras and homotopy double Poisson algebras. We generalize the result of Fernández and Herscovich in Theorem 1.2. Using the constructions from Chapter 9, we prove Theorem 1.6 (see Theorem 10.7). As a special case, we show that a dg module over an ultracyclic homotopy relative Rota–Baxter algebra naturally inherits a homotopy double Lie algebra structure. Moreover, we prove that the symmetric algebra of a homotopy double Lie algebra naturally carries a homotopy Poisson algebra structure. This yields a method for constructing homotopy Poisson structures from representations of dg homotopy Rota–Baxter algebras. As an application, we establish an equivalence between skew-symmetric solutions of the associative Yang–Baxter-infinity equations, ultracyclic homotopy Rota–Baxter algebra structures, a certain class of pre-Calabi—Yau algebras, and homotopy double Lie algebras, thus extending the results of Goncharov and Kolesnikov to the homotopical realm.

1.2.3 Part III

The purpose of the third part is to investigate the ideals of semidirect products of L-algebras, as well as the structure of simple L-algebras.

This part is organized as follows.

In Chapter 11, we recall the basic definitions and fundamental concepts related to L-algebras.

In Chapter 12, we prove that the self-similar closure is compatible with the semidirect product of L-algebras (see also Theorem 12.4):

Theorem 1.7. *Let X and Y be two* L*-algebras, and assume that Y acts on X via ρ. Then*

$$S(X \rtimes_{\rho} Y) = S(X) \rtimes_{\widetilde{\rho}} S(Y).$$

In Chapter 13, we study ideals and prime ideals of semidirect products of L-algebras, as well as symmetric semidirect products of CKL-algebras. We establish a decomposition criterion for ideals in semidirect products (see Theorem 13.8):

Theorem 1.8. *Let X and Y be* L*-algebras such that Y acts on X via ρ. Then K is an ideal of $X \rtimes_{\rho} Y$ if and only if ρ induces an action*

$$\tilde{\rho} : Y/K_Y \longrightarrow \operatorname{End}(X/K_X),$$

such that

$$(X \rtimes_{\rho} Y)\big/(K_X \rtimes_{\rho|_{K_Y}} K_Y) \;\cong\; X/K_X \rtimes_{\tilde{\rho}} Y/K_Y.$$

In this context, we introduce the notions of ρ-ideals, ρ-prime ideals, and the ρ-spectrum $\rho\operatorname{Spec}(X)$. A ρ-ideal is an ideal $I \subseteq X$ satisfying

$$\rho_v(I) \subseteq I, \qquad \forall v \in Y,$$

while a ρ-prime ideal is a prime element in the lattice of ρ-ideals. The ρ-spectrum $\rho\operatorname{Spec}(X)$ is defined as the topological space consisting of all ρ-prime ideals (see Definition 13.12).

Furthermore, we prove that the lattice $\rho\mathscr{I}(X)$ of ρ-ideals is distributive, and that the prime spectrum of $X \rtimes_{\rho} Y$ decomposes as a disjoint union of the ρ-spectrum of X and the spectrum of Y (see Theorem 13.17):

Theorem 1.9. *Let P be an ideal of X and Q an ideal of Y. Then $P \rtimes Q$ is a prime ideal of $X \rtimes_{\rho} Y$ if and only if one of the following holds:*

(i) $P = X$ and Q is a prime ideal of Y;

(ii) P is a ρ-prime ideal of X and $Q = \ker(\rho^P)$.

Moreover,

$$\operatorname{Spec}(X \rtimes_{\rho} Y) \cong \rho\operatorname{Spec}(X) \sqcup \operatorname{Spec}(Y).$$

This result can be regarded as a substantial extension of Proposition 1.3.

As an application, we establish a natural correspondence between ideals of semidirect products and those of symmetric semidirect products of CKL-algebras (see Proposition 13.20):

Proposition 1.10. *Let X and Y be* CKL*-algebras such that Y acts on X via ρ. Let L be an ideal of the symmetric semidirect product $X \rtimes\!\!\!\ltimes_{\rho} Y$. Define*

$$\widetilde{L} = L_X \rtimes_{\rho|_{L_Y}} L_Y \subseteq X \rtimes_{\rho} Y.$$

Then the correspondence

$$L \longmapsto \widetilde{L} \quad \textit{and} \quad K \cap (X \rtimes\!\!\!\ltimes_{\rho} Y) \longleftarrow\!\shortmid K$$

defines a bijection between ideals of $X \rtimes\!\!\!\ltimes_{\rho} Y$ and ideals of $X \rtimes_{\rho} Y$.

In Chapter 14, we introduce a family of simple CKL-algebras $\{\mathbf{A}_n\}_{n\geq 1}$ (see Proposition 14.5) and investigate linear L-algebras using ideal-theoretic methods. We prove that every simple linear L-algebra of size n is isomorphic to $\mathbf{A}_n$ (see Theorem 14.3).

We further introduce the notions of *tail* and *tail*$^+$ L-*algebras*. A tail is defined as the upward closure of minimal elements in an L-algebra, while a tail$^+$ L-algebra is a finite L-algebra X that either admits a tail or contains a subalgebra

$$Y \subseteq Y_0 \subseteq X,$$

such that Y has a tail, $Y_0 \setminus Y$ is a set of a minimal element of Y_0, and $X \setminus Y$ forms a partially ordered set (see Definition 14.10). We then prove that every simple tail$^+$ CKL-algebra is linear (see Theorem 14.15).

In Chapter 15, we study linear Hilbert algebras and their symmetric semidirect products in detail. We prove that a Hilbert algebra is simple if and only if it has at most two elements (see Proposition 15.3). Moreover, we construct a family of linear Hilbert algebras LH_n and show that every linear Hilbert algebra belongs to this class (see Proposition 15.5).

As an application of the previous results, we determine the number of ideals in symmetric semidirect products of Hilbert algebras (see Proposition 15.7). In particular, we compute that the number of ideals in the symmetric semidirect product of an n-element linear Hilbert algebra with a simple Hilbert algebra is $2n-k$ (see Example 15.9).

Part I

Deformations and cohomology theory of Rota–Baxter 3-Lie algebras and modified Rota–Baxter 3-Lie algebras

Chapter 2

Preliminaries

2.1 3-Lie algebras

In this section, we will recall some basic notions and facts about 3-Lie algebras from [10, 38, 55, 70].

Definition 2.1. A 3-*Lie algebra* is a vector space $\mathfrak{g}$ together with a skew-symmetric trilinear map

$$[-,-,-] : \wedge^3\mathfrak{g} \to \mathfrak{g}$$

satisfying the fundamental identity

$$[x,y,[u,v,w]] = [[x,y,u],v,w] + [u,[x,y,v],w] + [u,v,[x,y,w]], \qquad x,y,u,v,w \in \mathfrak{g}.$$

Definition 2.2. Let $(\mathfrak{g},[-,-,-])$ be a 3-Lie algebra and M a vector space. A *representation of* $\mathfrak{g}$ *on* M is a linear map

$$\rho : \wedge^2\mathfrak{g} \to \mathfrak{gl}(M)$$

such that for all $x_1,x_2,x_3,x_4 \in \mathfrak{g}$,

$$[\rho(x_1,x_2),\rho(x_3,x_4)] = \rho([x_1,x_2,x_3],x_4) + \rho(x_3,[x_1,x_2,x_4]), \tag{2.1}$$

$$\rho(x_1,[x_2,x_3,x_4]) = \rho(x_3,x_4)\rho(x_1,x_2) - \rho(x_2,x_4)\rho(x_1,x_3) + \rho(x_2,x_3)\rho(x_1,x_4). \tag{2.2}$$

Definition 2.3. Let $(\mathfrak{g},[-,-,-])$ be a 3-Lie algebra. Define

$$\mathrm{ad} : \wedge^2\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \qquad \mathrm{ad}_{x,y}(z) = [x,y,z].$$

Then ad is a representation of $\mathfrak{g}$ on $\mathfrak{g}$, called the *adjoint representation*.

Definition 2.4. Let $\mathfrak{g}$ be a 3-Lie algebra. The *derived algebra* is

$$\mathfrak{g}^1 = [\mathfrak{g},\mathfrak{g},\mathfrak{g}],$$

and the center is

$$\mathscr{C}(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x,y,z] = 0,\ \forall y,z \in \mathfrak{g}\}.$$

Definition 2.5. Let $(\mathfrak{g},[-,-,-])$ and $(\mathfrak{h},\{-,-,-\})$ be 3-Lie algebras, and $\rho : \wedge^2\mathfrak{g} \to \mathfrak{gl}(\mathfrak{h})$ a representation of $\mathfrak{g}$ on $\mathfrak{h}$. If

$$\rho(x,y)u \in \mathscr{C}(\mathfrak{h}), \qquad \rho(x,y)\{u,v,w\} = 0, \quad \forall x,y \in \mathfrak{g},\ u,v,w \in \mathfrak{h},$$

then ρ is called an *action of* $\mathfrak{g}$ *on* $\mathfrak{h}$.

Let (M,ρ) be a representation of a 3-Lie algebra $\mathfrak{g}$. We define the cochain complex $(\mathrm{C}^\bullet_{\text{3-Lie}}(\mathfrak{g},M),d^\bullet)$ of 3-Lie algebra with coefficients in (M,ρ), that is, let

$$\mathrm{C}^0_{\text{3-Lie}}(\mathfrak{g},M)=M,\qquad \mathrm{C}^n_{\text{3-Lie}}(\mathfrak{g},M)=\mathrm{Hom}_{\mathbf{k}}\big((\wedge^2\mathfrak{g})^{\otimes(n-1)}\wedge\mathfrak{g},\,M\big),\quad n\geq 1.$$

The coboundary operator

$$\partial^n_{\text{3-Lie}}:C^n_{\text{3-Lie}}(\mathfrak{g},M)\to C^{n+1}_{\text{3-Lie}}(\mathfrak{g},M)$$

is given, for $\mathfrak{X}_i=x_i\wedge y_i$, by

$$\begin{aligned}\partial^n_{\text{3-Lie}}f(\mathfrak{X}_1,\dots,\mathfrak{X}_n,x_{n+1})&=(-1)^{n+1}\rho(y_n,x_{n+1})f(\mathfrak{X}_1,\dots,\mathfrak{X}_{n-1},x_n)\\&+(-1)^{n+1}\rho(x_{n+1},x_n)f(\mathfrak{X}_1,\dots,\mathfrak{X}_{n-1},y_n)\\&+\sum_{j=1}^{n}(-1)^{j+1}\rho(x_j,y_j)f(\mathfrak{X}_1,\dots,\widehat{\mathfrak{X}_j},\dots,\mathfrak{X}_n,x_{n+1})\\&+\sum_{j=1}^{n}(-1)^{j}f(\mathfrak{X}_1,\dots,\widehat{\mathfrak{X}_j},\dots,\mathfrak{X}_n,[x_j,y_j,x_{n+1}])\\&+\sum_{1\leq j<k\leq n}(-1)^{j}f(\mathfrak{X}_1,\dots,\widehat{\mathfrak{X}_j},\dots,\mathfrak{X}_{k-1},\\&\qquad [x_j,y_j,x_k]\wedge y_k+x_k\wedge[x_j,y_j,y_k],\mathfrak{X}_{k+1},\dots,\mathfrak{X}_n,x_{n+1}).\end{aligned}$$

The corresponding cohomology is denoted by

$$\mathrm{H}^\bullet_{\text{3-Lie}}(\mathfrak{g},M).$$

When $M=\mathfrak{g}$ with the adjoint representation, we write

$$\mathrm{H}^n_{\text{3-Lie}}(\mathfrak{g})=\mathrm{H}^n_{\text{3-Lie}}(\mathfrak{g},\mathfrak{g}),\qquad n\geq 1.$$

2.2 $L_\infty[1]$-algebras and V-data

Next, we recall the basic notions on graded vector spaces. A (homologically) *graded space* is a $\mathbb{Z}$-indexed family of $\mathbf{k}$-vector spaces $V=\{V_n\}_{n\in\mathbb{Z}}$. Elements of $\bigcup_{n\in\mathbb{Z}}V_n$ are called *homogeneous*, and a homogeneous element v has degree $|v|=n$ if $v\in V_n$.

The *suspension* of a graded space V is the graded space sV, defined by $(sV)_n=V_{n-1}$ for all $n\in\mathbb{Z}$. For $v\in V_{n-1}$, we denote the corresponding element in $(sV)_n$ by sv. The map $s:V\to sV$, given by $v\mapsto sv$, is a graded map of degree 1.

Similarly, the *desuspension* of V, denoted $s^{-1}V$, is defined by $(s^{-1}V)_n=V_{n+1}$. For $v\in V_{n+1}$, the corresponding element in $(s^{-1}V)_n$ is written as $s^{-1}v$. The map $s^{-1}:V\to s^{-1}V$, given by $v\mapsto s^{-1}v$, is a graded map of degree -1.

Let V be a graded vector space. Define the graded symmetric algebra $\mathrm{Sym}(V)$ of V by $\mathrm{Sym}(V)=\mathrm{T}(V)/I$, where the two-sided ideal I is generated by

$$x\otimes y-(-1)^{|x||y|}y\otimes x$$

for all homogeneous elements $x,y\in V$. For homogeneous elements $x_1,\dots,x_n\in V$ and $\sigma\in\mathfrak{S}_n$, the *Koszul sign* $\varepsilon(\sigma;x_1,\dots,x_n)$ is defined by the identity

$$x_1\odot x_2\odot\cdots\odot x_n=\varepsilon(\sigma;x_1,\dots,x_n)x_{\sigma(1)}\odot x_{\sigma(2)}\odot\cdots\odot x_{\sigma(n)}\in\mathrm{Sym}(V),\tag{2.3}$$

where $\odot$ denotes the multiplication in $\mathrm{Sym}(V)$. For instance, $x\odot y=(-1)^{|x||y|}y\odot x$, hence $\varepsilon((1\,2);x,y)=(-1)^{|x||y|}$. We also define

$$\chi(\sigma;x_1,\dots,x_n)=\mathrm{sgn}(\sigma)\,\varepsilon(\sigma;x_1,\dots,x_n), \tag{2.4}$$

where $\mathrm{sgn}(\sigma)$ is the sign of the permutation σ.

Definition 2.6. An $L_\infty[1]$-*algebra* is a pair $(L=\bigoplus_{i\in\mathbb{Z}}L_i,\{l_k\}_{k\geq 1})$ consisting of a graded vector space $L=\bigoplus_{i\in\mathbb{Z}}L_i$ together with a family of degree-1 multilinear maps $l_k:L^{\otimes k}\to L$ $(k\geq 1)$ such that:

(i) (graded symmetry) for any $k\geq 1$, any homogeneous $x_1,\dots,x_k\in L$, and any $\sigma\in\mathfrak{S}_k$,

$$l_k(x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(k)})=\varepsilon(\sigma;x_1,\dots,x_k)\,l_k(x_1\otimes\cdots\otimes x_k);$$

(ii) (higher Jacobi identities) for any $n\geq 1$ and homogeneous $x_1,\dots,x_n\in L$,

$$\sum_{i+j=n+1}\ \sum_{\sigma\in\mathrm{Sh}(i,n-i)}\varepsilon(\sigma;x_1,\dots,x_n)\,l_j\Big(l_i(x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(i)})\otimes x_{\sigma(i+1)}\otimes\cdots\otimes x_{\sigma(n)}\Big)=0,$$

where $\mathrm{Sh}(i,n-i)\subset\mathfrak{S}_n$ denotes the set of $(i,n-i)$-shuffles, i.e., permutations $\sigma\in\mathfrak{S}_n$ such that

$$\sigma(1)<\cdots<\sigma(i)\quad\text{and}\quad\sigma(i+1)<\cdots<\sigma(n).$$

Throughout the paper, all $L_\infty[1]$-algebras are assumed to be weakly filtered, so that the infinite sums appearing below are convergent.

Definition 2.7. Let $(L,\{l_k\}_{k\geq 1})$ be an $L_\infty[1]$-algebra. An element $\alpha\in L_0$ is called a *Maurer–Cartan element* if it satisfies

$$\sum_{k=1}^{\infty}\frac{1}{k!}\,l_k(\alpha\otimes\cdots\otimes\alpha)=0,$$

whenever the series converges.

Proposition 2.8. *[40, Twisting procedure] Let α be a Maurer–Cartan element of an $L_\infty[1]$-algebra $(L,\{l_k\}_{k\geq 1})$. The twisted $L_\infty[1]$-algebra structure on L is given by multilinear maps $l_n^\alpha:L^{\otimes n}\to L$ defined by*

$$l_n^\alpha(x_1\otimes\cdots\otimes x_n)=\sum_{i=0}^{\infty}\frac{1}{i!}\,l_{n+i}\big(\underbrace{\alpha\otimes\cdots\otimes\alpha}_{i\ times}\otimes x_1\otimes\cdots\otimes x_n\big),\qquad\forall x_1,\dots,x_n\in L,$$

whenever the infinite sum converges (or terminates in a finite case). In particular, $\alpha\in L_0$ induces a degree-one differential

$$l_1^\alpha(x)=\sum_{i=0}^{\infty}\frac{1}{i!}\,l_{i+1}\big(\underbrace{\alpha\otimes\cdots\otimes\alpha}_{i\ times}\otimes x\big),$$

which turns (L,l_1^α) into a cochain complex. The corresponding cohomology groups are referred to as the cohomology induced by the Maurer–Cartan element α.

An important class of $L_\infty[1]$-algebras arise from V-datas [94]. Recall that a V-data is a quadrupe $(V,\mathfrak{a},\mathscr{P},\triangle)$ in which

(i) V is a graded Lie algebra (with the graded Lie bracket [,]);

(ii) $\mathfrak{a}\subset V$ is an abelian graded Lie subalgebra;

(iii) $\mathscr{P}:V\to\mathfrak{a}$ is a projection map with the property that $\ker(\mathscr{P})\subset V$ is a graded Lie subalgebra;

(iv) and $\triangle \in \ker(\mathscr{P})^1$ that satisfies $[\triangle, \triangle] = 0$.

The construction of L_∞-subalgebras using V-data has been studied in [23, 39, 72]. We summarize this method in the following simplified theorem.

Theorem 2.1. *Let $(V, \mathfrak{a}, \mathscr{P}, \triangle)$ be a V-data. Suppose $V' \subset V$ is a graded Lie subalgebra that satisfies $[\triangle, V'] \subset V'$. Then the graded vector space $sV' \oplus \mathfrak{a}$ carries an $L_\infty[1]$-algebra structure with the multilinear operations*

$$\begin{aligned} l_1(sx \otimes a) &= \big(-s[\triangle, x],\ \mathscr{P}(x + [\triangle, a])\big), \\ l_2(sx \otimes sy) &= (-1)^{|x|} s[x, y], \\ l_k(sx \otimes a_1 \otimes \cdots \otimes a_{k-1}) &= \mathscr{P}\left[\cdots[[x, a_1], a_2], \ldots, a_{k-1}\right], \quad \textit{for } k \geq 2, \\ l_k(a_1 \otimes \cdots \otimes a_k) &= \mathscr{P}\left[\cdots[[\triangle, a_1], a_2], \ldots, a_k\right], \quad \textit{for } k \geq 2. \end{aligned}$$

Here x, y are homogeneous elements in V' and $a_1, \ldots, a_k$ are homogeneous elements in $\mathfrak{a}$. Up to the permutations of the above entries, all other multilinear operations vanish.

Moreover, let $\iota : V'' \hookrightarrow V'$ be a monomorphism of graded Lie algebras such that $[\triangle, \iota(V'')] \subset \iota(V'')$. Then the graded vector space $sV'' \oplus \mathfrak{a}$ also carries an $L_\infty[1]$-algebra structure with the multilinear operations:

$$\begin{aligned} l'_1(sx \otimes a) &= \big(-s\iota^{-1}[\triangle, \iota(x)],\ \mathscr{P}(\iota(x) + [\triangle, a])\big), \\ l'_2(sx \otimes sy) &= (-1)^{|x|} \iota^{-1} s[\iota(x), \iota(y)], \\ l'_k(sx \otimes a_1 \otimes \cdots \otimes a_{k-1}) &= \mathscr{P}\left[\cdots[[\iota(x), a_1], a_2], \ldots, a_{k-1}\right], \quad \textit{for } k \geq 2, \\ l'_k(a_1 \otimes \cdots \otimes a_k) &= \mathscr{P}\left[\cdots[[\triangle, a_1], a_2], \ldots, a_k\right], \quad \textit{for } k \geq 2. \end{aligned}$$

Here x, y are homogeneous elements in V'' and $a_1, \ldots, a_k$ are homogeneous elements in $\mathfrak{a}$.

In addition, the map ι induces a monomorphism of $L_\infty[1]$-algebras:

$$\tilde{\iota} : \big(V''[1] \oplus \mathfrak{a}, \{l'_k\}_{k \geq 1}\big) \longrightarrow \big(V'[1] \oplus \mathfrak{a}, \{l_{k \geq 1}\big), \quad (f[1], \theta) \mapsto (\iota(f)[1], \theta).$$

Chapter 3

(Modified) Rota–Baxter 3-Lie algebras

3.1 (Modified) Rota–Baxter operators of weight λ

In this section, we introduce the notions of Rota–Baxter 3-Lie algebras of weight λ and relative (modified) Rota–Baxter 3-Lie algebras of weight λ, and give some of their basic properties.

Definition 3.1. [10] Let $(\mathfrak{g},[-,-,-])$ be a 3-Lie algebra and let $T:\mathfrak{g}\to\mathfrak{g}$ be a linear map. We call T an *(absolute) Rota–Baxter operator of weight* λ on $\mathfrak{g}$, or equivalently say that $(\mathfrak{g},[-,-,-],T)$ is a *(absolute) Rota–Baxter* 3*-Lie algebra of weight* λ, if for all $x,y,z\in\mathfrak{g}$,

$$\begin{aligned}[T(x),T(y),T(z)] = T\Big([T(x),T(y),z]+[x,T(y),T(z)]+[T(x),y,T(z)] \\ +\lambda[T(x),y,z]+\lambda[x,T(y),z]+\lambda[x,y,T(z)]+\lambda^2[x,y,z]\Big).\end{aligned} \tag{1.1}$$

Definition 3.2. [55] Let $(\mathfrak{g},[-,-,-])$ and $(\mathfrak{h},\{-,-,-\})$ be 3-Lie algebras. Let $\rho:\wedge^2\mathfrak{g}\to\mathfrak{gl}(\mathfrak{h})$ be an action of a 3-Lie algebra $(\mathfrak{g},[-,-,-])$ on a 3-Lie algebra $(\mathfrak{h},\{-,-,-\})$, and $\zeta:\wedge^2\mathfrak{h}\to\mathfrak{gl}(\mathfrak{g})$ an action of $(\mathfrak{h},\{-,-,-\})$ on $(\mathfrak{g},[-,-,-])$. A linear map $R:\mathfrak{h}\to\mathfrak{g}$ is called a *relative modified Rota–Baxter operator of weight* $\lambda\in\boldsymbol{k}$ from a 3-Lie algebra $\mathfrak{h}$ to a 3-Lie algebra $\mathfrak{g}$ with respect to actions ρ and ζ if $\forall u,v,w\in\mathfrak{h}$,

$$\begin{aligned}[R(u),R(v),R(w)] =& R\Big(\rho(R(u),R(v))w+\rho(R(v),R(w))u+\rho(R(w),R(u))v+\lambda\{u,v,w\}\Big) \\ &-\lambda\zeta(u,v)R(w)-\lambda\zeta(v,w)R(u)-\lambda\zeta(w,u)R(v).\end{aligned} \tag{1.2}$$

In this case, the quadruple $((\mathfrak{h},\{-,-,-\}),(\mathfrak{g},[-,-,-]),\rho,\zeta)$ is called a *relative* 3*-Lie algebra pair*. Moreover, the quintuple $((\mathfrak{h},\{-,-,-\}),(\mathfrak{g},[-,-,-]),\rho,\zeta,R)$ is called a *relative modified Rota–Baxter* 3*-Lie algebra of weight* λ.

Moreover, If

- $\mathfrak{h}=\mathfrak{g}$ and $\rho=\zeta=\mathrm{ad}$, we call $(\mathfrak{g},[-,-,-],R)$ an *absolute modified Rota–Baxter* 3*-Lie algebra of weight* λ, or simply a modified Rota–Baxter 3-Lie algebra of weight λ, or a modified Rota–Baxter 3-Lie algebra;
- $\zeta:\wedge^2\mathfrak{h}\to\mathfrak{gl}(\mathfrak{g})$ vanish, we call the quadruple $((\mathfrak{h},\{-,-,-\}),(\mathfrak{g},[-,-,-]),\rho,R)$ is called a *relative Rota–Baxter* 3*-Lie algebra of weight* λ.

Example 3.3. *Let $(\mathfrak{g},[-,-,-])$ be a 3-Lie algebra. The following statements hold simultaneously for Rota–Baxter operators and modified Rota–Baxter operators (with the corresponding weights).*

(i) The identity map $\mathrm{Id}_{\mathfrak{g}}$ is a modified Rota–Baxter operator of weight 1 and a Rota–Baxter operator of weight -1.

(ii) An operator R is a Rota–Baxter operator if and only if $-R$ is a Rota–Baxter operator. Similarly, R is a modified Rota–Baxter operator if and only if $-R$ is a modified Rota–Baxter operator.

(iii) Let R be a Rota–Baxter operator (respectively, a modified Rota–Baxter operator) on $\mathfrak{g}$ and let $\psi \in \mathfrak{gl}(\mathfrak{g})$ be an automorphism of the 3-Lie algebra $\mathfrak{g}$. Then the conjugate operator

$$\psi^{-1} \circ R \circ \psi$$

is again a Rota–Baxter operator (respectively, a modified Rota–Baxter operator).

Example 3.4. *[55] Let $(\mathfrak{g},[-,-,-]_{\mathfrak{g}})$ be a 4-dimensional 3-Lie algebra with a basis $\{e_1,e_2,e_3,e_4\}$ and the nonzero multiplication is given by*

$$[e_2,e_3,e_4] = e_1.$$

The center of $\mathfrak{g}$ is the subspace generated by $\{e_1\}$. It is obvious that the adjoint representation $\mathrm{ad}: \wedge^2\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is an action of $\mathfrak{g}$ on itself. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ generated by $\{e_3,e_4\}$.

The projection $P: \mathfrak{g} \to \mathfrak{g}$ given by $\begin{cases} P(e_1) = 0, \\ P(e_2) = 0, \\ P(e_3) = e_3, \\ P(e_4) = e_4, \end{cases}$ *is a relative Rota–Baxter operator of weight λ from $\mathfrak{g}$ to $\mathfrak{g}$ with respect to the adjoint action* ad.

Example 3.5. *Let $(\mathfrak{g},[-,-,-])$ be a 3-Lie algebra whose non-zero brackets are given with respect to a basis $\{e_1,e_2,e_3\}$ by*

$$[e_1,e_2,e_3] = e_1.$$

Then $R = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ *is a modified Rota–Baxter operator of weight 1 if and only if*

$$\begin{aligned}[R(e_1),R(e_2),R(e_3)] = R([R(e_1),R(e_2),e_3] + [e_1,R(e_2),R(e_3)] + [R(e_1),e_2,R(e_3)] + [e_1,e_2,e_3]) \\ - [R(e_1),e_2,e_3] - [e_1,R(e_2),e_3] - [e_1,e_2,R(e_3)].\end{aligned}$$

After calculation, we can obtain

$$\begin{aligned}[R(e_1),R(e_2),R(e_3)] &= [a_{11}e_1,a_{22}e_2,a_{33}e_3] + [a_{11}e_1,a_{32}e_3,a_{23}e_2] \\ &\quad + [a_{21}e_2,a_{12}e_1,a_{33}e_3] + [a_{21}e_2,a_{32}e_3,a_{13}e_1] \\ &\quad + [a_{31}e_3,a_{12}e_1,a_{23}e_2] + [a_{31}e_3,a_{22}e_2,a_{13}e_1] \\ &= (a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13})e_1,\end{aligned}$$

on the other hand,

$$\begin{aligned}
&R([R(e_1),R(e_2),e_3]+[e_1,R(e_2),R(e_3)]+[R(e_1),e_2,R(e_3)]+[e_1,e_2,e_3])\\
&\quad-[R(e_1),e_2,e_3]-[e_1,R(e_2),e_3]-[e_1,e_2,R(e_3)]\\
&=R([a_{11}e_1,a_{22}e_2,e_3])+R([a_{21}e_2,a_{12}e_1,e_3])+R([a_{11}e_1,e_2,a_{33}e_3])+R([a_{31}e_3,e_2,a_{13}e_1])\\
&\quad+R([e_1,a_{22}e_2,a_{33}e_3])+R([e_1,a_{32}e_3,a_{23}e_2])+R([e_1,e_2,e_3])\\
&\quad-[a_{11}e_1,e_2,e_3]-[e_1,a_{22}e_2,e_3]-[e_1,e_2,a_{33}e_3]\\
&=\big((a_{11}a_{22}-a_{21}a_{12}+a_{11}a_{33}-a_{31}a_{13}+a_{22}a_{33}-a_{32}a_{23})a_{11}-a_{22}-a_{33}\big)e_1\\
&\quad+(a_{11}a_{22}-a_{21}a_{12}+a_{11}a_{33}-a_{31}a_{13}+a_{22}a_{33}-a_{32}a_{23}+1)a_{21}e_2\\
&\quad+(a_{11}a_{22}-a_{21}a_{12}+a_{11}a_{33}-a_{31}a_{13}+a_{22}a_{33}-a_{32}a_{23}+1)a_{31}e_3.
\end{aligned}$$

Thus, R is a modified Rota–Baxter operator of weight 1 *if and only if*

$$\begin{aligned}
&a_{11}a_{22}a_{33}-a_{11}a_{32}a_{23}-a_{21}a_{12}a_{33}+a_{21}a_{32}a_{13}+a_{31}a_{12}a_{23}-a_{31}a_{22}a_{13}\\
&=(a_{11}a_{22}-a_{21}a_{12}+a_{11}a_{33}-a_{31}a_{13}+a_{22}a_{33}-a_{32}a_{23})a_{11}-a_{22}-a_{33},
\end{aligned}$$

and

$$\begin{aligned}
&(a_{11}a_{22}-a_{21}a_{12}+a_{11}a_{33}-a_{31}a_{13}+a_{22}a_{33}-a_{32}a_{23}+1)a_{21}\\
&=(a_{11}a_{22}-a_{21}a_{12}+a_{11}a_{33}-a_{31}a_{13}+a_{22}a_{33}-a_{32}a_{23}+1)a_{31}\\
&=0.
\end{aligned}$$

In particular, $R_1=\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$ *and* $R_2=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$ *are modified Rota–Baxter operators of weight* 1.

Example 3.6. *Let* $(\mathfrak{g},[-,-,-])$ *be a* 3*-Lie algebra whose non-zero brackets are given with respect to a basis* $\{e_1,e_2,e_3,e_4\}$ *by*

$$[e_2,e_3,e_4]=e_1.$$

Then $R=\begin{pmatrix}a_{11}&a_{12}&a_{13}&a_{14}\\a_{21}&a_{22}&a_{23}&a_{24}\\a_{31}&a_{32}&a_{33}&a_{34}\\a_{41}&a_{42}&a_{43}&a_{44}\end{pmatrix}$ *is a modified Rota–Baxter operator of weight* 1 *if and only if*

$$\begin{aligned}
[R(e_2),R(e_3),R(e_4)]=\;&R([R(e_2),R(e_3),e_4]+[e_2,R(e_3),R(e_4)]+[R(e_2),e_3,R(e_4)]+[e_2,e_3,e_4])\\
&-[R(e_2),e_3,e_4]-[e_2,R(e_3),e_4]-[e_2,e_3,R(e_4)].
\end{aligned}$$

After calculation, we can obtain

$$\begin{aligned}
[R(e_2),R(e_3),R(e_4)]&=[a_{22}e_2,a_{33}e_3,a_{44}e_4]+[a_{22}e_2,a_{43}e_4,a_{34}e_3]+[a_{32}e_3,a_{23}e_2,a_{44}e_4]\\
&\quad+[a_{32}e_3,a_{43}e_4,a_{24}e_2]+[a_{42}e_4,a_{23}e_2,a_{34}e_3]+[a_{42}e_4,a_{33}e_3,a_{24}e_2]\\
&=(a_{22}a_{33}a_{44}-a_{22}a_{43}a_{34}-a_{32}a_{23}a_{44}+a_{32}a_{43}a_{24}+a_{42}a_{23}a_{34}-a_{42}a_{33}a_{24})e_1,
\end{aligned}$$

on the other hand,

$$\begin{aligned}
&R([R(e_2),R(e_3),e_4]+[e_2,R(e_3),R(e_4)]+[R(e_2),e_3,R(e_4)]+[e_2,e_3,e_4])\\
&\quad-[R(e_2),e_3,e_4]-[e_2,R(e_3),e_4]-[e_2,e_3,R(e_4)]\\
&=R([a_{22}e_2,a_{33}e_3,e_4])+R([a_{32}e_3,a_{23}e_2,e_4])+R([e_2,a_{33}e_3,a_{44}e_4])\\
&\quad+R([e_2,a_{43}e_4,a_{34}e_3])+R([a_{22}e_2,e_3,a_{44}e_4])+R([a_{42}e_4,e_3,a_{24}e_2])\\
&\quad+R(e_1)-a_{22}e_1-a_{33}e_1-a_{44}e_1\\
&=(a_{22}a_{33}-a_{32}a_{23}+a_{33}a_{44}-a_{43}a_{34}+a_{22}a_{44}-a_{42}a_{24}+1)R(e_1)-a_{22}e_1-a_{33}e_1-a_{44}e_1\\
&=((a_{22}a_{33}-a_{32}a_{23}+a_{33}a_{44}-a_{43}a_{34}+a_{22}a_{44}-a_{42}a_{24}+1)a_{11}-a_{22}-a_{33}-a_{44})e_1\\
&\quad+(a_{22}a_{33}-a_{32}a_{23}+a_{33}a_{44}-a_{43}a_{34}+a_{22}a_{44}-a_{42}a_{24}+1)a_{21}e_2\\
&\quad+(a_{22}a_{33}-a_{32}a_{23}+a_{33}a_{44}-a_{43}a_{34}+a_{22}a_{44}-a_{42}a_{24}+1)a_{31}e_3\\
&\quad+(a_{22}a_{33}-a_{32}a_{23}+a_{33}a_{44}-a_{43}a_{34}+a_{22}a_{44}-a_{42}a_{24}+1)a_{41}e_4.
\end{aligned}$$

Thus, R is a modified Rota–Baxter operator of weight 1 *if and only if*

$$\begin{aligned}
&a_{22}a_{33}a_{44}-a_{22}a_{43}a_{34}-a_{32}a_{23}a_{44}+a_{32}a_{43}a_{24}+a_{42}a_{23}a_{34}-a_{42}a_{33}a_{24}\\
&=(a_{22}a_{33}-a_{32}a_{23}+a_{33}a_{44}-a_{43}a_{34}+a_{22}a_{44}-a_{42}a_{24}+1)a_{11}-a_{22}-a_{33}-a_{44},
\end{aligned}$$

and

$$\begin{aligned}
&(a_{22}a_{33}-a_{32}a_{23}+a_{33}a_{44}-a_{43}a_{34}+a_{22}a_{44}-a_{42}a_{24}+1)a_{21}\\
&=(a_{22}a_{33}-a_{32}a_{23}+a_{33}a_{44}-a_{43}a_{34}+a_{22}a_{44}-a_{42}a_{24}+1)a_{31}\\
&=(a_{22}a_{33}-a_{32}a_{23}+a_{33}a_{44}-a_{43}a_{34}+a_{22}a_{44}-a_{42}a_{24}+1)a_{41}=0.
\end{aligned}$$

In particular, $R_1=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$ *and* $R_2=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}$ *are modified Rota–Baxter operators of weight* 1.

Proposition 3.7. *Let* $(\mathfrak{g},[-,-,-])$ *be a* 3*-Lie algebra. Then* R *is a Rota–Baxter operator of weight* λ *if and only if* $2R+\lambda\,\mathrm{id}$ *is a modified Rota–Baxter operator of weight* λ^2.

Proof. For any $x,y,z\in\mathfrak{g}$, we have

$$\begin{aligned}
&[(2R+\lambda\ \mathrm{id})(x),(2R+\lambda\ \mathrm{id})(y),(2R+\lambda\ \mathrm{id})(z)]\\
&=[2Rx+\lambda x,2Ry+\lambda y,2Rz+\lambda z]\\
&=8[Rx,Ry,Rz]+4\lambda[Rx,Ry,z]+4\lambda[R(x),y,Rz]+4\lambda[x,Ry,Rz]\\
&\quad+2\lambda^2[x,y,Rz]+2\lambda^2[x,Ry,z]+2\lambda^2[Rx,y,z]+\lambda^3[x,y,z]\\
&=8R\Big([Rx,Ry,z]+[Rx,y,Rz]+[x,Ry,Rz]+\lambda[Rx,y,z]+\lambda[x,Ry,z]+\lambda[x,y,Rz]\\
&\quad+\lambda^2[x,y,z]\Big)+4\lambda[Rx,Ry,z]+4\lambda[R(x),y,Rz]+4\lambda[x,Ry,Rz]\\
&\quad+2\lambda^2[x,y,Rz]+2\lambda^2[x,Ry,z]+2\lambda^2[Rx,y,z]+\lambda^3[x,y,z]\\
&=(2R+\lambda\ \mathrm{id})\Big([(2R+\lambda\ \mathrm{id})x,(2R+\lambda\ \mathrm{id})y,z]+[x,(2R+\lambda\ \mathrm{id})y,(2R+\lambda\ \mathrm{id})z]\\
&\quad+[(2R+\lambda\ \mathrm{id})x,y,(2R+\lambda\ \mathrm{id})z]+\lambda^2[x,y,z]\Big)-\lambda^2[(2R+\lambda\ \mathrm{id})x,y,z]\\
&\quad-\lambda^2[x,(2R+\lambda\ \mathrm{id})y,z]-\lambda^2[x,y,(2R+\lambda\ \mathrm{id})z].
\end{aligned}$$

And the proof is finished. □

3.2 The constructions of modified Rota–Baxter operators of weight λ

In [10], Bai, Guo, Li, and Wu. constructed Rota–Baxter 3-Lie algebras from Rota–Baxter Lie algebras and Rota–Baxter pre-Lie algebras. In this section, inspired by these constructions of 3-Lie algebras, we present several constructions of modified Rota–Baxter operators of weight λ.

Lemma 3.8. *([9]) Let $(\mathfrak{g},[\cdot,\cdot])$ be a Lie algebra and $\mathfrak{g}^*$ be the dual space of $\mathfrak{g}$. Suppose that $f\in\mathfrak{g}^*$ satisfies $f([x,y])=0$ for all $x,y\in\mathfrak{g}$. Then there is a 3-Lie algebra structure on $\mathfrak{g}$ given by*

$$[x,y,z]_f=f(x)[y,z]+f(y)[z,x]+f(z)[x,y],\qquad \forall x,y,z\in\mathfrak{g}.$$

Definition 3.9. A linear map $R:\mathfrak{g}\to\mathfrak{g}$ is called a *modified Rota–Baxter operator of weight $\lambda\in\mathbf{k}$*, or simply a *modified Rota–Baxter operator on a Lie algebra* $(\mathfrak{g},[\cdot,\cdot])$ if R satisfies the following condition

$$[R(x),R(y)]=R\Big([R(x),y]+[x,R(y)]\Big)+\lambda[x,y],\qquad \forall x,y\in\mathfrak{g}.$$

Theorem 3.1. *Let R be a modified Rota–Baxter operator of weight λ on a Lie algebra $(\mathfrak{g},[\cdot,\cdot])$, $f\in\mathfrak{g}^*$ satisfies $f([x,y])=0$ for any $x,y\in\mathfrak{g}$. Then R is a modified Rota–Baxter operator of weight λ on a 3-Lie algebra $(\mathfrak{g},[-,-,-]_f)$ if and only if R satisfies*

$$\begin{aligned}&R\Big(f(x)[Ry,Rz]+f(y)[Rz,Rx]+f(z)[Rx,Ry]+\lambda f(x)[y,z]+\lambda f(y)[z,x]+\lambda f(z)[x,y]\Big)\\&-\lambda f(x)[Ry,z]-\lambda f(x)[y,Rz]-\lambda f(y)[Rz,x]-\lambda f(y)[z,Rx]-\lambda f(z)[Rx,y]-\lambda f(z)[x,Ry]\\&-2\lambda f(Ry)[z,x]-2\lambda f(Ry)[z,x]-2\lambda f(Rz)[x,y]=0.\end{aligned}$$

Proof. For any $x,y,z\in\mathfrak{g}$, we have

$$\begin{aligned}[Rx,Ry,Rz]_f&=f(Rx)[Ry,Rz]+f(Ry)[Rz,Rx]+f(Rz)[Rx,Ry]\\&=R(f(Rx)[Ry,z]+f(Rx)[y,Rz])+R(f(Ry)[Rz,x]+f(Ry)[z,Rx])\\&\quad+R(f(Rz)[Rx,y]+f(Rz)[Rx,y])+\lambda f(Rx)[y,z]+\lambda f(Ry)[z,x]\\&\quad+\lambda f(Rz)[x,y].\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&R([Rx,Ry,z]_f+[x,Ry,Rz]_f+[Rx,y,Rz]_f+\lambda[x,y,z]_f)-\lambda[Rx,y,z]_f-\lambda[x,Ry,z]_f-\lambda[x,y,Rz]_f\\&=R\Big(f(Rx)[Ry,z]+f(Ry)[z,Rx]+f(z)[Rx,Ry]+f(x)[Ry,Rz]+f(Ry)[Rz,x]+f(Rz)[x,Ry]\\&\quad+f(Rx)[y,Rz]+f(y)[Rz,Rx]+f(Rz)[Rx,y]+\lambda f(x)[y,z]+\lambda f(y)[z,x]+\lambda f(z)[x,y]\Big)\\&\quad-\lambda f(Rx)[y,z]-\lambda f(y)[z,Rx]-\lambda f(z)[Rx,y]-\lambda f(x)[Ry,z]-\lambda f(Ry)[z,x]-\lambda f(z)[x,Ry]\\&\quad-\lambda f(x)[y,Rz]-\lambda f(y)[Rz,x]-\lambda f(Rz)[x,y]\\&=R\Big(f(x)[Ry,Rz]+f(y)[Rz,Rx]+f(z)[Rx,Ry]+\lambda f(x)[y,z]+\lambda f(y)[z,x]+\lambda f(z)[x,y]\Big)\\&\quad-\lambda f(x)[Ry,z]-\lambda f(x)[y,Rz]-\lambda f(y)[Rz,x]-\lambda f(y)[z,Rx]-\lambda f(z)[Rx,y]-\lambda f(z)[x,Ry]\\&\quad+[Rx,Ry,Rz]_f-2\lambda f(Ry)[z,x]-2\lambda f(Rx)[y,z]-2\lambda f(Rz)[x,y].\end{aligned}$$

Then R is a modified Rota–Baxter operator of weight λ on a 3-Lie algebra $(\mathfrak{g},[-,-,-]_f)$ if and only if R satisfies

$$\begin{aligned}&R\Big(f(x)[Ry,Rz]+f(y)[Rz,Rx]+f(z)[Rx,Ry]+\lambda f(x)[y,z]+\lambda f(y)[z,x]+\lambda f(z)[x,y]\Big)\\&-\lambda f(x)[Ry,z]-\lambda f(x)[y,Rz]-\lambda f(y)[Rz,x]-\lambda f(y)[z,Rx]-\lambda f(z)[Rx,y]-\lambda f(z)[x,Ry]\\&-2\lambda f(Ry)[z,x]-2\lambda f(Ry)[z,x]-2\lambda f(Rz)[x,y]=0.\end{aligned}$$

□

Example 3.10. *Let* $(\mathfrak{g},[-,-])$ *be the 3-dimensional Lie algebra given by*

$$[e_1,e_2]=e_2,$$

where $\{e_1,e_2,e_3\}$ *is a basis of* $\mathfrak{g}$. *By Lemma 3.8, the trace function* $f\in\mathfrak{g}^*$, *where* $\begin{cases} f(e_1)=1, \\ f(e_2)=0, \\ f(e_3)=1, \end{cases}$ *induces a 3-Lie algebra* $(\mathfrak{g},[-,-,-]_f)$ *defined with the same basis by*

$$[e_1,e_2,e_3]_f=e_2.$$

Consider a linear map $R:\mathfrak{g}\to\mathfrak{g}$ *defined by* $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ *with respect to the basis* $\{e_1,e_2,e_3\}$. *Define*

$$Re_1=a_{11}e_1+a_{21}e_2+a_{31}e_3,\ \ Re_2=a_{12}e_1+a_{22}e_2+a_{32}e_3,\ \ Re_3=a_{13}e_1+a_{23}e_2+a_{33}e_3.$$

In order to obtain that R is a modified Rota–Baxter operator of weight 1 *on the Lie algebra* $\mathfrak{g}$, *we need*

$$[Re_i,Re_j]=R([Re_i,e_j]+[e_i,Re_j])+[e_i,e_j],\ \ i,j=1,2,3.$$

By a straightforward computation, we conclude that R is a modified Rota–Baxter operator of weight 1 *on the Lie algebra* $\mathfrak{g}$ *if and only if*

$$a_{11}=-a_{22},\quad a_{22}^2+a_{12}a_{21}+1=0.$$

By Theorem 3.1, R is a modified Rota–Baxter operator of weight 1 *on the 3-Lie algebra* $(\mathfrak{g},[-,-,-]_f)$ *if and only if*

$$2a_{33}+2a_{22}a_{32}a_{23}+a_{21}a_{32}a_{13}+a_{31}a_{12}a_{23}-2a_{22}=0,$$

and

$$a_{31}a_{13}+a_{32}a_{23}=2.$$

Definition 3.11. [49] A linear map $R:\mathfrak{g}\to\mathfrak{g}$ is called a modified Rota–Baxter operator of weight λ on a pre-Lie algebra $(\mathfrak{g},*)$ if R satisfies the following condition

$$R(x)*R(y)=R(R(x)*y+x*R(y))+\lambda x*y,\ \ \forall x,y\in\mathfrak{g},$$

The following conclusions can be directly verified by definition, so the proof is omitted.

Lemma 3.12. *Let R be a modified Rota–Baxter operator of weight* λ *on a pre-Lie algebra* $(\mathfrak{g},*)$. *Then R is a modified Rota–Baxter operator of weight* λ *on a Lie algebra* $(\mathfrak{g},[\cdot,\cdot]_*)$, *where* $[\cdot,\cdot]_*$ *is defined by*

$$[x,y]_*=x*y-y*x,\ \ \forall x,y\in\mathfrak{g}.$$

Lemma 3.13. *Let R be a modified Rota–Baxter operator of weight* λ *on a commutative associative algebra* $(\mathfrak{g},\cdot)$, *D be a derivation with* $D\circ R=R\circ D$. *Then R is a modified Rota–Baxter operator of weight* λ *on a pre-Lie algebra* $(\mathfrak{g},*)$, *where*

$$x*y=Dx\cdot y,\ \ \forall x,y\in\mathfrak{g}.$$

According to Theorem 3.1 and Lemma 3.12, we can know

Theorem 3.2. *Let R be a modified Rota–Baxter operator of weight λ on a pre-Lie algebra $(\mathfrak{g},*)$, $f \in \mathfrak{g}^*$ satisfies $f(x*y-y*x)=0$ for any $x,y \in \mathfrak{g}$. Define*

$$[x,y,z]_f = f(x)(y*z-z*y)+f(y)(z*x-x*z)+f(z)(x*y-y*x), \forall x,y,z \in \mathfrak{g},$$

Then R is a modified Rota–Baxter operator of weight λ on a 3-Lie algebra $(\mathfrak{g},[-,-,-]_f)$ if and only if R satisfies

$$\begin{aligned} R\Big(& f(x)(Ry*Rz-Rz*Ry)+f(y)(Rz*Rx-Rx*Rz)+f(z)(Rx*Ry-Ry*Rx)+\lambda f(x)(y*z-z*y) \\ & +\lambda f(y)(z*x-x*z)+\lambda f(z)(x*y-y*x)\Big)-\lambda f(x)(Ry*z-z*Ry) \\ & -\lambda f(x)(y*Rz-Rz*y)-\lambda f(y)(Rz*x-x*Rz)-\lambda f(y)(z*Rx-Rx*z) \\ & -\lambda f(z)(Rx*y-y*Rx)-\lambda f(z)(x*Ry-Ry*x)-2\lambda f(Ry)(z*x-x*z) \\ & -2\lambda f(Rx)(y*z-z*y)-2\lambda f(Rz)(x*y-y*x)=0. \end{aligned}$$

Lemma 3.14. *[10] Let $(\mathfrak{g},\cdot)$ be a commutative associative algebra and D be a derivation and $f \in \mathfrak{g}^*$ satisfies $f(D(x)\cdot y)=f(x\cdot D(y))$ for all $x,y \in \mathfrak{g}$. Then $(\mathfrak{g},\{-,-,-\}_{f,D})$ is a 3-Lie algebra, where the bracket is given*

$$\{x,y,z\}_{f,D} = \begin{vmatrix} f(x) & f(y) & f(z) \\ D(x) & D(y) & D(z) \\ x & y & z \end{vmatrix}, \tag{2.1}$$

for any $x,y,z \in \mathfrak{g}$.

Theorem 3.3. *Let $(\mathfrak{g},\cdot,R)$ be a commutative modified Rota–Baxter algebra of weight λ and D be a derivation with $D\circ R = R\circ D$ and $f \in \mathfrak{g}^*$ satisfies $f(D(x)\cdot y)=f(x\cdot D(y))$. Then R is a modified Rota–Baxter operator weight λ on the 3-Lie algebra $(\mathfrak{g},\{-,-,-\}_{f,D})$ if and only if R satisfies*

$$\begin{aligned} R\Big(& f(x)(DRy\cdot Rz-DRz\cdot Ry)+f(y)(DRz\cdot Rx-DRx\cdot Rz)+f(z)(DRx\cdot Ry-DRy\cdot Rx) \\ & +\lambda f(x)(Dy\cdot z-Dz\cdot y)+\lambda f(y)(Dz\cdot x-Dx\cdot z)+\lambda f(z)(Dx\cdot y-Dy\cdot x)\Big) \\ & -\lambda f(x)(DRy\cdot z-Dz\cdot Ry)-\lambda f(x)(Dy\cdot Rz-DRz\cdot y)-\lambda f(y)(DRz\cdot x-Dx\cdot Rz) \\ & -\lambda f(y)(Dz\cdot Rx-DRx\cdot z)-\lambda f(z)(DRx\cdot y-Dy\cdot Rx)-\lambda f(z)(Dx\cdot Ry-DRy\cdot x) \\ & -2\lambda f(Ry)(Dz\cdot x-Dx\cdot z)-2\lambda f(Rx)(Dy\cdot z-Dz\cdot y)-2\lambda f(Rz)(Dx\cdot y-Dy\cdot x)=0. \end{aligned}$$

Proof. The result follows directly from Lemma 3.12 and Theorem 3.2. □

Chapter 4

Cohomology and deformations of Rota–Baxter 3-Lie algebras

4.1 Representations of Rota–Baxter 3-Lie algebras

In this section, we introduce representations of Rota–Baxter 3-Lie algebras of arbitrary weight. We also provide various examples and new constructions.

Definition 4.1. Let

$$(\mathfrak{g},[-,-,-]_{\mathfrak{g}},T) \text{ and } (\mathfrak{g}',[-,-,-]_{\mathfrak{g}'},T')$$

be two Rota–Baxter 3-Lie algebras of weight λ. A morphism between them is a 3-Lie algebra homomorphism $\phi:\mathfrak{g}\to\mathfrak{g}'$ such that $\phi\circ T = T'\circ\phi$.

Denote by $\mathsf{RB3\text{-}Lie}^{\lambda}$ the category of Rota–Baxter 3-Lie algebras of weight λ with morphisms defined above.

Definition 4.2. Let $(\mathfrak{g},[-,-,-]_{\mathfrak{g}},T)$ be a Rota–Baxter 3-Lie algebra of weight λ and (M,ρ) be a representation over the 3-Lie algebra $(\mathfrak{g},[-,-,-]_{\mathfrak{g}})$. We say that (M,ρ,T_M) is a *representation* over the Rota–Baxter 3-Lie algebra $(\mathfrak{g},[-,-,-]_{\mathfrak{g}},T)$ of weight λ if (M,ρ) is endowed with a linear operator $T_M: M\to M$ such that the following equation

$$\begin{aligned}\rho\big(T(x),T(y)\big)(T_M(m)) \quad = \quad & T_M\Big(\rho(T(x),T(y))(m)+\rho(T(x),y)(T_M(m))+\rho(x,T(y))(T_M(m))\\ &+\lambda\rho(T(x),y)(m)+\lambda\rho(x,T(y))(m)+\lambda\rho(x,y)(T_M(m))\\ &+\lambda^2\rho(x,y)(m)\Big)\end{aligned}$$

holds for any $x,y\in\mathfrak{g}$ and $m\in M$.

Example 4.3. *It is obvious that $(\mathfrak{g},[-,-,-]_{\mathfrak{g}},T)$ itself is a representation over the Rota–Baxter 3-Lie algebra $(\mathfrak{g},[-,-,-]_{\mathfrak{g}},T)$ of weight λ, called the regular representation or the adjoint representation.*

Example 4.4. *Let $(\mathfrak{g},[-,-,-]_{\mathfrak{g}})$ be a 3-Lie algebra and (M,ρ) be a representation over it. Then the triple (M,ρ,Id_M) is a representation of the Rota–Baxter 3-Lie algebra $(\mathfrak{g},[-,-,-]_{\mathfrak{g}},\mathrm{Id}_{\mathfrak{g}})$ of weight -1.*

Example 4.5. *Let $(\mathfrak{g},[-,-,-]_{\mathfrak{g}},T)$ be a Rota–Baxter 3-Lie algebra of weight λ and (M,ρ,T_M) be a representation over it. Then for arbitrary scalar $\mu\in\mathbf{k}$, the triple $(M,\rho,\mu T_M)$ is a representation of the Rota–Baxter 3-Lie algebra $(\mathfrak{g},[-,-,-]_{\mathfrak{g}},\mu T)$ of weight $(\mu\lambda)$.*

Proposition 4.6. *Let $(\mathfrak{g},[-,-,-]_{\mathfrak{g}},T)$ be a Rota–Baxter 3-Lie algebra of weight λ and $\{(M_i,\rho_i,T_{M_i})\}_{i\in I}$ be a family of representation of it. Then the triple $(\oplus_{i\in I}M_i,(\rho_i)_{i\in I},\oplus_{i\in I}T_{M_i})$ is a representation of the Rota–Baxter 3-Lie algebra $(\mathfrak{g},[-,-,-]_{\mathfrak{g}},T)$ of weight λ.*

Proof. For arbitrary $x,y\in\mathfrak{g}$ and $m=(m_i)_{i\in I}\in\oplus_{i\in I}M_i$, we have

$$\begin{aligned}
&(\rho_i)_{i\in I}(T(x),T(y))(\oplus_{i\in I}T_{M_i})(m_i)_{i\in I}\\
=\;&\big(\rho_i(T(x),T(y))(T_{M_i})(m_i)\big)_{i\in I}\\
=\;&\big(T_{M_i}(\rho_i(T(x),T(y))(m)+\rho_i(T(x),y)(T_M(m))+\rho_i(x,T(y))(T_M(m))\\
&+\lambda\ \rho_i(T(x),y)(m)+\lambda\ \rho_i(x,T(y))(m)+\lambda\ \rho_i(x,y)(T_M(m))+\lambda^2\rho_i(x,y)(m))\big)_{i\in I}\\
=\;&(\oplus_{i\in I}T_{M_i})\big((\rho_i)_{i\in I}(T(x),T(y))(m)+(\rho_i)_{i\in I}(T(x),y)(T_M(m))+(\rho_i)_{i\in I}(x,T(y))(T_M(m))\\
&+\lambda\ (\rho_i)_{i\in I}(T(x),y)(m)+\lambda\ (\rho_i)_{i\in I}(x,T(y))(m)+\lambda\ (\rho_i)_{i\in I}(x,y)(T_M(m))\\
&+\lambda^2(\rho_i)_{i\in I}(x,y)(m)\big).
\end{aligned}$$

Hence, $(\oplus_{i\in I}M_i,(\rho_i)_{i\in I},\oplus_{i\in I}T_{M_i})$ is a representation of the Rota–Baxter 3-Lie algebra $(\mathfrak{g},[-,-,-]_{\mathfrak{g}},T)$ of weight λ. □

In the following, we construct the semidirect product in the context of Rota–Baxter 3-Lie algebras of weight λ.

Proposition 4.7. *Let $(\mathfrak{g},[-,-,-]_{\mathfrak{g}},T)$ be a Rota–Baxter 3-Lie algebra of weight λ and (M,ρ,T_M) be a representation of it. Then $(\mathfrak{g}\oplus M,T\oplus T_M)$ is a Rota–Baxter 3-Lie algebra of weight λ, where the Lie bracket on $\mathfrak{g}\oplus M$ is given by the semidirect product*

$$[x+u,y+v,z+w]_{\mathfrak{g}\oplus M}\;=\;[x,y,z]_{\mathfrak{g}}+\rho(x,y)w+\rho(y,z)u+\rho(z,x)v,$$

for arbitrary $x,y,z\in\mathfrak{g}$ and $u,v,w\in M$.

Proof. For arbitrary $x,y,z\in\mathfrak{g}$ and $u,v,w\in M$, we have

$$\begin{aligned}
&[(T\oplus T_M)(x+u),(T\oplus T_M)(y+v),(T\oplus T_M)(z+w)]_{\mathfrak{g}\oplus M}\\
=\;&[T(x),T(y),T(z)]_{\mathfrak{g}}+\rho(T(x),T(y))T_M(w)+\rho(T(y),T(z))T_M(u)+\rho(T(z),T(x))T_M(v)\\
=\;&T\big([T(x),T(y),z]_{\mathfrak{g}}+[T(x),y,T(z)]_{\mathfrak{g}}+[x,T(y),T(z)]_{\mathfrak{g}}+\lambda\ [T(x),y,z]_{\mathfrak{g}}\\
&+\lambda\ [x,T(y),z]_{\mathfrak{g}}+\lambda\ [x,y,T(z)]_{\mathfrak{g}}+\lambda^2[x,y,z]_{\mathfrak{g}}\big)\\
&+T_M\big(\rho(T(x),T(y))w+\rho(T(x),y)T_M(w)+\rho(x,T(y))(T_M(w))+\lambda\ \rho(T(x),y)(w)\\
&+\lambda\ \rho(x,T(y))(w)+\lambda\ \rho(x,y)(T_M(w))+\lambda^2\rho(x,y)(w)\big)\\
&+T_M\big(\rho(T(y),T(z))(u)+\rho(T(y),z)(T_M(u))+\rho(y,T(z))(T_M(u))+\lambda\ \rho(T(y),z)(u)\\
&+\lambda\ \rho(y,T(z))(u)+\lambda\ \rho(y,z)(T_M(u))+\lambda^2\rho(y,z)(u)\big)\\
&+T_M\big(\rho(T(z),T(x))(v)+\rho(T(z),x)T_M(v)+\rho(z,T(x))(T_M(v))\\
&+\lambda\ \rho(T(z),x)(v)+\lambda\ \rho(z,T(x))(v)+\lambda\ \rho(z,x)(T_M(v))+\lambda^2\rho(z,x)(v)\big)\\
=\;&(T\oplus T_M)\big([(T\oplus T_M)(x+u),(T\oplus T_M)(y+v),z+w]_{\mathfrak{g}\oplus M}\\
&+[(T\oplus T_M)(x+u),y+v,(T\oplus T_M)(z+w)]_{\mathfrak{g}\oplus M}+[x+u,(T\oplus T_M)(y+v),(T\oplus T_M)(z+w)]_{\mathfrak{g}\oplus M}\\
&+\lambda\ [(T\oplus T_M)(x+u),y+v,z+w]_{\mathfrak{g}\oplus M}+\lambda\ [x+u,(T\oplus T_M)(y+v),z+w]_{\mathfrak{g}\oplus M}\\
&+\lambda\ [x+u,y+v,(T\oplus T_M)(z+w)]_{\mathfrak{g}\oplus M}+\lambda^2[x+u,y+v,z+w]_{\mathfrak{g}\oplus M}\big).
\end{aligned}$$

This shows that $T\oplus T_M$ is a Rota–Baxter operator of weight λ on the semidirect product 3-Lie algebra. Hence the result follows. □

Proposition 4.8. *[10] Let $(\mathfrak{g},[-,-,-]_{\mathfrak{g}},T)$ be a Rota–Baxter 3-Lie algebra of weight λ. Define a new trinary operation as:*

$$\begin{aligned}[x,y,z]_T: &= [T(x),T(y),z]_{\mathfrak{g}}+[T(x),y,T(z)]_{\mathfrak{g}}+[x,T(y),T(z)]_{\mathfrak{g}}\\ &+\lambda\ [T(x),y,z]_{\mathfrak{g}}+\lambda\ [x,T(y),z]_{\mathfrak{g}}+\lambda\ [x,y,T(z)]_{\mathfrak{g}}+\lambda^2[x,y,z]_{\mathfrak{g}},\end{aligned}$$

for arbitrary $x,y,z\in\mathfrak{g}$. Then

(a) $(\mathfrak{g},[-,-,-]_T)$ is a new 3-Lie algebra, we denote this 3-Lie algebra by $\mathfrak{g}_T$;

(b) the triple $(\mathfrak{g}_T,[-,-,-]_T,T)$ is a Rota–Baxter 3-Lie algebra of weight λ;

(c) the map $T:(\mathfrak{g}_T,[-,-,-]_T,T)\to(\mathfrak{g},[-,-,-]_{\mathfrak{g}},T)$ is a morphism of Rota–Baxter 3-Lie algebras of weight λ.

Proof. (a) On can show it directly by a tedious computation;

(b) We observe that

$$\begin{aligned}&[T(x),T(y),T(z)]_T\\ =\ &[T^2(x),T^2(y),T(z)]_{\mathfrak{g}}+[T^2(x),T(y),T^2(z)]_{\mathfrak{g}}+[T(x),T^2(y),T^2(z)]_{\mathfrak{g}}\\ &+\lambda\ [T^2(x),T(y),T(z)]_{\mathfrak{g}}+\lambda\ [T(x),T^2(y),T(z)]_{\mathfrak{g}}+\lambda\ [T(x),T(y),T^2(z)]_{\mathfrak{g}}+\lambda^2[T(x),T(y),T(z)]_{\mathfrak{g}}\\ =\ &T\Big([T(x),T(y),z]_T+[T(x),y,T(z)]_T+[x,T(y),T(z)]_T\\ &+\lambda\ [T(x),y,z]_T+\lambda\ [x,T(y),z]_T+\lambda\ [x,y,T(z)]_T+\lambda^2[x,y,z]_T\Big),\end{aligned}$$

which shows that T is a Rota–Baxter operator of weight λ on the 3-Lie algebra $\mathfrak{g}_T$.

(c) Since T is a Rota–Baxter operator of weight λ on $\mathfrak{g}$, it follows from Equation (1.1) that

$$T([x,y,z]_T)=[T(x),T(y),T(z)]_{\mathfrak{g}},\ \text{ for }\ x,y,z\in\mathfrak{g}.$$

This implies that $T:(\mathfrak{g}_T,[-,-,-]_T,T)\to(\mathfrak{g},[-,-,-]_{\mathfrak{g}},T)$ is a morphism of Rota–Baxter 3-Lie algebra of weight λ. ☐

In the following, we will introduce new Rota–Baxter representations that will be useful in the next section to construct the cohomology of Rota–Baxter 3-Lie algebras of arbitrary weights.

Theorem 4.9. *Let $(\mathfrak{g},[-,-,-]_{\mathfrak{g}},T)$ be a Rota–Baxter 3-Lie algebra of weight λ and (M,ρ,T_M) be a representation of it. Define a map $\rho_T:\wedge^2\mathfrak{g}\to End(M)$ by*

$$\rho_T(x,y)m:=\rho\big(T(x),T(y)\big)m-T_M\big(\rho(T(x),y)m+\rho(x,T(y))m+\lambda\rho(x,y)m\big),$$

for arbitrary $x,y\in\mathfrak{g},m\in M$. Then ρ_T defines a representation of the 3-Lie algebra $(\mathfrak{g}_T,[-,-,-]_T)$ on M. Moreover, (M,ρ_T,T_M) is a representation of the Rota–Baxter 3-Lie algebra $(\mathfrak{g}_T,[-,-,-]_T,T)$ of weight λ.

Proof. One can show that ρ_T defines a representation of the 3-Lie algebra $(\mathfrak{g}_T,[-,-,-]_T)$ on M directly by a tedious computation. Moreover, we have

$$\begin{aligned}
& \rho_T(T(x),T(y))T_M(m) \\
= \ & \rho(T^2(x),T^2(y))T_M(m) - T_M(\rho(T^2(x),T(y))T_M(m) + \rho(T(x),T^2(y))T_M(m) \\
& +\lambda\rho(T(x),T(y))T_M(m)) \\
= \ & T_M\big(\rho(T^2(x),T^2(y))m + \rho(T^2(x),T(y))T_M(m) + \rho(T(x),T^2(y))T_M(m) \\
& +\lambda\ \rho(T^2(x),T(y))m + \lambda\ \rho(T(x),T^2(y))m + \lambda\ \rho(T(x),T(y))T_M(m) + \lambda^2\rho(T(x),T(y))m\big) \\
& -T_M^2\big(\rho(T^2(x),T(y))m + \rho(T^2(x),y)T_M(m) + \rho(T(x),T(y))T_M(m) + \lambda\ \rho(T^2(x),y)m \\
& +\lambda\ \rho(T(x),T(y))m + \lambda\ \rho(T(x),y)T_M(m) + \lambda^2\rho(T(x),y)m + \rho(T(x),T^2(y))m \\
& +\rho(T(x),T(y))T_M(m) + \rho(x,T^2(y))T_M(m) + \lambda\ \rho(T(x),T(y))m + \lambda\ \rho(x,T^2(y))m \\
& +\lambda\ \rho(x,T(y))T_M(m) + \lambda^2\rho(x,T(y))m + \lambda\rho(T(x),T(y))m + \lambda\rho(T(x),y)T_M(m) \\
& +\lambda\rho(x,T(y))T_M(m) + \lambda^2\rho(T(x),y)m + \lambda^2\rho(x,T(y))m + \lambda^2\rho(x,y)T_M(m) + \lambda^3\rho(x,y)m\big) \\
= \ & T_M\big(\rho_T(T(x),T(y))m + \rho_T(T(x),y)T_M(m) + \rho_T(x,T(y))T_M(m) \\
& +\lambda\ \rho_T(T(x),y)m + \lambda\ \rho_T(x,T(y))m + \lambda\ \rho_T(x,y)T_M(m) + \lambda^2\rho_T(x,y)m\big),
\end{aligned}$$

which shows (M,ρ_T,T_M) is a representation of the Rota–Baxter 3-Lie algebra $(\mathfrak{g}_T,[-,-,-]_T,T)$ of weight λ. □

4.2 Cohomology of Rota–Baxter operators

Firstly, let's introduce the cohomology of Rota–Baxter operators of arbitrary weights.

Let $(\mathfrak{g},[-,-,-]_{\mathfrak{g}},T)$ be a Rota–Baxter 3-Lie algebra of weight λ and (M,ρ,T_M) be a representation over it. Recall that Proposition 4.8 and Proposition 4.9 give a new 3-Lie algebra $\mathfrak{g}_T$ and a new representation M over $\mathfrak{g}_T$. Consider the cochain complex of $\mathfrak{g}_T$ with coefficients in M:

$$\mathrm{C}^{\bullet}_{\text{3-Lie}}(\mathfrak{g}_T,M) = \bigoplus_{n=0}^{\infty}\mathrm{C}^{n}_{\text{3-Lie}}(\mathfrak{g}_T,M).$$

More precisely, for $n \geqslant 1$, $\mathrm{C}^{n}_{\text{3-Lie}}(\mathfrak{g}_T,M) = \mathrm{Hom}_{\mathbf{k}}(\underbrace{\wedge^2\mathfrak{g}\otimes\cdots\otimes\wedge^2\mathfrak{g}}_{n-1}\wedge\mathfrak{g},M)$ and its differential

$$\partial^n : \mathrm{C}^{n}_{\text{3-Lie}}(\mathfrak{g}_T,\ M) \to \mathrm{C}^{n+1}_{\text{3-Lie}}(\mathfrak{g}_T,M)$$

is defined as:

$$
\begin{aligned}
& (\partial^n f)(\mathfrak{X}_1,...\mathfrak{X}_n,x_{n+1}) \\
= & \sum_{1\le j<k\le n}(-1)^j f\big(\mathfrak{X}_1,\cdots,\hat{\mathfrak{X}}_j,\cdots,\mathfrak{X}_{k-1},[x_j,y_j,x_k]_T\wedge y_k+x_k\wedge[x_j,y_j,y_k]_T,\mathfrak{X}_{k+1},\cdots,\mathfrak{X}_n,x_{n+1}\big) \\
& +\sum_{j=1}^n(-1)^j f\big(\mathfrak{X}_1,\cdots,\hat{\mathfrak{X}}_j,\cdots,\mathfrak{X}_n,[x_j,y_j,x_{n+1}]_T\big)+\sum_{j=1}^n(-1)^{j+1}\rho_T(x_j,y_j)f\big(\mathfrak{X}_1,\cdots,\hat{\mathfrak{X}}_j,\cdots,\mathfrak{X}_n,x_{n+1}\big) \\
& +(-1)^{n+1}\Big(\rho_T(y_n,x_{n+1})f\big(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n-1},x_n\big)+\rho_T(x_{n+1},x_n)f\big(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n-1},y_n\big)\Big), \\
= & \sum_{1\le j<k\le n}(-1)^j f(\mathfrak{X}_1,\cdots,\hat{\mathfrak{X}}_j,\cdots,\mathfrak{X}_{k-1},[x_j,y_j,x_k]_T\wedge y_k+x_k\wedge[x_j,y_j,y_k]_T,\mathfrak{X}_{k+1},\cdots,\mathfrak{X}_n,x_{n+1}) \\
& +\sum_{j=1}^n(-1)^j f(\mathfrak{X}_1,\cdots,\hat{\mathfrak{X}}_j,\cdots,\mathfrak{X}_n,[x_j,y_j,x_{n+1}]_T)+\sum_{j=1}^n(-1)^{j+1}\rho(T(x_j),T(y_j))f(\mathfrak{X}_1,\cdots,\hat{\mathfrak{X}}_j,\cdots,\mathfrak{X}_n,x_{n+1}) \\
& -\sum_{j=1}^n(-1)^{j+1}T_M\left(\rho(T(x_j),y_j)f(\mathfrak{X}_1,\cdots,\hat{\mathfrak{X}}_j,\cdots,\mathfrak{X}_n,x_{n+1})\right) \\
& -\sum_{j=1}^n(-1)^{j+1}T_M\left(\rho(x_j,T(y_j))f(\mathfrak{X}_1,\cdots,\hat{\mathfrak{X}}_j,\cdots,\mathfrak{X}_n,x_{n+1})\right) \\
& -\sum_{j=1}^n(-1)^{j+1}\lambda\ T_M\left(\rho(x_j,y_j)f(\mathfrak{X}_1,\cdots,\hat{\mathfrak{X}}_j,\cdots,\mathfrak{X}_n,x_{n+1})\right) \\
& +(-1)^{n+1}\left(\rho(T(y_n),T(x_{n+1}))f(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n-1},x_n)+\rho(T(x_{n+1}),T(x_n))f(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n-1},y_n)\right) \\
& -(-1)^{n+1}T_M\left(\rho(T(y_n),x_{n+1})f(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n-1},x_n)+\rho(T(x_{n+1}),x_n)f(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n-1},y_n)\right) \\
& -(-1)^{n+1}T_M\left(\rho(y_n,T(x_{n+1}))f(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n-1},x_n)+\rho(x_{n+1},T(x_n))f(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n-1},y_n)\right) \\
& -(-1)^{n+1}\lambda\ T_M\left(\rho(y_n,x_{n+1})f(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n-1},x_n)+\rho(x_{n+1},x_n)f(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n-1},y_n)\right)
\end{aligned}
$$

for $\mathfrak{X}_i=x_i\wedge y_i\in\wedge^2\mathfrak{g}, i=1,2\cdots n$ and $x_{n+1}\in\mathfrak{g}$.

Definition 4.10. Let $(\mathfrak{g},[-,-,-]_{\mathfrak{g}},T)$ be a Rota–Baxter 3-Lie algebra of weight λ and (M,ρ,T_M) be a representation over it. Then the cochain complex $(\mathrm{C}^\bullet_{\text{3-Lie}}(\mathfrak{g}_T,M),\partial)$ is called the *cochain complex of Rota–Baxter operator T* of weight λ with coefficients in (M,ρ_T,T_M), denoted by $C^\bullet_{\mathsf{RBO}^\lambda}(\mathfrak{g},M)$. The cohomology of $C^\bullet_{\mathsf{RBO}^\lambda}(\mathfrak{g},M)$, denoted by $\mathrm{H}^\bullet_{\mathsf{RBO}^\lambda}(\mathfrak{g},M)$, are called the *cohomology of Rota–Baxter operator T* of weight λ with coefficients in (M,ρ_T,T_M).

When (M,ρ,T_M) is the regular representation

$$(\mathfrak{g},[-,-,-]_{\mathfrak{g}},T),$$

we denote $\mathrm{C}^\bullet_{\mathsf{RBO}^\lambda}(\mathfrak{g},\mathfrak{g})$ by $\mathrm{C}^\bullet_{\mathsf{RBO}^\lambda}(\mathfrak{g})$ and call it the cochain complex of Rota–Baxter operator of weight λ, and denote $\mathrm{H}^\bullet_{\mathsf{RBO}^\lambda}(\mathfrak{g},\mathfrak{g})$ by $\mathrm{H}^\bullet_{\mathsf{RBO}^\lambda}(\mathfrak{g})$ and call it the cohomology of Rota–Baxter operator T of weight λ.

4.3 Cohomology of Rota–Baxter 3-Lie algebras

In this section, we will combine the cohomology of 3-Lie algebras and the cohomology of Rota–Baxter operators of arbitrary weights to define a cohomology theory for Rota–Baxter 3-Lie algebras of arbitrary weights.

Let $M=(M,\rho,T_M)$ be a representation over a Rota–Baxter 3-Lie algebra $\mathfrak{g}=(\mathfrak{g},\mu=[-,-,-]_{\mathfrak{g}},T)$ of weight λ. Now, let's construct a chain map

$$\Phi^\bullet:\mathrm{C}^\bullet_{\text{3-Lie}}(\mathfrak{g},M)\to C^\bullet_{\mathsf{RBO}^\lambda}(\mathfrak{g},M),$$

i.e., the following commutative diagram:

$$\begin{array}{ccccccc}
C^0_{3\text{-Lie}}(\mathfrak{g},M) & \xrightarrow{\delta^0} & C^1_{3\text{-Lie}}(\mathfrak{g},M) & \cdots\cdots & C^n_{3\text{-Lie}}(\mathfrak{g},M) & \xrightarrow{\delta^n} & C^{n+1}_{3\text{-Lie}}(\mathfrak{g},M) \ \cdots\cdots \\
\downarrow \Phi^0 & & \downarrow \Phi^1 & & \downarrow \Phi^n & & \downarrow \Phi^{n+1} \\
C^0_{\mathsf{RBO}^\lambda}(\mathfrak{g},M) & \xrightarrow{\partial^0} & C^1_{\mathsf{RBO}^\lambda}(\mathfrak{g},M) & \cdots\cdots & C^n_{\mathsf{RBO}^\lambda}(\mathfrak{g},M) & \xrightarrow{\partial^n} & C^{n+1}_{\mathsf{RBO}^\lambda}(\mathfrak{g},M) \ \cdots\cdots .
\end{array}$$

Define $\Phi^0 = \mathrm{Id}_{\mathrm{Hom}_{\mathbf{k}}(\mathbf{k},\mathbf{M})} = \mathrm{Id}_M$, $\Phi^1 = f\circ T - T_M\circ f$ and for $n\geqslant 2$ and $f\in C^n_{3\text{-Lie}}(\mathfrak{g},M)$, define $\Phi^n(f)\in C^n_{\mathsf{RBO}^\lambda}(\mathfrak{g},M)$ as:

$$\begin{aligned}
\Phi^n(f) = & f\circ(T,\cdots,T,T)\circ\left((\mathrm{Id}\wedge\mathrm{Id})^{\otimes n-2}\otimes \mathrm{Id}^{\wedge 3}\right)\\
& -\sum_{k=0}^{2n-2}\lambda^{2n-k-2}\sum_{1\leqslant i_1<i_2<\cdots<i_{k-1}\leqslant 2n-1}\\
& T_M\circ f\circ(\mathrm{Id}^{(i_1-1)},T,\mathrm{Id}^{(i_2-i_1-1)},T,\cdots,T,\mathrm{Id}^{(2n-1-i_{k-1})})\circ\left((\mathrm{Id}\wedge\mathrm{Id})^{\otimes n-2}\otimes\mathrm{Id}^{\wedge 3}\right),
\end{aligned}$$

where the operators

$$\mathrm{Id}^{(i_1-1)},\ T,\ \mathrm{Id}^{(i_2-i_1-1)},\ T,\ \ldots,\ T,\ \mathrm{Id}^{(2n-1-i_k)}$$

act successively on the elements

$$x_1, y_1, \ldots, x_n, y_n, x_{n+1}.$$

Proposition 4.11. *The map $\Phi^\bullet : C^\bullet_{3\text{-Lie}}(\mathfrak{g},M)\to C^\bullet_{\mathsf{RBO}^\lambda}(\mathfrak{g},M)$ is a chain map.*

We leave the long proof of this result to Appendix A.

Definition 4.12. Let $M=(M,\rho,T_M)$ be a representation over a Rota–Baxter 3-Lie algebra $\mathfrak{g}=(\mathfrak{g},\mu=[-,-,-]_{\mathfrak{g}},T)$ of weight λ. We define the cochain complex $(C^\bullet_{\mathsf{RB3\text{-}Lie}^\lambda}(\mathfrak{g},M),d^\bullet)$ of Rota–Baxter 3-Lie algebra $(\mathfrak{g},\mu,T)$ of weight λ with coefficients in (M,ρ,T_M) to the negative shift of the mapping cone of $\Phi^\bullet$, that is, let

$$C^0_{\mathsf{RB3\text{-}Lie}^\lambda}(\mathfrak{g},M)=C^0_{3\text{-Lie}}(\mathfrak{g},M)\quad\text{and}\quad C^n_{\mathsf{RB3\text{-}Lie}^\lambda}(\mathfrak{g},M)=C^n_{3\text{-Lie}}(\mathfrak{g},M)\oplus C^{n-1}_{\mathsf{RBO}^\lambda}(\mathfrak{g},M),\forall n\geqslant 1,$$

and the differential $d^n : C^n_{\mathsf{RB3\text{-}Lie}^\lambda}(\mathfrak{g},M)\to C^{n+1}_{\mathsf{RB3\text{-}Lie}^\lambda}(\mathfrak{g},M)$ is given by

$$d^n(f,g)=(\delta^n(f),-\partial^{n-1}(g)-\Phi^n(f))$$

for arbitrary $f\in C^n_{3\text{-Lie}}(\mathfrak{g},M)$ and $g\in C^{n-1}_{\mathsf{RBO}^\lambda}(\mathfrak{g},M)$. The cohomology of $(C^\bullet_{\mathsf{RB3\text{-}Lie}^\lambda}(\mathfrak{g},M),d^\bullet)$, denoted by $H^\bullet_{\mathsf{RB3\text{-}Lie}^\lambda}(\mathfrak{g},M)$, is called the *cohomology of the Rota–Baxter 3-Lie algebra* $(\mathfrak{g},\mu,T)$ of weight λ with coefficients in (M,ρ,T_M). When $(M,\rho,T_M)=(\mathfrak{g},\mu,T)$, we just denote $C^\bullet_{\mathsf{RB3\text{-}Lie}^\lambda}(\mathfrak{g},\mathfrak{g})$, $H^\bullet_{\mathsf{RB3\text{-}Lie}^\lambda}(\mathfrak{g},\mathfrak{g})$ by $C^\bullet_{\mathsf{RB3\text{-}Lie}^\lambda}(\mathfrak{g})$, $H^\bullet_{\mathsf{RB3\text{-}Lie}^\lambda}(\mathfrak{g})$ respectively, and call them the *cochain complex, the cohomology of Rota–Baxter* 3*-Lie algebra* $(\mathfrak{g},\mu,T)$ of weight λ respectively.

There is an obvious short exact sequence of complexes:

$$0\to s^{-1}C^\bullet_{\mathsf{RBO}^\lambda}(\mathfrak{g},M)\to C^\bullet_{\mathsf{RB3\text{-}Lie}^\lambda}(\mathfrak{g},M)\to C^\bullet_{3\text{-Lie}}(\mathfrak{g},M)\to 0,$$

which induces a long exact sequence of cohomology groups

$$0\to H^0_{\mathsf{RB3\text{-}Lie}^\lambda}(\mathfrak{g},M)\to H^0_{3-\mathrm{Lie}}(\mathfrak{g},M)\to H^0_{\mathsf{RBO}^\lambda}(\mathfrak{g},M)\to H^1_{\mathsf{RB3\text{-}Lie}^\lambda}(\mathfrak{g},M)\to H^1_{3-\mathrm{Lie}}(\mathfrak{g},M)\to\cdots$$

$$\cdots\to H^p_{3-\mathrm{Lie}}(\mathfrak{g},M)\to H^p_{\mathsf{RBO}^\lambda}(\mathfrak{g},M)\to H^{p+1}_{\mathsf{RB3\text{-}Lie}^\lambda}(\mathfrak{g},M)\to H^{p+1}_{3-\mathrm{Lie}}(\mathfrak{g},M)\to\cdots$$

4.4 Formal deformations of Rota–Baxter 3-Lie algebras

In this section, we will study formal deformations of Rota–Baxter 3-Lie algebras of arbitrary weights and interpret them via lower degree cohomology groups of Rota–Baxter 3-Lie algebras defined in last section.

4.4.1 Formal deformations of Rota–Baxter operator with 3-Lie bracket fixed

Let $(\mathfrak{g},\mu,T)$ be a Rota–Baxter 3-Lie algebra of weight λ. Let us consider the case where we only deform the Rota–Baxter operator with the 3-Lie bracket fixed. In this case, $\mathfrak{g}[[t]]=\{\sum_{i=0}^{\infty}a_it^i \mid a_i\in\mathfrak{g},\forall i\geqslant 0\}$ is endowed with the 3-Lie bracket induced from that of $\mathfrak{g}$, say,

$$\mu\Big(\sum_{i=0}^{\infty}a_it^i,\sum_{j=0}^{\infty}b_jt^j,\sum_{k=0}^{\infty}c_kt^k\Big)=\sum_{n=0}^{\infty}\Big(\sum_{\substack{i+j+k=n\\ i,j,k\geqslant 0}}\mu(a_i,b_j,c_k)\Big)t^n.$$

Then $\mathfrak{g}[[t]]$ becomes a 3-Lie algebra over $\mathbf{k}[[t]]$, whose 3-Lie bracket is still denoted by μ.

Consider a 1-parameterized family:

$$T_t=\sum_{i=0}^{\infty}T_it^i,\ T_i\in \mathrm{C}^1_{\mathsf{RBO}^\lambda}(\mathfrak{g}).$$

Definition 4.13. A 1*-parameter formal deformation of the Rota–Baxter operator* T on a 3-Lie algebra $(\mathfrak{g},\mu)$ is a family T_t which is a $\mathbf{k}[[t]]$-linear Rota–Baxter operator on the 3-Lie algebra $\mathfrak{g}[[t]]$ such that $T_0=T$. The operator T_1 is called the *infinitesimal* of the 1-parameter formal deformation $(\mathfrak{g}[[t]],T_t)$ of Rota–Baxter operators on the 3-Lie algebra $(\mathfrak{g},\mu)$.

Power series T_t determine a 1-parameter formal deformation of the Rota–Baxter operator T on a 3-Lie algebra $(\mathfrak{g},\mu)$ if and only if the following equation holds:

$$\begin{aligned}
&\mu(T_t(x_1),T_t(x_2),T_t(x_3))\\
=\ &T_t\big(\mu(T_t(x_1),T_t(x_2),x_3)+\mu(T_t(x_1),x_2,T_t(x_3))+\mu(x_1,T_t(x_2),T_t(x_3))\\
&+\lambda\ \mu(T_t(x_1),x_2,x_3)+\lambda\ \mu(x_1,T_t(x_2),x_3)+\lambda\ \mu(x_1,x_2,T_t(x_3))+\lambda^2\mu(x_1,x_2,x_3)\big),
\end{aligned}$$

for arbitrary $x_1,x_2,x_3,x_4,x_5\in\mathfrak{g}$. Expanding this equation and comparing the coefficient of t^n, we obtain that $\{T_i\}_{i\geqslant 0}$ have to satisfy: for arbitrary $n\geqslant 0$,

$$\begin{aligned}
&\sum_{\substack{i+j+k=n\\ i,j,k\geqslant 0}}\mu(T_i(x_1),T_j(x_2),T_k(x_3))\\
=&\sum_{\substack{i+j+k=n\\ i,j,k\geqslant 0}}\big(T_i(\mu(x_1,T_j(x_2),T_k(x_3)))+T_i(\mu(T_j(x_1),x_2,T_k(x_3)))+T_k(\mu(T_i(x_1),T_j(x_2),x_3))\big)\\
&+\lambda\sum_{\substack{i+j=n\\ i,j\geqslant 0}}\big(T_i(\mu(T_j(x_1),x_2,x_3))+T_i(\mu(x_1,T_j(x_2),x_3))+T_i(\mu(x_1,x_2,T_j(x_3)))\big)\\
&+\lambda^2T_n(\mu(x_1,x_2,x_3)).
\end{aligned}\tag{4.1}$$

When $n=0$, Equation (4.1) reduces to the defining identity of a Rota–Baxter operator, namely that for $T=T_0$.

When $n=1$, Equation (4.1) has the form

$$\begin{aligned}
&\mu\left(T_1(x_1),T(x_2),T(x_3)\right)+\mu\left(T(x_1),T_1(x_2),T(x_3)\right)+\mu\left(T(x_1),T(x_2),T_1(x_3)\right)\\
=\;& T\left(\mu(x_1,T_1(x_2),T(x_3))+\mu(x_1,T(x_2),T_1(x_3))\right)\\
&+T\left(\mu(T_1(x_1),x_2,T(x_3))+\mu(T(x_1),x_2,T_1(x_3))\right)\\
&+T\left(\mu(T_1(x_1),T(x_2),x_3)+\mu(T(x_1),T_1(x_2),x_3)\right)\\
&+T_1\left(\mu(x_1,T(x_2),T(x_3))+\mu(T(x_1),x_2,T(x_3))+\mu(T(x_1),T(x_2),x_3)\right)\\
&+\lambda T\left(\mu(T_1(x_1),x_2,x_3)+\mu(x_1,T(x_2),x_3)+\mu(x_1,x_2,T_1(x_3))\right)\\
&+\lambda T_1\left(\mu(x_1,T(x_2),x_3)+\mu(T(x_1),x_2,x_3)+\mu(x_1,x_2,T(x_3))\right)\\
&+\lambda^2 T_1(\mu(x_1,x_2,x_3))
\end{aligned}\tag{4.2}$$

which says exactly that $\partial^1(T_1)=0\in \mathrm{C}^\bullet_{\mathsf{RBO}^\lambda}(\mathfrak{g})$. This proves the following result:

Proposition 4.14. *Let T_t be a 1-parameter formal deformation of Rota–Baxter operator T of weight λ. Then T_1 is a 1-cocycle in the cochain complex $\mathrm{C}^\bullet_{\mathsf{RBO}^\lambda}(\mathfrak{g})$.*

Remark 4.15. *The above result shows that the cochain complex $\mathrm{C}^\bullet_{\mathsf{RBO}^\lambda}(\mathfrak{g})$ controls formal deformations of Rota–Baxter operators, which justifies the name "cochain complex of Rota–Baxter operators". In fact, we were inspired by Equation* (4.2) *while defining $\mathrm{C}^\bullet_{\mathsf{RBO}^\lambda}(\mathfrak{g})$.*

4.4.2 Formal deformations of Rota–Baxter 3-Lie algebras

Let $(\mathfrak{g},\mu,T)$ be a Rota–Baxter 3-Lie algebra of weight λ. Consider a 1-parameterized family:

$$\mu_t=\sum_{i=0}^{\infty}\mu_i t^i,\ \mu_i\in \mathrm{C}^2_{\text{3-Lie}}(\mathfrak{g}),\quad T_t=\sum_{i=0}^{\infty}T_i t^i,\ T_i\in \mathrm{C}^1_{\mathsf{RBO}^\lambda}(\mathfrak{g}).$$

Definition 4.16. A *1-parameter formal deformation of Rota–Baxter* 3*-Lie algebra* $(\mathfrak{g},\mu,T)$ of weight λ is a pair (μ_t,T_t) which endows the flat $\mathbf{k}[[t]]$-module $\mathfrak{g}[[t]]$ with a Rota–Baxter 3-Lie algebra structure of weight λ over $\mathbf{k}[[t]]$ such that $(\mu_0,T_0)=(\mu,T)$. The pair (μ_1,T_1) is called the *infinitesimal* of the 1-parameter formal deformation $(\mathfrak{g}[[t]],\mu_t,T_t)$ of Rota–Baxter 3-Lie algebra $(\mathfrak{g},\mu,T)$ of weight λ.

Power series μ_t and T_t determine a 1-parameter formal deformation of Rota–Baxter 3-Lie algebra $(\mathfrak{g},\mu,T)$ of weight λ if and only if for arbitrary $x_1,x_2,x_3,x_4,x_5\in\mathfrak{g}$, the following equations hold :

$$\begin{aligned}
&\mu_t(x_1,x_2,\mu_t(x_3,x_4,x_5))\\
=\;&\mu_t(\mu_t(x_1,x_2,x_3),x_4,x_5)+\mu_t(x_3,\mu_t(x_1,x_2,x_4),x_5)+\mu_t(x_3,x_4,\mu_t(x_1,x_2,x_5))\\
&\mu_t(T_t(x_1),T_t(x_2),T_t(x_3))\\
=\;&T_t\big(\mu_t(T_t(x_1),T_t(x_2),x_3)+\mu_t(T_t(x_1),x_2,T_t(x_3))+\mu_t(x_1,T_t(x_2),T_t(x_3))\\
&+\lambda\ \mu_t(T_t(x_1),x_2,x_3)+\lambda\ \mu_t(x_1,T_t(x_2),x_3)+\lambda\ \mu_t(x_1,x_2,T_t(x_3))\\
&+\lambda^2\mu_t(x_1,x_2,x_3)\big).
\end{aligned}$$

By expanding these equations and comparing the coefficient of t^n, we obtain that $\{\mu_i\}_{i\geqslant 0}$ and $\{T_i\}_{i\geqslant 0}$ have to satisfy: for arbitrary $n\geqslant 0$,

$$\begin{aligned}
&\sum_{i+j=n}\mu_i(x_1,x_2,\mu_j(x_3,x_4,x_5))\\
=&\sum_{i+j=n}\mu_i(\mu_j(x_1,x_2,x_3),x_4,x_5)+\mu_i(x_3,\mu_j(x_1,x_2,x_4),x_5)+\mu_i(x_3,x_4,\mu_j(x_1,x_2,x_5)),
\end{aligned}\tag{4.3}$$

$$\begin{aligned}
&\sum_{i+j+k+l=n} \mu_i(T_j(x_1),T_k(x_2),T_l(x_3)) \\
= &\sum_{i+j+k+l=n} T_i\big(\mu_j(T_k(x_1),T_l(x_2),x_3)+\mu_j(T_k(x_1),x_2,T_l(x_3))+\mu_j(x_1,T_k(x_2),T_l(x_3))\big) \\
&+\lambda \sum_{i+j+k=n} T_i\big(\mu_j(T_k(x_1),x_2,x_3)+\mu_j(x_1,T_k(x_2),x_3)+\mu_j(x_1,x_2,T_k(x_3))\big) \\
&+\lambda^2 \sum_{i+j=n} T_i\big(\mu_j(x_1,x_2,x_3)\big).
\end{aligned} \tag{4.4}$$

When $n=0$, the above conditions reduce precisely to the original 3-Lie bracket $\mu=\mu_0$ together with the defining equation of a Rota–Baxter operator $T=T_0$ of weight λ on $(\mathfrak{g},\mu_0)$.

Proposition 4.17. *Let $(\mathfrak{g}[[t]],\mu_t,T_t)$ be a 1-parameter formal deformation of Rota–Baxter 3-Lie algebra $(\mathfrak{g},\mu,T)$ of weight λ. Then (μ_1,T_1) is a 2-cocycle in the cochain complex $C^{\bullet}_{\mathrm{RB3\text{-}Lie}^{\lambda}}(\mathfrak{g})$.*

Proof. When $n=1$, Equations (4.3) and (4.4) become

$$\begin{aligned}
&[x_1,x_2,\mu_1(x_3,x_4,x_5)]_{\mathfrak{g}}+\mu_1(x_1,x_2,[x_3,x_4,x_5]_{\mathfrak{g}}) \\
= &[\mu_1(x_1,x_2,x_3),x_4,x_5]_{\mathfrak{g}}+[x_3,\mu_1(x_1,x_2,x_4),x_5]_{\mathfrak{g}}+[x_3,x_4,\mu_1(x_1,x_2,x_5)]_{\mathfrak{g}} \\
&+\mu_1([x_1,x_2,x_3]_{\mathfrak{g}},x_4,x_5)+\mu_1(x_3,[x_1,x_2,x_4]_{\mathfrak{g}},x_5)+\mu_1(x_3,x_4,[x_1,x_2,x_5]_{\mathfrak{g}})
\end{aligned}$$

and

$$\begin{aligned}
&\mu_1(T(x_1),T(x_2),T(x_3))-T\big(\mu_1(T(x_1),T(x_2),x_3)-\mu_1(T(x_1),x_2,T(x_3))-\mu_1(x_1,T(x_2),T(x_3))\big) \\
&-\lambda\ T\big(\mu_1(T(x_1),x_2,x_3)+\mu_1(x_1,T(x_2),x_3)+\mu_1(x_1,x_2,T(x_3))\big)-\lambda^2 T\big(\mu_1(x_1,x_2,x_3)\big) \\
= &T_1\big([T(x_1),T(x_2),x_3]_{\mathfrak{g}}+[T(x_1),x_2,T(x_3)]_{\mathfrak{g}}+[x_1,T(x_2),T(x_3)]_{\mathfrak{g}}\big) \\
&+\lambda\ T_1\big([T(x_1),x_2,x_3]_{\mathfrak{g}}+[x_1,T(x_2),x_3]_{\mathfrak{g}}+[x_1,x_2,T(x_3)]_{\mathfrak{g}}\big)+\lambda^2 T_1\big([x_1,x_2,x_3]_{\mathfrak{g}}\big) \\
&-[T_1(x_1),T(x_2),T(x_3)]_{\mathfrak{g}}-[T(x_1),T_1(x_2),T(x_3)]_{\mathfrak{g}}-[T(x_1),T(x_2),T_1(x_3)]_{\mathfrak{g}} \\
&+T\big([T_1(x_1),T(x_2),x_3]_{\mathfrak{g}}+[T_1(x_1),x_2,T(x_3)]_{\mathfrak{g}}+[x_1,T_1(x_2),T(x_3)]_{\mathfrak{g}}\big) \\
&+T\big([T(x_1),T_1(x_2),x_3]_{\mathfrak{g}}+[T(x_1),x_2,T_1(x_3)]_{\mathfrak{g}}+[x_1,T(x_2),T_1(x_3)]_{\mathfrak{g}}\big) \\
&+\lambda\ T\big([T_1(x_1),x_2,x_3]_{\mathfrak{g}}+[x_1,T_1(x_2),x_3]_{\mathfrak{g}}+[x_1,x_2,T_1(x_3)]_{\mathfrak{g}}\big).
\end{aligned} \tag{4.5}$$

Note that the first equation is exactly $\delta^2(\mu_1)=0\in \mathrm{C}^{\bullet}_{3\text{-Lie}}(\mathfrak{g})$ and that second equation is exactly to

$$\Phi^2(\mu_1)=-\partial^1(T_1)\in \mathrm{C}^{\bullet}_{\mathrm{RBO}^{\lambda}}(\mathfrak{g}).$$

So (μ_1,T_1) is a 2-cocycle in $\mathrm{C}^{\bullet}_{\mathrm{RB3\text{-}Lie}^{\lambda}}(\mathfrak{g})$. ☐

Remark 4.18. *Equation* (4.5) *inspired us to introduce the chain map $\Phi^{\bullet}:\mathrm{C}^{\bullet}_{3\text{-Lie}}(\mathfrak{g},M)\to C^{\bullet}_{\mathrm{RBO}^{\lambda}}(\mathfrak{g},M)$ as well the cochain complex $\mathrm{C}^{\bullet}_{\mathrm{RB3\text{-}Lie}^{\lambda}}(\mathfrak{g},M)$.*

Definition 4.19. Let $(\mathfrak{g}[[t]],\mu_t,T_t)$ and $(\mathfrak{g}[[t]],\mu'_t,T'_t)$ be two 1-parameter formal deformations of Rota–Baxter 3-Lie algebra $(\mathfrak{g},\mu,T)$ of weight λ. A formal isomorphism from $(\mathfrak{g}[[t]],\mu'_t,T'_t)$ to $(\mathfrak{g}[[t]],\mu_t,T_t)$ is a power series $\psi_t=\sum_{i=0}\psi_i t^i:\mathfrak{g}[[t]]\to\mathfrak{g}[[t]]$, where $\psi_i:\mathfrak{g}\to\mathfrak{g}$ are linear maps with $\psi_0=\mathrm{Id}_{\mathfrak{g}}$, such that:

$$\psi_t\circ\mu'_t = \mu_t\circ(\psi_t\otimes\psi_t\otimes\psi_t), \tag{4.6}$$
$$\psi_t\circ T'_t = T_t\circ\psi_t. \tag{4.7}$$

In this case, we say that the two 1-parameter formal deformations $(\mathfrak{g}[[t]],\mu_t,T_t)$ and $(\mathfrak{g}[[t]],\mu'_t,T'_t)$ are equivalent.

Given a Rota–Baxter 3-Lie algebra $(\mathfrak{g},\mu,T)$ of weight λ, the power series μ_t, T_t with $\mu_i = \delta_{i,0}\mu, T_i = \delta_{i,0}T$ makes $(\mathfrak{g}[[t]],\mu_t,T_t)$ into a 1-parameter formal deformation of $(\mathfrak{g},\mu,T)$. Formal deformations is said to be trivial if it is equivalent to $(\mathfrak{g},\mu,T)$.

Theorem 4.20. *The infinitesimals of two equivalent 1-parameter formal deformations of* $(\mathfrak{g},\mu,T)$ *are in the same cohomology class in* $\mathrm{H}^\bullet_{\mathsf{RB3\text{-}Lie}^\lambda}(\mathfrak{g})$.

Proof. Let $\psi_t : (\mathfrak{g}[[t]],\mu_t',T_t') \to (\mathfrak{g}[[t]],\mu_t,T_t)$ be a formal isomorphism. Expanding the identities and collecting coefficients of t, we get from Equations (4.6) and (4.7):

$$\begin{aligned}\mu_1' &= \mu_1 + \mu\circ(\mathrm{Id}\otimes\mathrm{Id}\otimes\psi_1) - \psi_1\circ\mu + \mu\circ(\psi_1\otimes\mathrm{Id}\otimes\mathrm{Id}) + \mu\circ(\mathrm{Id}\otimes\psi_1\otimes\mathrm{Id}),\\ T_1' &= T_1 + T\circ\psi_1 - \psi_1\circ T,\end{aligned}$$

that is, we have

$$(\mu_1',T_1') - (\mu_1,T_1) = (\delta^1(\psi_1), -\Phi^1(\psi_1)) = d^1(\psi_1,0) \in \mathrm{C}^\bullet_{\mathsf{RB3\text{-}Lie}^\lambda}(\mathfrak{g}).$$

□

Definition 4.21. A Rota–Baxter 3-Lie algebra $(\mathfrak{g},\mu,T)$ of weight λ is said to be rigid if every 1-parameter formal deformation is trivial.

Theorem 4.22. *Let* $(\mathfrak{g},\mu,T)$ *be a Rota–Baxter* 3*-Lie algebra of weight* λ. *If* $\mathrm{H}^2_{\mathsf{RB3\text{-}Lie}^\lambda}(\mathfrak{g}) = 0$, *then* $(\mathfrak{g},\mu,T)$ *is rigid.*

Proof. Let $(\mathfrak{g}[[t]],\mu_t,T_t)$ be a 1-parameter formal deformation of $(\mathfrak{g},\mu,T)$. By Proposition 4.17, (μ_1,T_1) is a 2-cocycle. By $\mathrm{H}^2_{\mathsf{RB3\text{-}Lie}^\lambda}(\mathfrak{g}) = 0$, there exists a 1-cochain

$$(\psi_1',x) \in \mathrm{C}^1_{\mathsf{RB3\text{-}Lie}^\lambda}(\mathfrak{g}) = \mathrm{C}^1_{\mathsf{3\text{-}Lie}}(\mathfrak{g}) \oplus \mathrm{Hom}_{\mathbf{k}}(k,\mathfrak{g})$$

such that $(\mu_1,T_1) = d^1(\psi_1',x)$, that is, $\mu_1 = \delta^1(\psi_1')$ and $T_1 = -\partial^0(x) - \Phi^1(\psi_1')$. Let $\psi_1 = \psi_1' + \delta^0(x)$. Then $\mu_1 = \delta^1(\psi_1)$ and $T_1 = -\Phi^1(\psi_1)$, as it can be readily seen that $\Phi^1(\delta^0(x)) = \partial^0(x)$.

Setting $\psi_t = \mathrm{Id}_{\mathfrak{g}} - \psi_1 t$, we have a deformation $(\mathfrak{g}[[t]],\overline{\mu}_t,\overline{T}_t)$, where

$$\overline{\mu}_t = \psi_t^{-1}\circ\mu_t\circ(\psi_t\times\psi_t\times\psi_t)$$

and

$$\overline{T}_t = \psi_t^{-1}\circ T_t\circ\psi_t.$$

It is easy to verify that $\overline{\mu}_1 = 0, \overline{T}_1 = 0$. Then

$$\begin{aligned}\overline{\mu}_t &= \mu + \overline{\mu}_2 t^2 + \cdots,\\ T_t &= T + \overline{T}_2 t^2 + \cdots.\end{aligned}$$

By Equations (4.3) and (4.4), we see that $(\overline{\mu}_2,\overline{T}_2)$ is still a 2-cocycle, so by induction, we can show that $(\mathfrak{g}[[t]],\mu_t,T_t)$ is equivalent to $(\mathfrak{g}[[t]],\mu,T)$. Thus, $(\mathfrak{g},\mu,T)$ is rigid.

□

Chapter 5

Cohomology and deformations of modified Rota–Baxter 3-Lie algebras

5.1 Representations of modified Rota–Baxter 3-Lie algebras

In this section, we introduce representations of modified Rota–Baxter 3-Lie algebras of arbitrary weight. We also provide various examples and new constructions.

Definition 5.1. Let

$$(\mathfrak{g},[-,-,-]_{\mathfrak{g}},R) \text{ and } (\mathfrak{g}',[-,-,-]_{\mathfrak{g}'},R')$$

be two Rota–Baxter 3-Lie algebras of weight λ. A morphism between them is a 3-Lie algebra homomorphism $\phi:\mathfrak{g}\to\mathfrak{g}'$ satisfying $\phi\circ R=R'\circ\phi$.

Denote by $\mathsf{mRB3\text{-}Lie}^{\lambda}$ the category of modified Rota–Baxter 3-Lie algebras of weight λ with morphisms defined above.

Definition 5.2. Let $(\mathfrak{g},[-,-,-]_{\mathfrak{g}},R)$ be a modified Rota–Baxter 3-Lie algebra of weight λ and (M,ρ) be a representation over the 3-Lie algebra $(\mathfrak{g},[-,-,-]_{\mathfrak{g}})$. We say that (M,ρ,R_M) is a *representation* over the modified Rota–Baxter 3-Lie algebra $(\mathfrak{g},[-,-,-]_{\mathfrak{g}},R)$ of weight λ if (M,ρ) is endowed with a linear operator $R_M:M\to M$ such that the following equation

$$\begin{aligned}&\rho\big(R(x),R(y)\big)(R_M(m))\\ =\;& R_M\Big(\rho(R(x),R(y))(m)+\rho(R(x),y)(R_M(m))+\rho(x,R(y))(R_M(m))+\lambda\rho(x,y)(m)\Big)\\ &-\lambda\rho(R(x),y)(m)-\lambda\rho(x,R(y))(m)-\lambda\rho(x,y)(R_M(m))\end{aligned}$$

holds for any $x,y\in\mathfrak{g}$ and $m\in M$.

Similar to the case of representations of Rota–Baxter 3-Lie algebras, we have the following examples and properties.

Example 5.3. $(\mathfrak{g},[-,-,-]_{\mathfrak{g}},R)$ *itself is a representation over the modified Rota–Baxter* 3*-Lie algebra* $(\mathfrak{g},[-,-,-]_{\mathfrak{g}},R)$ *of weight* λ*, called the regular representation or the adjoint representation.*

Example 5.4. *Let* $(\mathfrak{g},[-,-,-]_{\mathfrak{g}})$ *be a* 3*-Lie algebra and* (M,ρ) *be a representation over it. Then the triple* (M,ρ,Id_M) *is a representation of the modified Rota–Baxter* 3*-Lie algebra* $(\mathfrak{g},[-,-,-]_{\mathfrak{g}},\mathrm{Id}_{\mathfrak{g}})$ *of weight* 1.

Example 5.5. *Let $(\mathfrak{g},[-,-,-]_{\mathfrak{g}},R)$ be a modified Rota–Baxter 3-Lie algebra of weight λ and (M,ρ,R_M) be a representation over it. Then for arbitrary scalar $\mu\in\mathbf{k}$, the triple $(M,\rho,\mu R_M)$ is a representation of the modified Rota–Baxter 3-Lie algebra $(\mathfrak{g},[-,-,-]_{\mathfrak{g}},\mu R)$ of weight $(\mu^2\lambda)$.*

Proposition 5.6. *Let*

$$(\mathfrak{g},[-,-,-]_{\mathfrak{g}},R)$$

be a modified Rota–Baxter 3-Lie algebra of weight λ and $\{(M_i,\rho_i,R_{M_i})\}_{i\in I}$ be a family of representation of it. Then the triple $(\oplus_{i\in I}M_i,(\rho_i)_{i\in I},\oplus_{i\in I}R_{M_i})$ is a representation of the modified Rota–Baxter 3-Lie algebra $(\mathfrak{g},[-,-,-]_{\mathfrak{g}},R)$ of weight λ.

Proposition 5.7. *Let*

$$(\mathfrak{g},[-,-,-]_{\mathfrak{g}},R)$$

be a modified Rota–Baxter 3-Lie algebra of weight λ and (M,ρ,R_M) be a representation of it. Then $(\mathfrak{g}\oplus M,R\oplus R_M)$ is a modified Rota–Baxter 3-Lie algebra of weight λ, where the Lie bracket on $\mathfrak{g}\oplus M$ is given by the semidirect product

$$[x+u,y+v,z+w]_{\mathfrak{g}\oplus M} = [x,y,z]_{\mathfrak{g}}+\rho(x,y)w+\rho(y,z)u+\rho(z,x)v,$$

for arbitrary $x,y,z\in\mathfrak{g}$ and $u,v,w\in M$.

Let $(\mathfrak{g},[-,-,-])$ be a 3-Lie algebra and R be a modified Rota–Baxter operator of weight λ, by [55], in order to make $(\mathfrak{g},[-,-,-]_R)$ a 3-Lie algebra, we always assume $\mathfrak{g}^1\subset\mathscr{C}(\mathfrak{g})$, where

$$[x,y,z]_R=[Rx,Ry,z]+[x,Ry,Rz]+[Rx,y,Rz]+\lambda[x,y,z],\quad \forall x,y,z\in\mathfrak{g}. \tag{1.1}$$

Proposition 5.8. *Let R be a modified Rota–Baxter operator of weight λ on a 3-Lie algebra $(\mathfrak{g},[-,-,-])$. Then R is a modified Rota–Baxter operator of weight λ on a 3-Lie algebra $(\mathfrak{g},[-,-,-]_R)$.*

Proof. For any $x,y,z\in\mathfrak{g}$, we have

$$\begin{aligned}
&[R(x),R(y),R(z)]_R\\
&=[R^2(x),R^2(y),R(z)]+[R(x),R^2(y),R^2(z)]+[R^2(x),R(y),R^2(z)]+\lambda[R(x),R(y),R(z)]\\
&=R([R^2(x),R^2(y),z]+[R(x),R^2(y),R(z)]+[R^2(x),R(y),R(z)]+\lambda[R(x),R(y),z])-\lambda[R^2(x),R(y),z]\\
&\quad-\lambda[R(x),R^2(y),z]-\lambda[Rx,R(y),R(z)]+R([R(x),R^2(y),R(z)]+[x,R^2(y),R^2z]+[R(x),R(y),R^2(z)]\\
&\quad+\lambda[x,R(y),R(z)])-\lambda[R(x),R(y),R(z)]-\lambda[x,R^2(y),R(z)]-\lambda[x,R(y),R^2(z)]+R([R^2(x),R(y),R(z)]\\
&\quad+[R(x),R(y),R^2(z)]+[R^2(x),y,R^2(z)]+\lambda[R(x),y,R(z)])-\lambda[R^2(x),y,R(z)]-\lambda[R(x),R(y),R(z)]\\
&\quad-\lambda[R(x),y,R^2(z)]+\lambda[R(x),R(y),R(z)]\\
&=R([R^2(x),R^2(y),z]+[R(x),R^2(y),R(z)]+[R^2(x),R(y),R(z)]+\lambda[R(x),R(y),z])-\lambda[R^2(x),R(y),z]\\
&\quad-\lambda[R(x),R^2(y),z]+R([R(x),R^2(y),R(z)]+[x,R^2(y),R^2z]+[R(x),R(y),R^2(z)]+\lambda[x,R(y),R(z)])\\
&\quad-\lambda[R(x),R(y),R(z)]-\lambda[x,R^2(y),R(z)]-\lambda[x,R(y),R^2(z)]+R([R^2(x),R(y),R(z)]+[R(x),R(y),R^2(z)]\\
&\quad+[R^2(x),y,R^2(z)]+\lambda[R(x),y,R(z)])-\lambda[R^2(x),y,R(z)]-\lambda[R(x),R(y),R(z)]-\lambda[R(x),y,R^2(z)]\\
&=R([R(x),R(y),z]_R+[x,R(y),R(z)]_R+[R(x),y,R(z)]_R+\lambda[x,y,z]_R)\\
&\quad-\lambda[R(x),y,z]_R-\lambda[x,R(y),z]_R-\lambda[x,y,R(z)]_R.
\end{aligned}$$

Therefore, R is a modified Rota–Baxter operator of weight λ on a 3-Lie algebra $(\mathfrak{g},[-,-,-]_R)$. ☐

In analogy with Theorem 4.9, we obtain the following result.

Theorem 5.9. *Let $R\colon \mathfrak{g}\to\mathfrak{g}$ be a modified Rota–Baxter operator of weight λ on a 3-Lie algebra $(\mathfrak{g},[\cdot,\cdot,\cdot])$ and (M,ρ,R_M) be a representation on it. Define $\rho_R\colon \mathfrak{g}\wedge\mathfrak{g}\to\mathfrak{gl}(M)$ by*

$$\rho_R(x,y)m=\rho(Rx,Ry)m-R_M\big(\rho(Rx,y)m+\rho(x,Ry)m\big)+\lambda\,\rho(x,y)m,\qquad \forall x,y\in\mathfrak{g},\ m\in M. \tag{1.2}$$

Then $(\mathfrak{g},\rho_R)$ is a representation of the 3-Lie algebra $(\mathfrak{g},[\cdot,\cdot,\cdot]_R)$. Moreover, (M,ρ_R,R_M) is a representation of the modified Rota–Baxter 3-Lie algebra $(\mathfrak{g}_R,[\cdot,\cdot,\cdot]_R,R)$ of weight λ.

5.2 Cohomology of modified Rota–Baxter operator on 3-Lie algebras

In this section, we will define the cohomology of modified Rota–Baxter operators of weight λ on 3-Lie algebras.

Let $\partial_R:\mathrm{C}^n_{\text{3-Lie}}(\mathfrak{g}_R,M)\to\mathrm{C}^{n+1}_{\text{3-Lie}}(\mathfrak{g}_R,M)$ be the corresponding coboundary operator of the 3-Lie algebra $(\mathfrak{g},[-,-,-]_R)$ with coefficients in the representation (M,ρ,R_M). More precisely,

$$\partial_R:\mathrm{C}^n_{\text{3-Lie}}(\mathfrak{g}_R,M)\to\mathrm{C}^{n+1}_{\text{3-Lie}}(\mathfrak{g}_R,M)$$

is given by

$$\begin{aligned}
&(\partial_R f)(\mathfrak{X}_1,\dots,\mathfrak{X}_n,x_{n+1})\\
&=\sum_{1\le j<k\le n}(-1)^j f(\mathfrak{X}_1,\dots,\hat{\mathfrak{X}}_j,\dots,\mathfrak{X}_{k-1},[x_j,y_j,x_k]_R\wedge y_k+x_k\wedge[x_j,y_j,y_k]_R,\mathfrak{X}_{k+1},\dots,\mathfrak{X}_n,x_{n+1})\\
&\quad+\sum_{j=1}^{n}(-1)^{j-1}\rho_R(x_j,y_j)f(\mathfrak{X}_1,\dots,\hat{\mathfrak{X}}_j,\dots,\mathfrak{X}_n,x_{n+1})\\
&\quad+(-1)^{n+1}(\rho_R(y_n,x_{n+1})f(\mathfrak{X}_1,\dots,\mathfrak{X}_{n-1},x_n)+\rho_R(x_{n+1},x_n)f(\mathfrak{X}_1,\dots,\mathfrak{X}_{n-1},y_n)),
\end{aligned}$$

for $\mathfrak{X}_i=x_i\wedge y_i\in\wedge^2\mathfrak{g}, i=1,\cdots,n$ and $x_{n+1}\in\mathfrak{g}$.

Obviously $f\in C^1_{\text{3-Lie}}(\mathfrak{g},M)$ is closed if and only if

$$\begin{aligned}
&\rho(Ry,Rz)f(x)+\rho(Rz,Rx)f(y)+\rho(Rx,Ry)f(z)\\
&=R(\rho(Ry,z)f(x)+\rho(z,Rx)f(y)+\rho(Rz,x)f(y)+\rho(x,Ry)f(z)+\rho(Rx,y)f(z)]+\rho(y,Rz)f(x),)\\
&\quad+f([Rx,Ry,z]+[x,Ry,Rz]+[Rx,y,Rz]+\lambda[x,y,z])-\lambda[f(x),y,z]-\lambda[x,f(y),z]-\lambda[x,y,f(z)].
\end{aligned}$$

Definition 5.10. Let

$$(\mathfrak{g},[-,-,-]_{\mathfrak{g}},R)$$

be a modified Rota–Baxter 3-Lie algebra of weight λ and (M,ρ,R_M) be a representation over it. Then the cochain complex $(\mathrm{C}^\bullet_{\text{3-Lie}}(\mathfrak{g}_R,M),\partial_R)$ is called the *cochain complex of modified Rota–Baxter operator T* of weight λ with coefficients in (M,ρ_R,R_M), denoted by $\mathrm{C}^\bullet_{\mathsf{mRBO}^\lambda}(\mathfrak{g},M)$. The cohomology of $\mathrm{C}^\bullet_{\mathsf{mRBO}^\lambda}(\mathfrak{g},M)$, denoted by $\mathrm{H}^\bullet_{\mathsf{mRBO}^\lambda}(\mathfrak{g},M)$, are called the *cohomology of modified Rota–Baxter operator R* of weight λ with coefficients in (M,ρ_R,R_M).

When (M,ρ,R_M) is the regular representation

$$(\mathfrak{g},[-,-,-]_{\mathfrak{g}},R),$$

we denote $\mathrm{C}^\bullet_{\mathsf{mRBO}^\lambda}(\mathfrak{g},\mathfrak{g})$ by $\mathrm{C}^\bullet_{\mathsf{mRBO}^\lambda}(\mathfrak{g})$ and call it the cochain complex of modified Rota–Baxter operator of weight λ, and denote $\mathrm{H}^\bullet_{\mathsf{mRBO}^\lambda}(\mathfrak{g},\mathfrak{g})$ by $\mathrm{H}^\bullet_{\mathsf{mRBO}^\lambda}(\mathfrak{g})$ and call it the cohomology of modified Rota–Baxter operator T of weight λ.

5.3 Cohomology of modified Rota–Baxter 3-Lie algebras

In this section, we will combine the cohomology of 3-Lie algebras and the cohomology of modified Rota–Baxter operators of arbitrary weights to define a cohomology theory for modified Rota–Baxter 3-Lie algebras of arbitrary weights.

Let $M=(M,\rho,R_M)$ be a representation over a modified Rota–Baxter 3-Lie algebra $\mathfrak{g}=(\mathfrak{g},\mu=[-,-,-]_{\mathfrak{g}},R)$ of weight λ. Now, let's construct a chain map

$$\Psi^\bullet : \mathrm{C}^\bullet_{\text{3-Lie}}(\mathfrak{g},M)\to C^\bullet_{\mathrm{mRBO}^\lambda}(\mathfrak{g},M),$$

i.e., the following commutative diagram:

$$\begin{array}{ccccccccc}
\mathrm{C}^0_{\text{3-Lie}}(\mathfrak{g},M) & \xrightarrow{\delta^0} & \mathrm{C}^1_{\text{3-Lie}}(\mathfrak{g},M) & \cdots\cdots & \mathrm{C}^n_{\text{3-Lie}}(\mathfrak{g},M) & \xrightarrow{\delta^n} & \mathrm{C}^{n+1}_{\text{3-Lie}}(\mathfrak{g},M) & \cdots\cdots & \\
\downarrow{\scriptstyle\Phi^0} & & \downarrow{\scriptstyle\Phi^1} & & \downarrow{\scriptstyle\Phi^n} & & \downarrow{\scriptstyle\Phi^{n+1}} & & \\
\mathrm{C}^0_{\mathrm{RBO}^\lambda}(\mathfrak{g},M) & \xrightarrow{\partial^0} & \mathrm{C}^1_{\mathrm{RBO}^\lambda}(\mathfrak{g},M) & \cdots\cdots & \mathrm{C}^n_{\mathrm{RBO}^\lambda}(\mathfrak{g},M) & \xrightarrow{\partial^n} & \mathrm{C}^{n+1}_{\mathrm{RBO}^\lambda}(\mathfrak{g},M) & \cdots\cdots & .
\end{array}$$

Define $\Psi^0=\mathrm{Id}_{\mathrm{Hom}(k,M)}=\mathrm{Id}_M$. For $f\in \mathrm{C}^1_{\text{3-Lie}}(\mathfrak{g},M)$, set

$$\Psi^1(f)=f\circ R-R_M\circ f.$$

For $n\geqslant 2$ and $f\in \mathrm{C}^n_{\text{3-Lie}}(\mathfrak{g},M)$, define $\Psi^n(f)\in \mathrm{C}^n_{\mathrm{mRBO}^\lambda}(\mathfrak{g},M)$ by

$$\begin{aligned}
&\Psi^n(f)=\\
&\sum_{k=1}^{n}\Big(\lambda^{n-k-1}\sum_{1\leqslant i_1<i_2<\cdots<i_{2k-1}\leqslant 2n-1} f\circ(\mathrm{Id}^{(i_1-1)},R,\mathrm{Id}^{(i_2-i_1-1)},R,\cdots,R,\mathrm{Id}^{(2n-1-i_{2k-1})})\circ\Big((\mathrm{Id}\wedge\mathrm{Id})^{\otimes n-2}\otimes \mathrm{Id}^{\wedge 3}\Big)\\
&-\lambda^{n-k-1}\sum_{1\leqslant i_1<i_2<\cdots<i_{2k-2}\leqslant 2n-1} R_M\circ f\circ(\mathrm{Id}^{(i_1-1)},R,\mathrm{Id}^{(i_2-i_1-1)},R,\cdots,R,\mathrm{Id}^{(2n-1-i_{2k-2})})\circ\Big((\mathrm{Id}\wedge\mathrm{Id})^{\otimes n-2}\otimes \mathrm{Id}^{\wedge 3}\Big)\Big).
\end{aligned}$$

Here the operators

$$\mathrm{Id}^{(i_1-1)},\ R,\ \mathrm{Id}^{(i_2-i_1-1)},\ T,\ \ldots,\ R,\ \mathrm{Id}^{(2n-1-i_{2k-1})}$$

(or $\mathrm{Id}^{(2n-1-i_{2k-2})}$ in the second summand) act successively on

$$x_1,y_1,\ldots,x_n,y_n,x_{n+1}.$$

Proposition 5.11. *The map $\Psi^\bullet : \mathrm{C}^\bullet_{\text{3-Lie}}(\mathfrak{g},M)\to \mathrm{C}^\bullet_{\mathrm{mRBO}^\lambda}(\mathfrak{g},M)$ is a chain map.*

It can be proved by arguments similar to those in Appendix A; hence we omit the details.

Similarly, we give the definition of cohomology for modified Rota–Baxter algebras.

Definition 5.12. Let $M=(M,\rho,R_M)$ be a representation over a modified Rota–Baxter 3-Lie algebra $\mathfrak{g}=(\mathfrak{g},\mu=[-,-,-]_{\mathfrak{g}},R)$ of weight λ. We define the cochain complex $(\mathrm{C}^\bullet_{\text{RB3-Lie}^\lambda}(\mathfrak{g},M),d^\bullet)$ of modified Rota–Baxter 3-Lie algebra $(\mathfrak{g},\mu,R)$ of weight λ with coefficients in (M,ρ,R_M) to the negative shift of the mapping cone of $\Psi^\bullet$, that is, let

$$\mathrm{C}^0_{\text{RB3-Lie}^\lambda}(\mathfrak{g},M)=\mathrm{C}^0_{\text{3-Lie}}(\mathfrak{g},M)\quad\text{and}\quad \mathrm{C}^n_{\text{mRB3-Lie}^\lambda}(\mathfrak{g},M)=\mathrm{C}^n_{\text{3-Lie}}(\mathfrak{g},M)\oplus \mathrm{C}^{n-1}_{\mathrm{mRBO}^\lambda}(\mathfrak{g},M),\forall n\geqslant 1,$$

and the differential $d^n : \mathrm{C}^n_{\mathrm{mRB3\text{-}Lie}^\lambda}(\mathfrak{g},M) \to \mathrm{C}^{n+1}_{\mathrm{mRB3\text{-}Lie}^\lambda}(\mathfrak{g},M)$ is given by

$$d^n(f,g) = (\delta^n(f), -\partial^{n-1}(g) - \Phi^n(f))$$

for arbitrary $f \in \mathrm{C}^n_{\text{3-Lie}}(\mathfrak{g},M)$ and $g \in \mathrm{C}^{n-1}_{\mathrm{mRBO}^\lambda}(\mathfrak{g},M)$. The cohomology of $(\mathrm{C}^\bullet_{\mathrm{mRB3\text{-}Lie}^\lambda}(\mathfrak{g},M), d^\bullet)$, denoted by $\mathrm{H}^\bullet_{\mathrm{mRB3\text{-}Lie}^\lambda}(\mathfrak{g},M)$, is called the *cohomology of the Rota–Baxter* 3*-Lie algebra* $(\mathfrak{g},\mu,R)$ of weight λ with coefficients in (M,ρ,R_M). When $(M,\rho,R_M) = (\mathfrak{g},\mu,R)$, we just denote $\mathrm{C}^\bullet_{\mathrm{mRB3\text{-}Lie}^\lambda}(\mathfrak{g},\mathfrak{g})$, $\mathrm{H}^\bullet_{\mathrm{mRB3\text{-}Lie}^\lambda}(\mathfrak{g},\mathfrak{g})$ by $\mathrm{C}^\bullet_{\mathrm{mRB3\text{-}Lie}^\lambda}(\mathfrak{g})$, $\mathrm{H}^\bullet_{\mathrm{mRB3\text{-}Lie}^\lambda}(\mathfrak{g})$ respectively, and call them the *cochain complex, the cohomology of modified Rota–Baxter* 3*-Lie algebra* $(\mathfrak{g},\mu,T)$ of weight λ respectively.

There is also a natural short exact sequence of cochain complexes

$$0 \longrightarrow s^{-1}\mathrm{C}^\bullet_{\mathrm{mRBO}^\lambda}(\mathfrak{g},M) \longrightarrow \mathrm{C}^\bullet_{\mathrm{mRB3\text{-}Lie}^\lambda}(\mathfrak{g},M) \longrightarrow \mathrm{C}^\bullet_{\text{3-Lie}}(\mathfrak{g},M) \longrightarrow 0,$$

which gives rise to the associated long exact sequence in cohomology:

$$\begin{aligned} 0 &\longrightarrow \mathrm{H}^0_{\mathrm{mRB3\text{-}Lie}^\lambda}(\mathfrak{g},M) \longrightarrow \mathrm{H}^0_{\text{3-Lie}}(\mathfrak{g},M) \longrightarrow \mathrm{H}^0_{\mathrm{mRBO}^\lambda}(\mathfrak{g},M) \\ &\longrightarrow \mathrm{H}^1_{\mathrm{mRB3\text{-}Lie}^\lambda}(\mathfrak{g},M) \longrightarrow \mathrm{H}^1_{\text{3-Lie}}(\mathfrak{g},M) \longrightarrow \cdots \\ \cdots &\longrightarrow \mathrm{H}^p_{\text{3-Lie}}(\mathfrak{g},M) \longrightarrow \mathrm{H}^p_{\mathrm{mRBO}^\lambda}(\mathfrak{g},M) \longrightarrow \mathrm{H}^{p+1}_{\mathrm{mRB3\text{-}Lie}^\lambda}(\mathfrak{g},M) \\ &\longrightarrow \mathrm{H}^{p+1}_{\text{3-Lie}}(\mathfrak{g},M) \longrightarrow \cdots. \end{aligned}$$

5.4 Formal deformations of modified Rota–Baxter algebras

In this section, we study formal deformations of modified Rota–Baxter 3-Lie algebras of weight λ.

5.4.1 Formal deformations of modified Rota–Baxter operator with 3-Lie bracket fixed

Let $(\mathfrak{g},\mu,R)$ be a modified Rota–Baxter 3-Lie algebra of weight λ. Let us consider the case where we only deform the modified Rota–Baxter operator with the 3-Lie bracket fixed. As in Section 4.4.1, we consider a one-parameter family of the form

$$R_t = \sum_{i=0}^{\infty} R_i t^i, \qquad R_i \in \mathrm{C}^1_{\mathrm{mRBO}^\lambda}(\mathfrak{g}).$$

Definition 5.13. A *1-parameter formal deformation* of a modified Rota–Baxter operator R on a 3-Lie algebra $(\mathfrak{g},\mu)$ is a formal power series $R_t \in \mathrm{End}_{\mathbf{k}[[t]]}(\mathfrak{g}[[t]])$ such that:

- R_t is a $\mathbf{k}[[t]]$-linear modified Rota–Baxter operator on the 3-Lie algebra $\mathfrak{g}[[t]]$;
- R_t reduces to R at $t = 0$, i.e. $R_0 = R$.

The coefficient R_1 is called the *infinitesimal* of the deformation.

Power series R_t determine a 1-parameter formal deformation of the modified Rota–Baxter operator R on a 3-Lie algebra $(\mathfrak{g},\mu)$ if and only if the following equation holds:

$$\begin{aligned} & \mu(R_t(x_1),R_t(x_2),R_t(x_3)) \\ = \; & R_t\big(\mu(R_t(x_1),R_t(x_2),x_3) + \mu(R_t(x_1),x_2,R_t(x_3)) + \mu(x_1,R_t(x_2),R_t(x_3)) + \lambda\mu(x_1,x_2,x_3)\big) \\ & -\lambda\ \mu(R_t(x_1),x_2,x_3) - \lambda\ \mu(x_1,R_t(x_2),x_3) - \lambda\ \mu(x_1,x_2,R_t(x_3)), \end{aligned}$$

for arbitrary $x_1,x_2,x_3,x_4,x_5 \in \mathfrak{g}$. Expanding this equation and comparing the coefficient of t^n, we obtain that $\{T_i\}_{i\geqslant 0}$ have to satisfy: for arbitrary $n \geqslant 0$,

$$\begin{aligned}&\sum_{\substack{i+j+k=n\\ i,j,k\geqslant 0}} \mu(R_i(x_1),R_j(x_2),R_k(x_3))\\ =&\sum_{\substack{i+j+k=n\\ i,j,k\geqslant 0}} \big(R_i(\mu(x_1,R_j(x_2),R_k(x_3)))+R_i(\mu(R_j(x_1),x_2,R_k(x_3)))+R_k(\mu(R_i(x_1),R_j(x_2),x_3))\big)\\ &-\lambda\sum_{\substack{i+j=n\\ i,j\geqslant 0}} \big(R_i(\mu(R_j(x_1),x_2,x_3))+R_i(\mu(x_1,R_j(x_2),x_3))+R_i(\mu(x_1,x_2,R_j(x_3)))\big)+\lambda R_n\mu(x_1,x_2,x_3).\end{aligned} \tag{4.3}$$

Obviously, when $n=0$, Equation (4.3) becomes exactly Equation (1.2) defining modified Rota–Baxter operator $T=T_0$.

When $n=1$, Equation (4.3) has the form

$$\begin{aligned}&\mu\,(R_1(x_1),R(x_2),R(x_3))+\mu\,(R(x_1),R_1(x_2),R(x_3))+\mu\,(R(x_1),R(x_2),R_1(x_3))\\ =&\;\; R\,(\mu(x_1,R_1(x_2),R(x_3))+\mu(x_1,R(x_2),R_1(x_3)))\\ &+R\,(\mu(R_1(x_1),x_2,R(x_3))+\mu(R(x_1),x_2,R_1(x_3)))\\ &+R\,(\mu(R_1(x_1),R(x_2),x_3)+\mu(R(x_1),R_1(x_2),x_3))\\ &+R_1\,(\mu(x_1,R(x_2),R(x_3))+\mu(R(x_1),x_2,R(x_3))+\mu(R(x_1),R(x_2),x_3))\\ &-\lambda\,(\mu(R_1(x_1),x_2,x_3)+\mu(x_1,R(x_2),x_3)+\mu(x_1,x_2,R_1(x_3)))\\ &-\lambda\,(\mu(x_1,R(x_2),x_3)+\mu(R(x_1),x_2,x_3)+\mu(x_1,x_2,R(x_3)))\\ &+\lambda R_1\mu(x_1,x_2,x_3)\end{aligned} \tag{4.4}$$

which says exactly that $\partial^1(R_1)=0\in \mathrm{C}^\bullet_{\mathsf{RBO}^\lambda}(\mathfrak{g})$. This proves the following result:

Proposition 5.14. *Let R_t be a 1-parameter formal deformation of modified Rota–Baxter operator R of weight λ. Then T_1 is a 1-cocycle in the cochain complex $\mathrm{C}^\bullet_{\mathsf{mRBO}^\lambda}(\mathfrak{g})$.*

Remark 5.15. *The above result implies that $\mathrm{C}^\bullet_{\mathsf{mRBO}^\lambda}(\mathfrak{g})$ controls the formal deformation theory of Rota–Baxter operators.*

5.4.2 Formal deformations of modified Rota–Baxter 3-Lie algebras

Let $(\mathfrak{g},\mu,R)$ be a modified Rota–Baxter 3-Lie algebra of weight λ. Consider a 1-parameterized family:

$$\mu_t=\sum_{i=0}^{\infty}\mu_i t^i,\ \mu_i\in \mathrm{C}^2_{\text{3-Lie}}(\mathfrak{g}),\quad R_t=\sum_{i=0}^{\infty}R_i t^i,\ R_i\in \mathrm{C}^1_{\mathsf{mRBO}^\lambda}(\mathfrak{g}).$$

Definition 5.16. A *1-parameter formal deformation* of a modified Rota–Baxter 3-Lie algebra $(\mathfrak{g},\mu,R)$ of weight λ is a pair (μ_t,R_t) such that the flat $\mathbf{k}[[t]]$-module $\mathfrak{g}[[t]]$ becomes a modified Rota–Baxter 3-Lie algebra of weight λ over $\mathbf{k}[[t]]$, and such that $(\mu_0,R_0)=(\mu,R)$. The pair (μ_1,R_1) is called the *infinitesimal* of the deformation.

Power series μ_t and R_t determine a 1-parameter formal deformation of Rota–Baxter 3-Lie algebra $(\mathfrak{g},\mu,R)$ of weight λ if and only if for arbitrary $x_1,x_2,x_3,x_4,x_5\in\mathfrak{g}$, the following equations hold :

$$\begin{aligned}&\mu_t(x_1,x_2,\mu_t(x_3,x_4,x_5))\\ =&\;\;\mu_t(\mu_t(x_1,x_2,x_3),x_4,x_5)+\mu_t(x_3,\mu_t(x_1,x_2,x_4),x_5)+\mu_t(x_3,x_4,\mu_t(x_1,x_2,x_5))\\ &\mu_t(R_t(x_1),R_t(x_2),R_t(x_3))\\ =&\;\;R_t\big(\mu_t(R_t(x_1),R_t(x_2),x_3)+\mu_t(R_t(x_1),x_2,R_t(x_3))+\mu_t(x_1,R_t(x_2),R_t(x_3))+\lambda\mu_t(x_1,x_2,x_3)\big)\\ &-\lambda\;\mu_t(R_t(x_1),x_2,x_3)-\lambda\;\mu_t(x_1,R_t(x_2),x_3)-\lambda\;\mu_t(x_1,x_2,R_t(x_3)).\end{aligned}$$

By expanding these equations and comparing the coefficient of t^n, we obtain that $\{\mu_i\}_{i\geqslant 0}$ and $\{T_i\}_{i\geqslant 0}$ have to satisfy: for arbitrary $n \geqslant 0$,

$$\begin{aligned}&\sum_{i+j=n} \mu_i(x_1,x_2,\mu_j(x_3,x_4,x_5))\\ =&\sum_{i+j=n} \mu_i(\mu_j(x_1,x_2,x_3),x_4,x_5)+\mu_i(x_3,\mu_j(x_1,x_2,x_4),x_5)+\mu_i(x_3,x_4,\mu_j(x_1,x_2,x_5)),\end{aligned} \tag{4.5}$$

$$\begin{aligned}&\sum_{i+j+k+l=n} \mu_i(R_j(x_1),R_k(x_2),R_l(x_3))\\ =&\sum_{i+j+k+l=n} T_i\big(\mu_j(R_k(x_1),R_l(x_2),x_3)+\mu_j(R_k(x_1),x_2,R_l(x_3))+\mu_j(x_1,R_k(x_2),R_l(x_3))\big)\\ &-\lambda\sum_{j+k=n}\big(\mu_j(R_k(x_1),x_2,x_3)+\mu_j(x_1,R_k(x_2),x_3)+\mu_j(x_1,x_2,R_k(x_3))\big)\\ &+\lambda\sum_{i+j=n} R_i\big(\mu_j(x_1,x_2,x_3)\big).\end{aligned} \tag{4.6}$$

Obviously, when $n=0$, the above conditions are exactly the 3-Lie bracket $\mu=\mu_0$ and Equation (1.2) which is the defining relation of modified Rota–Baxter operator $R=R_0$ of weight λ.

Proposition 5.17. *Let $(\mathfrak{g}[[t]],\mu_t,T_t)$ be a 1-parameter formal deformation of modified Rota–Baxter 3-Lie algebra $(\mathfrak{g},\mu,T)$ of weight λ. Then (μ_1,T_1) is a 2-cocycle in the cochain complex $C^\bullet_{\mathrm{mRB3\text{-}Lie}^\lambda}(\mathfrak{g})$.*

Proof. When $n=1$, Equations (4.6) become

$$\begin{aligned}&\mu_1(R(x_1),R(x_2),R(x_3))-R\big(\mu_1(R(x_1),R(x_2),x_3)-\mu_1(R(x_1),x_2,R(x_3))-\mu_1(x_1,R(x_2),R(x_3))\big)\\ &+\lambda\big(\mu_1(R(x_1),x_2,x_3)+\mu_1(x_1,R(x_2),x_3)+\mu_1(x_1,x_2,R(x_3))\big)-\lambda R\big(\mu_1(x_1,x_2,x_3)\big)\\ =\;&R_1\big([R(x_1),R(x_2),x_3]_{\mathfrak{g}}+[R(x_1),x_2,R(x_3)]_{\mathfrak{g}}+[x_1,R(x_2),R(x_3)]_{\mathfrak{g}}+\lambda[x_1,x_2,x_3]_{\mathfrak{g}}\big)\\ &-[R_1(x_1),R(x_2),R(x_3)]_{\mathfrak{g}}-[R(x_1),R_1(x_2),R(x_3)]_{\mathfrak{g}}-[R(x_1),R(x_2),R_1(x_3)]_{\mathfrak{g}}\\ &+R\big([R_1(x_1),R(x_2),x_3]_{\mathfrak{g}}+[R_1(x_1),x_2,R(x_3)]_{\mathfrak{g}}+[x_1,R_1(x_2),R(x_3)]_{\mathfrak{g}}\big)\\ &+R\big([R(x_1),R_1(x_2),x_3]_{\mathfrak{g}}+[R(x_1),x_2,R_1(x_3)]_{\mathfrak{g}}+[x_1,R(x_2),R_1(x_3)]_{\mathfrak{g}}\big)\\ &-\lambda\big([R_1(x_1),x_2,x_3]_{\mathfrak{g}}+[x_1,R_1(x_2),x_3]_{\mathfrak{g}}+[x_1,x_2,R_1(x_3)]_{\mathfrak{g}}\big).\end{aligned} \tag{4.7}$$

As in the proof of Proposition 4.17, we have

$$\delta^2(\mu_1)=0\in \mathrm{C}^\bullet_{\text{3-Lie}}(\mathfrak{g}),$$

and Equation 4.7 is equivalent to

$$\Phi^2(\mu_1)=-\partial^1(R_1)\in \mathrm{C}^\bullet_{\mathrm{mRBO}^\lambda}(\mathfrak{g}).$$

So (μ_1,R_1) is a 2-cocycle in $\mathrm{C}^\bullet_{\mathrm{mRB3\text{-}Lie}^\lambda}(\mathfrak{g})$. □

Remark 5.18. *Equation* (4.7) *inspired us to introduce the chain map $\Phi^\bullet:\mathrm{C}^\bullet_{\text{3-Lie}}(\mathfrak{g},M)\to C^\bullet_{\mathrm{mRBO}^\lambda}(\mathfrak{g},M)$ as well the cochain complex $\mathrm{C}^\bullet_{\mathrm{mRB3\text{-}Lie}^\lambda}(\mathfrak{g},M)$.*

Definition 5.19. Let $(\mathfrak{g}[[t]],\mu_t,R_t)$ and $(\mathfrak{g}[[t]],\mu_t',R_t')$ be two 1-parameter formal deformations of the modified Rota–Baxter 3-Lie algebra $(\mathfrak{g},\mu,R)$ of weight λ. A formal isomorphism from $(\mathfrak{g}[[t]],\mu_t',R_t')$ to $(\mathfrak{g}[[t]],\mu_t,R_t)$ is a power series $\psi_t=\sum_{i=0}\psi_i t^i:\mathfrak{g}[[t]]\to\mathfrak{g}[[t]]$, where $\psi_i:\mathfrak{g}\to\mathfrak{g}$ are linear maps with $\psi_0=\mathrm{Id}_{\mathfrak{g}}$, such that:

$$\psi_t\circ\mu_t' = \mu_t\circ(\psi_t\otimes\psi_t\otimes\psi_t), \tag{4.8}$$

$$\psi_t\circ R_t' = R_t\circ\psi_t. \tag{4.9}$$

Given a modified Rota–Baxter 3-Lie algebra $(\mathfrak{g},\mu,R)$ of weight λ, the power series μ_t, R_t with $\mu_i = \delta_{i,0}\mu, R_i = \delta_{i,0}T$ makes $(\mathfrak{g}[[t]],\mu_t,R_t)$ into a 1-parameter formal deformation of $(\mathfrak{g},\mu,R)$. Formal deformations is said to be trivial if it is equivalent to $(\mathfrak{g},\mu,R)$.

Definition 5.20. A modified Rota–Baxter 3-Lie algebra $(\mathfrak{g},\mu,T)$ of weight λ is said to be rigid if every 1-parameter formal deformation is trivial.

Arguing as in Section 4.4.2, we can establish the following theorems. We omit the proofs, since they follow the same pattern.

Theorem 5.21. *The infinitesimals of two equivalent 1-parameter formal deformations of* $(\mathfrak{g},\mu,R)$ *are in the same cohomology class in* $\mathrm{H}^{\bullet}_{\mathsf{mRB3\text{-}Lie}^{\lambda}}(\mathfrak{g})$.

Theorem 5.22. *Let* $(\mathfrak{g},\mu,R)$ *be a modified Rota–Baxter* 3*-Lie algebra of weight* λ. *If* $\mathrm{H}^{2}_{\mathsf{mRB3\text{-}Lie}^{\lambda}}(\mathfrak{g}) = 0$, *then* $(\mathfrak{g},\mu,R)$ *is rigid.*

Chapter 6

$L_\infty[1]$-structure for (relative and absolute) modified Rota–Baxter 3-Lie algebras

6.1 $L_\infty[1]$-structure for relative modified Rota–Baxter 3-Lie algebras

In this section, by using the derived bracket technique, we construct an $L_\infty[1]$-algebra whose Maurer-Cartan elements are in bijection with the set of structures of relative modified Rota–Baxter 3-Lie algebras of weight λ.

Theorem 6.1. *[79] The graded vector space $\mathrm{C}^\bullet_{\text{3-Lie}}(\mathfrak{g},\mathfrak{g})$ equipped with the graded commutator bracket*

$$[P,Q]_{\mathrm{R}} = P\circ Q-(-1)^{pq}Q\circ P,\quad \forall P\in \mathrm{C}^p_{\text{3-Lie}}(\mathfrak{g},\mathfrak{g}), Q\in \mathrm{C}^q_{\text{3-Lie}}(\mathfrak{g},\mathfrak{g}),$$

is a graded Lie algebra, where $P\circ Q\in \mathrm{C}^{p+q}_{\text{3-Lie}}(\mathfrak{g},\mathfrak{g})$ is defined by

$$\begin{aligned}
&(P\circ Q)\left(\mathfrak{X}_1,\cdots,\mathfrak{X}_{p+q},x\right)\\
=&\sum_{k=1}^{p}(-1)^{(k-1)q}\sum_{\sigma\in\mathbb{S}(k-1,q)}(-1)^{\sigma}P\Big(\mathfrak{X}_{\sigma(1)},\cdots,\mathfrak{X}_{\sigma(k-1)},\\
&\qquad Q\left(\mathfrak{X}_{\sigma(k)},\cdots,\mathfrak{X}_{\sigma(k+q-1)},x_{k+q}\right)\wedge y_{k+q},\mathfrak{X}_{k+q+1},\cdots,\mathfrak{X}_{p+q},x\Big)\\
&+\sum_{k=1}^{p}(-1)^{(k-1)q}\sum_{\sigma\in\mathbb{S}(k-1,q)}(-1)^{\sigma}P\Big(\mathfrak{X}_{\sigma(1)},\cdots,\mathfrak{X}_{\sigma(k-1)},\\
&\qquad x_{k+q}\wedge Q\left(\mathfrak{X}_{\sigma(k)},\cdots,\mathfrak{X}_{\sigma(k+q-1)},y_{k+q}\right),\mathfrak{X}_{k+q+1},\cdots,\mathfrak{X}_{p+q},x\Big)\\
&+\sum_{\sigma\in\mathbb{S}(p,q)}(-1)^{pq}(-1)^{\sigma}P\Big(\mathfrak{X}_{\sigma(1)},\cdots,\mathfrak{X}_{\sigma(p)},\\
&\qquad Q\left(\mathfrak{X}_{\sigma(p+1)},\cdots,\mathfrak{X}_{\sigma(p+q-1)},\mathfrak{X}_{\sigma(p+q)},x\right)\Big),
\end{aligned}$$

for all $\mathfrak{X}_i=x_i\wedge y_i\in\wedge^2\mathfrak{g}, i=1,2,\cdots,p+q$ and $x\in\mathfrak{g}$. Moreover, $\mu:\wedge^3\mathfrak{g}\longrightarrow\mathfrak{g}$ is a 3-Lie bracket if and only if $\mu\in \mathrm{C}^2_{\text{3-Lie}}(\mathfrak{g},\mathfrak{g})$ such that $[\mu,\mu]_{\mathrm{R}}=0$.

Proposition 6.1. *Denote the graded vector space* $\mathrm{C}^{\bullet}_{3\mathrm{Lie-act}}(\mathfrak{h},\mathfrak{g})$ *by*

$$\bigoplus_{n\geq 0}\bigoplus_{i_1,\dots,i_n\in\{0,1\}}\begin{pmatrix}\mathrm{Hom}_{\mathbf{k}}\big(V_{i_1}\otimes\cdots\otimes V_{i_n}\otimes\big(\wedge^3\mathfrak{g}\oplus(\wedge^2\mathfrak{h}\otimes\mathfrak{g})\big),\mathfrak{g}\big)\\ \oplus\ \mathrm{Hom}_{\mathbf{k}}\big(V_{i_1}\otimes\cdots\otimes V_{i_n}\otimes\big(\wedge^3\mathfrak{h}\oplus(\wedge^2\mathfrak{g}\otimes\mathfrak{h})\big),\mathfrak{h}\big)\end{pmatrix}$$

where $V_0=\wedge^2\mathfrak{h}$ *and* $V_1=\wedge^2\mathfrak{g}$*. Then* $V_{r\mathrm{Pair}}:=\mathrm{C}^{\bullet}_{3\mathrm{Lie-act}}(\mathfrak{h},\mathfrak{g})$ *is a subalgebra of the graded Lie algebra*

$$(\mathrm{C}^{\bullet}_{3\text{-Lie}}(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{g}\oplus\mathfrak{h}),[\cdot,\cdot]_{\mathrm{R}}).$$

Moreover, a quadruple

$$((\mathfrak{h},\mu),(\mathfrak{g},\pi),\rho,\zeta)$$

forms a relative 3*-Lie algebra pair if and only if* $\delta:=\pi+\rho+\mu+\zeta\in(V_{r\mathrm{Pair}})^1$ *satisfies*

$$[\delta,\delta]_{\mathrm{R}}=0,$$

where

$$\pi\in\mathrm{Hom}_{\mathbf{k}}(\wedge^3\mathfrak{g},\mathfrak{g}),\quad \mu\in\mathrm{Hom}_{\mathbf{k}}(\wedge^3\mathfrak{h},\mathfrak{h}),\quad \rho\in\mathrm{Hom}_{\mathbf{k}}(\wedge^2\mathfrak{g}\otimes\mathfrak{h},\mathfrak{h}),\quad \zeta\in\mathrm{Hom}_{\mathbf{k}}(\wedge^2\mathfrak{h}\otimes\mathfrak{g},\mathfrak{g}).$$

Proof. Denote

$$\bigoplus_{n\geq 0}\bigoplus_{i_1,\dots,i_n\in\{0,1\}}\big(\mathrm{Hom}_{\mathbf{k}}(V_{i_1}\otimes\cdots\otimes V_{i_n}\otimes\big((\wedge^3\mathfrak{g})\oplus(\wedge^2\mathfrak{h}\otimes\mathfrak{g})\big),\mathfrak{g})\big)$$

by $V_{\mathfrak{g}}$ and

$$\bigoplus_{n\geq 0}\bigoplus_{i_1,\dots,i_n\in\{0,1\}}\big(\mathrm{Hom}_{\mathbf{k}}(V_{i_1}\otimes\cdots\otimes V_{i_n}\otimes\big((\wedge^3\mathfrak{h})\oplus(\wedge^2\mathfrak{g}\otimes\mathfrak{h})\big),\mathfrak{h})\big)$$

by $V_{\mathfrak{h}}$, respectively. For each $P\in V_{\mathfrak{g}}$, $Q\in V_{\mathfrak{h}}$ and $R\in V_{r\mathrm{Pair}}$, we can show that $P\circ R\in V_{\mathfrak{g}}$ and $Q\circ R\in V_{\mathfrak{h}}$. Thus, $V_{r\mathrm{Pair}}$ is a subalgebra of $\mathrm{C}^{\bullet}_{3\mathrm{Lie-act}}(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{g}\oplus\mathfrak{h})$.

For all $\delta\in(V_{r\mathrm{Pair}})^1$, can decompose δ as $\delta=\pi+\rho+\mu+\zeta$, where

$$\pi\in\mathrm{Hom}_{\mathbf{k}}(\wedge^3\mathfrak{g},\mathfrak{g}),$$

$$\mu\in\mathrm{Hom}_{\mathbf{k}}(\wedge^3\mathfrak{h},\mathfrak{h}),$$

$$\rho\in\mathrm{Hom}_{\mathbf{k}}(\wedge^2\mathfrak{g}\otimes\mathfrak{h},\mathfrak{h})$$

and $\zeta\in\mathrm{Hom}_{\mathbf{k}}(\wedge^2\mathfrak{h}\otimes\mathfrak{g},\mathfrak{g})$. Since $[\delta,\delta]_{\mathrm{R}}=0$, we have

$$[\mu,\mu]_{\mathrm{R}}=0,\ [\pi,\pi]_{\mathrm{R}}=0,\ \rho\circ\rho=\rho\circ\pi \text{ and } \zeta\circ\zeta=\zeta\circ\mu.$$

By Theorem 6.1, μ and π define 3-Lie brackets on $\mathfrak{h}$ and $\mathfrak{g}$, respectively. $\rho\circ\rho=\rho\circ\pi$ is equivalent to that ρ and π satisfy the Equations (2.1)(2.2). Therefore, $\rho:\wedge^2\mathfrak{g}\otimes\mathfrak{h}\to\mathfrak{h}$ is action of $(\mathfrak{g},\pi)$ on $(\mathfrak{h},\mu)$. Similarly, $\mu:\wedge^2\mathfrak{h}\otimes\mathfrak{g}\to\mathfrak{g}$ is action of $(\mathfrak{h},\mu)$ on $(\mathfrak{g},\pi)$.

Vice versa. □

By Theorem 2.1 and Proposition 6.1, we can construct the following $L_\infty[1]$-algebra.

Proposition 6.2. *Let* $\mathfrak{g}$ *and* $\mathfrak{h}$ *are two vector space. Then we have a* V*-data* $(V,\mathfrak{a},\mathscr{P},\Delta)$ *as follows:*

(a) the graded Lie algebra $(V,[-,-])$ *is given by* $\big(C^{*}_{3\text{-Lie}}(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{g}\oplus\mathfrak{h}),[\cdot,\cdot]_{\mathrm{R}}\big)$*;*

(b) the abelian graded Lie subalgebra $\mathfrak{a}$ *is given by*

$$\mathfrak{a} = \mathrm{C}^\bullet_{\text{3-Lie}}(\mathfrak{h},\mathfrak{g}) = \oplus_{n\geq 0}\mathrm{C}^n_{\text{3-Lie}}(\mathfrak{h},\mathfrak{g}) = \bigoplus_{n\geq 0}\mathrm{Hom}_{\mathbf{k}}\left(\underbrace{\wedge^2\mathfrak{h}\otimes\cdots\otimes\wedge^2\mathfrak{h}}_{n\ times}\wedge\mathfrak{h},\mathfrak{g}\right);$$

(c) $\mathscr{P}: L\to L$ *is the projection onto the subspace* $\mathfrak{a}$*;*

(d) $\Delta = 0\in\ker(\mathscr{P})^1$.

We obtain an L_∞*-algebra* $(V_{r\mathrm{Pair}}[1]\oplus\mathfrak{a},\{l_k\}_{k\geq 1})$*, where*

$$l_2^{m\mathrm{RB}}(sf\otimes sg) = s[f,g]_{\mathrm{R}}$$
$$l_k^{m\mathrm{RB}}(sf\otimes\theta_1\otimes\ldots\otimes\theta_{k-1}) = \mathscr{P}\left[\cdots[[f,\theta_1]_{\mathrm{R}},\theta_2]_{\mathrm{R}},\ldots,\theta_{k-1}\right]_{\mathrm{R}},\ for\ k\geq 2$$

for all $\theta_1,\ldots,\theta_{k-1}\in\mathfrak{a}$ *and* $f,g\in V_{r\mathrm{Pair}}$.

Theorem 6.2. *With all the above notations. Suppose there are maps*

$$\pi\in\mathrm{Hom}_{\mathbf{k}}(\wedge^3\mathfrak{g},\mathfrak{g}),$$
$$\mu\in\mathrm{Hom}_{\mathbf{k}}(\wedge^3\mathfrak{h},\mathfrak{h}),$$
$$\rho\in\mathrm{Hom}_{\mathbf{k}}(\wedge^2\mathfrak{g}\otimes\mathfrak{h},\mathfrak{h}),$$
$$\zeta\in\mathrm{Hom}_{\mathbf{k}}(\wedge^2\mathfrak{h}\otimes\mathfrak{g},\mathfrak{g}),$$
$$T\in\mathrm{Hom}_{\mathbf{k}}(\mathfrak{h},\mathfrak{g})$$

and $\delta = \pi+\rho+\lambda(\mu+\zeta)$ *with nonzero* λ.

Then

$$(\delta[1],T)\in(V_{r\mathrm{Pair}}[1]\oplus\mathfrak{a})^0$$

is a Maurer-Cartan element in $L_\infty[1]$*-algebra*

$$(V_{r\mathrm{Pair}}[1]\oplus\mathfrak{a},\{l_k^{m\mathrm{RB}}\}_{k\geq 1})$$

if and only

$$((\mathfrak{h},\pi),(\mathfrak{g},\mu),\rho,\zeta,T)$$

is a relative modified Rota–Baxter 3*-Lie algebra structure.*

Proof. Let $(\delta[1],T)\in(V_{r\mathrm{Pair}})^0$. Then δ can be decomposed as $\delta = \pi+\rho+\lambda(\mu+\zeta)$ and we have that

$$l_2^{m\mathrm{RB}}\big((\delta[1],T)\otimes(\delta[1],T)\big) = ([\delta,\delta]_{\mathrm{R}}[1],2\mathscr{P}[\delta,T]_{\mathrm{R}}),$$
$$l_4^{m\mathrm{RB}}\big((\delta[1],T)\otimes(\delta[1],T)\otimes\delta[1],T)\otimes(\delta[1],T)\big)$$
$$= (0,24\mathscr{P}[[[\delta,T]_{\mathrm{R}},T]_{\mathrm{R}},T]_{\mathrm{R}}),$$

where for all $(u,v,w)\in\wedge^2\mathfrak{h}$

$$\mathscr{P}[\delta,T]_{\mathrm{R}}(u,v,w)$$
$$= -\lambda T\mu(u,v,w)+\lambda\left(\zeta(u,v)Tw+\zeta(v,w)Tu+\zeta(w,u)Tv\right),$$
$$\mathscr{P}[[[\delta,T]_{\mathrm{R}},T]_{\mathrm{R}},T]_{\mathrm{R}}(u,v,w)$$
$$= \pi(Tu,Tv,Tw)-T\left(\rho(Tu,Tv)w+\rho(Tv,Tw)u+\rho(Tw,Tu)v\right).$$

Then, the element $(\delta[1],T)\in(V_{r\text{Pair}})^0$ is a Maurer-Cartan element in $(V_{r\text{Pair}}[1]\oplus\mathfrak{a},\{l_k^{m\text{RB}}\}_{k\geq 1})$, if and only if

$$\begin{aligned}
&\sum_{k=1}^{\infty}\frac{1}{k!}l_k^{m\text{RB}}\Big((\delta[1],T)\otimes\cdots\otimes(\delta[1],T)\Big)\\
&=\frac{1}{2!}l_2^{m\text{RB}}\big((\delta[1],T)\otimes(\delta[1]\otimes T)+\frac{1}{4!}l_4^{m\text{RB}}\big((\delta[1],T)\otimes(\delta[1],T)\otimes(\delta[1],T),(\delta[1],T)\big)\\
&=0,
\end{aligned}$$

that is,

$$\Big(\frac{1}{2!}[\delta,\delta]_{\text{R}}[1],\frac{1}{2!}\mathscr{P}[\delta,T]_{\text{R}}+\frac{1}{4!}\mathscr{P}[[[\delta,T]_{\text{R}},T]_{\text{R}},T]_{\text{R}}\Big)=0. \tag{1.1}$$

By the first component of Equation 1.1, we have

$$[\delta,\delta]_{\text{R}}=0.$$

Thus, by Proposition 6.1, this implies that the quadruple $((\mathfrak{h},\mu),(\mathfrak{g},\pi),\rho,\zeta)$ defines a relative 3-Lie algebra pair structure.

From the second component, we obtain

$$\begin{aligned}
\pi(Tu,Tv,Tw)=&T\left(\rho(Tu,Tv)w+\rho(Tv,Tw)u+\rho(Tw,Tu)v\right)\\
&\lambda\left(-\zeta(u,v)Tw-\zeta(v,w)Tu-\zeta(w,u)Tv+T\mu(u,v,w)\right),
\end{aligned}$$

which shows that T is a relative modified Rota–Baxter operator of weight λ on the relative 3-Lie algebra pair $((\mathfrak{h},\mu),(\mathfrak{g},\pi),\rho,\zeta)$.

Therefore, the element $(\delta[1],T)\in(V_{r\text{Pair}})^0$ is a Maurer-Cartan element if and only if $((\mathfrak{h},\pi),(\mathfrak{g},\mu),\rho,\zeta,T)$ is defines a relative modified Rota–Baxter 3-Lie algebra structure. □

6.2 $L_\infty[1]$-structure for (absolute) modified Rota–Baxter 3-Lie algebras

In this subsection, we will construct an $L_\infty[1]$-structure whose Maurer-Cartan elements are in bijection with the set of structures of modified Rota–Baxter 3-Lie algebras of weight λ. The cohomology induced by the Maurer-Cartan element of this $L_\infty[1]$-algebra constructed via the twisting procedure in Proposition 2.8 coincide with the cohomologies associated to the modified Rota–Baxter algebras, as described in Definition 5.12.

Let $\mathfrak{g}$ be a vector space. Consider another copy of $\mathfrak{g}$, denoted by $\mathfrak{g}'$, and set $\mathfrak{h}:=\mathfrak{g}'$. Define the following cochain complexes:

$$L:=\mathrm{C}^\bullet_{3\text{-Lie}}(\mathfrak{g}\oplus\mathfrak{g}',\,\mathfrak{g}\oplus\mathfrak{g}'),\qquad V_{r\text{Pair}}:=\mathrm{C}^\bullet_{3\text{-Lie}}(\mathfrak{g}',\,\mathfrak{g}),$$

$$\mathfrak{a}:=\mathrm{C}^\bullet_{3\text{-Lie}}(\mathfrak{g}',\,\mathfrak{g})=\bigoplus_{n\geq 0}\mathrm{Hom}_{\mathbf{k}}\left(\underbrace{\wedge^2\mathfrak{g}'\otimes\cdots\otimes\wedge^2\mathfrak{g}'}_{n\text{ times}}\wedge\mathfrak{g}',\,\mathfrak{g}\right).$$

As shown in Section 6.1, the graded vector space

$$V_{r\text{Pair}}[1]\oplus\mathfrak{a}$$

admits an $L_\infty[1]$-algebra structure, with higher operations denoted by $\{l_k^{m\mathrm{RB}}\}_{k\geq 1}$. More precisely, we have the decomposition

$$V_{r\mathrm{Pair}} = \mathfrak{V}_1 \oplus \mathfrak{V}_2 \oplus \mathfrak{V}_3 \oplus \mathfrak{V}_4,$$

where:

$$\begin{aligned}
\mathfrak{V}_1 &= \bigoplus_{n\geq 0} \mathfrak{V}_1(n), \quad \mathfrak{V}_1(n+1) = \bigoplus_{i_1,\dots,i_n\in\{0,1\}} \mathrm{Hom}_{\mathbf{k}}\left(V_{i_1}\otimes\cdots\otimes V_{i_n}\otimes\wedge^3\mathfrak{g},\, \mathfrak{g}\right),\\
\mathfrak{V}_2 &= \bigoplus_{n\geq 0} \mathfrak{V}_2(n), \quad \mathfrak{V}_2(n+1) = \bigoplus_{i_1,\dots,i_n\in\{0,1\}} \mathrm{Hom}_{\mathbf{k}}\left(V_{i_1}\otimes\cdots\otimes V_{i_n}\otimes\wedge^2\mathfrak{g}'\otimes\mathfrak{g},\, \mathfrak{g}\right),\\
\mathfrak{V}_3 &= \bigoplus_{n\geq 0} \mathfrak{V}_3(n), \quad \mathfrak{V}_3(n+1) = \bigoplus_{i_1,\dots,i_n\in\{0,1\}} \mathrm{Hom}_{\mathbf{k}}\left(V_{i_1}\otimes\cdots\otimes V_{i_n}\otimes\wedge^3\mathfrak{g}',\, \mathfrak{g}'\right),\\
\mathfrak{V}_4 &= \bigoplus_{n\geq 0} \mathfrak{V}_4(n), \quad \mathfrak{V}_4(n+1) = \bigoplus_{i_1,\dots,i_n\in\{0,1\}} \mathrm{Hom}_{\mathbf{k}}\left(V_{i_1}\otimes\cdots\otimes V_{i_n}\otimes\wedge^2\mathfrak{g}\otimes\mathfrak{g}',\, \mathfrak{g}'\right),
\end{aligned}$$

with $V_0 := \wedge^2\mathfrak{g}'$ and $V_1 := \wedge^2\mathfrak{g}$.

Denote

$$V_{\mathrm{Pair}} := \bigoplus_{n\geq 0} V_{\mathrm{Pair}}(n) \text{ and } V_{\mathrm{Pair}}(n) := \mathrm{Hom}_{\mathbf{k}}\left(\underbrace{\wedge^2\mathfrak{g}\otimes\cdots\otimes\wedge^2\mathfrak{g}}_{n \text{ times}}\wedge\mathfrak{g},\mathfrak{g}\right).$$

Consider the embedding map of graded spaces, defined for each $n\geq 1$ by

$$\begin{aligned}
\nu\colon V_{\mathrm{Pair}}(n) &\longrightarrow V_{r\mathrm{Pair}}(n) = \mathfrak{V}_1(n)\oplus\mathfrak{V}_2(n)\oplus\mathfrak{V}_3(n)\oplus\mathfrak{V}_4(n),\\
f &\longmapsto \sum_{i_1,\dots,i_{n-1},i_n,j\in\{0,1\}} f^j_{i_1,\dots,i_{n-1},i_n},
\end{aligned}$$

where the components are defined as follows:

For $x_1\wedge y_1\in V_{i_1},\dots,x_{n-1}\wedge y_{n-1}\in V_{i_{n-1}}$ and:

(i) $x,y,z\in\mathfrak{g}$, define $f^1_{i_1,\dots,i_{n-1},1}\in\mathfrak{V}_1(n)$ by

$$f^1_{i_1,\dots,i_{n-1},1}(x_1\wedge y_1\otimes\cdots\otimes x_n\wedge y_n\otimes x\wedge y\wedge z) := f(x_1\wedge y_1\otimes\cdots\otimes x_n\wedge y_n\otimes x\wedge y\wedge z),$$

(ii) $x,y\in\mathfrak{g}'$ and $z\in\mathfrak{g}$, define $f^1_{i_1,\dots,i_{n-1},0}\in\mathfrak{V}_2(n)$ by

$$f^1_{i_1,\dots,i_{n-1},0}(x_1\wedge y_1\otimes\cdots\otimes x_n\wedge y_n\otimes x\wedge y\otimes z) := f(x_1\wedge y_1\otimes\cdots\otimes x_n\wedge y_n\otimes x\wedge y\wedge z),$$

(iii) $x,y,z\in\mathfrak{g}'$, define $f^0_{i_1,\dots,i_{n-1},0}\in\mathfrak{V}_3(n)$ by

$$f^0_{i_1,\dots,i_{n-1},0}(x_1\wedge y_1\otimes\cdots\otimes x_n\wedge y_n\otimes x\wedge y\wedge z) := f(x_1\wedge y_1\otimes\cdots\otimes x_n\wedge y_n\otimes x\wedge y\wedge z),$$

(iv) $x,y\in\mathfrak{g}$ and $z\in\mathfrak{g}'$, define $f^0_{i_1,\dots,i_{n-1},1}\in\mathfrak{V}_4(n)$ by

$$f^0_{i_1,\dots,i_{n-1},1}(x_1\wedge y_1\otimes\cdots\otimes x_n\wedge y_n\otimes x\wedge y\otimes z) := f(x_1\wedge y_1\otimes\cdots\otimes x_n\wedge y_n\otimes x\wedge y\wedge z).$$

Proposition 6.3. *The embedding map $\nu\colon V_{\mathrm{Pair}}\to V_{r\mathrm{Pair}}$ is a monomorphism of graded Lie algebra.*

Proof. For $f \in V_{\text{Pair}}(n)$ and $g \in V_{\text{Pair}}(m)$, we compute:

$$\begin{aligned}[\nu(f),\nu(g)]_{\mathrm{R}} &= \left[\sum_{i_1,\dots,i_n,\, i\in\{0,1\}} f^i_{i_1,\dots,i_n},\ \sum_{j_1,\dots,j_m,\, j\in\{0,1\}} g^j_{j_1,\dots,j_m}\right]_{\mathrm{R}} \\ &= \sum_{i_1,\dots,i_{m+n},\, i,j\in\{0,1\}} \left(f^i_{i_1,\dots,i_n}\circ g^j_{i_{n+1},\dots,i_{m+n}} - (-1)^{mn} g^i_{i_1,\dots,i_m}\circ f^j_{i_{m+1},\dots,i_{m+n}}\right),\end{aligned}$$

where, for each $i_1,\dots,i_{m+n-1}\in\{0,1\}$, the components lie in the following spaces:

$$\sum_{i\in\{0,1\}} \left(f^i_{i_1,\dots,i_n}\circ g^1_{i_{n+1},\dots,i_{m+n-1},1} - (-1)^{mn} g^i_{i_1,\dots,i_m}\circ f^1_{i_{m+1},\dots,i_{m+n-1},1}\right) \in \mathrm{Hom}_{\mathbf{k}}\left(V_{i_1}\otimes\cdots\otimes V_{i_{m+n-1}}\otimes\wedge^3\mathfrak{g},\, \mathfrak{g}\right),$$

$$\sum_{i\in\{0,1\}} \left(f^i_{i_1,\dots,i_n}\circ g^1_{i_{n+1},\dots,i_{m+n-1},0} - (-1)^{mn} g^i_{i_1,\dots,i_m}\circ f^1_{i_{m+1},\dots,i_{m+n-1},0}\right) \in \mathrm{Hom}_{\mathbf{k}}\left(V_{i_1}\otimes\cdots\otimes V_{i_{m+n-1}}\otimes\wedge^2\mathfrak{g}'\otimes\mathfrak{g},\, \mathfrak{g}\right),$$

$$\sum_{i\in\{0,1\}} \left(f^i_{i_1,\dots,i_n}\circ g^0_{i_{n+1},\dots,i_{m+n-1},0} - (-1)^{mn} g^i_{i_1,\dots,i_m}\circ f^0_{i_{m+1},\dots,i_{m+n-1},0}\right) \in \mathrm{Hom}_{\mathbf{k}}\left(V_{i_1}\otimes\cdots\otimes V_{i_{m+n-1}}\otimes\wedge^3\mathfrak{g}',\, \mathfrak{g}'\right),$$

$$\sum_{i\in\{0,1\}} \left(f^i_{i_1,\dots,i_n}\circ g^0_{i_{n+1},\dots,i_{m+n-1},1} - (-1)^{mn} g^i_{i_1,\dots,i_m}\circ f^0_{i_{m+1},\dots,i_{m+n-1},1}\right) \in \mathrm{Hom}_{\mathbf{k}}\left(V_{i_1}\otimes\cdots\otimes V_{i_{m+n-1}}\otimes\wedge^2\mathfrak{g}\otimes\mathfrak{g}',\, \mathfrak{g}'\right).$$

Thus, we have

$$[\nu(f),\nu(g)]_{\mathrm{R}} = \nu\left(f\circ g - (-1)^{mn} g\circ f\right)[1] = \nu\left([f,g]_{\mathrm{R}}\right),$$

that is,

$$\nu\colon V_{\text{Pair}} \to V_{r\text{Pair}}$$

is a monomorphism of graded Lie algebras. □

By Theorem 2.1 and Proposition 6.3, we have the following Proposition:

Proposition 6.4.

$$\left(sV_{\text{Pair}}\oplus\mathfrak{a},\{l^{\text{ab}}_k\}_{k\geq 1}\right)$$

is an $L_\infty[1]$-algebra with operations

$$\begin{aligned} &l^{\text{ab}}_2(sf\otimes sg) = s\nu^{-1}[\nu(f),\nu(g)]_{\mathrm{R}},\\ &l^{\text{ab}}_k(sf\otimes\theta_1\otimes\cdots\otimes\theta_{k-1}) = \mathscr{P}\left[\cdots[[\nu(f),\theta_1]_{\mathrm{R}},\theta_2]_{\mathrm{R}},\dots,\theta_{k-1}\right]_{\mathrm{R}},\quad \textit{for } k\geq 2,\end{aligned}$$

for all $\theta_1,\dots,\theta_{k-1}\in\mathfrak{a}$ and $f,g\in V_{\text{Pair}}$.

Moreover, ν induces a monomorphism of $L_\infty[1]$-algebras:

$$\begin{aligned}\tilde{\nu}:\left(sV_{\text{Pair}}\oplus\mathfrak{a},\{l^{\text{ab}}_k\}_{k\geq 1}\right) &\longrightarrow \left(sV_{r\text{Pair}}\oplus\mathfrak{a},\{l^{m\text{RB}}_k\}_{k\geq 1}\right),\\ (sf,\theta) &\longmapsto (s\nu(f),\theta).\end{aligned}$$

Theorem 6.3. *With all the above notations. Suppose there are maps $\pi\in\mathrm{Hom}_{\mathbf{k}}(\wedge^3\mathfrak{g},\mathfrak{g})$, $T\in\mathrm{Hom}_{\mathbf{k}}(\mathfrak{g},\mathfrak{g})$ and λ is non-zero square.*

Then $(s\pi,\lambda^{-1/2}T)\in(V_{\text{Pair}}[1]\oplus\mathfrak{a})^0$ is a Maurer-Cartan element in $L_\infty[1]$-algebra $(sV_{\text{Pair}}\oplus\mathfrak{a},\{l^{\text{ab}}_k\}_{k\geq 1})$ if and only $((\mathfrak{g},\pi),\rho,\zeta,T)$ is a modified Rota–Baxter 3-Lie algebra structure.

Proof. Let $(s\pi,\lambda^{-1/2}T)\in(V_{r\text{Pair}})^0$. Then we have

$$\begin{aligned} l^{\text{ab}}_2\left((s\pi,\lambda^{-1/2}T)\otimes(s\pi,\lambda^{-1/2}T)\right) &= \left(\nu^{-1}s[\nu(\pi),\nu(\pi)]_{\mathrm{R}},\, 2\lambda^{-1/2}\mathscr{P}[\nu(\pi),T]_{\mathrm{R}}\right),\\ l^{\text{ab}}_4\left((s\nu(\pi),\lambda^{-1/2}T)\otimes\cdots\otimes(s\nu(\pi),\lambda^{-1/2}T)\right) &= \left(0,\, 24\lambda^{-3/2}\mathscr{P}[[[\nu(\pi),T]_{\mathrm{R}},T]_{\mathrm{R}},T]_{\mathrm{R}}\right).\end{aligned}$$

For all $(u,v,w) \in \wedge^2\mathfrak{h}$, we compute:

$$\mathscr{P}[\nu(\pi),T]_{\mathrm{R}}(u,v,w) = -T\pi(u,v,w) + (\pi(u,v,Tw) + \pi(v,w,Tu) + \pi(w,u,Tv)),$$
$$\mathscr{P}[[[\nu(\pi),T]_{\mathrm{R}},T]_{\mathrm{R}},T]_{\mathrm{R}}(u,v,w) = \pi(Tu,Tv,Tw) - T\,(\pi(Tu,Tv,w) + \pi(Tv,Tw,u) + \pi(Tw,Tu,v)).$$

Therefore, the element $(s\pi, \lambda^{-1/2}T) \in (V_{r\mathrm{Pair}})^0$ is a Maurer-Cartan element in the $L_\infty[1]$-algebra $(V_{\mathrm{Pair}}[1] \oplus \mathfrak{a}, \{l_k^{\mathrm{ab}}\}_{k\geq 1})$ if and only if

$$\sum_{k=1}^{\infty} \frac{1}{k!}\, l_k^{\mathrm{ab}}\left((s\pi, \lambda^{-1/2}T)^{\otimes k}\right) = \frac{1}{2} l_2^{\mathrm{ab}}\left((s\pi, \lambda^{-1/2}T)^{\otimes 2}\right) + \frac{1}{24} l_4^{\mathrm{ab}}\left((\pi[1], \lambda^{-1/2}T)^{\otimes 4}\right) = 0.$$

That is,

$$[\pi,\pi]_{\mathrm{R}} = 0,$$

and

$$\begin{aligned}\pi(Tu,Tv,Tw) = {} & T\,(\pi(Tu,Tv,w) + \pi(Tv,Tw,u) + \pi(Tw,Tu,v)) \\ & + \lambda\,(-\pi(u,v,Tw) - \pi(v,w,Tu) - \pi(w,u,Tv) + T\pi(u,v,w)).\end{aligned}$$

Hence, the element $(s\pi, \lambda^{-1/2}T) \in (V_{r\mathrm{Pair}})^0$ is a Maurer-Cartan element if and only if the triple $(\mathfrak{g}, \pi, T)$ defines a modified Rota–Baxter 3-Lie algebra structure. □

Remark 6.5. *Let R be a modified Rota–Baxter operator of weight λ on a 3-Lie algebra $(\mathfrak{g},\pi)$. Then the element*

$$\alpha := (s\pi,\, \lambda^{-1/2}T)$$

is a Maurer–Cartan element in the $L_\infty[1]$-algebra $(sV_{\mathrm{Pair}} \oplus \mathfrak{a}, \{l_k^{\mathrm{ab}}\}_{k\geq 1})$. Moreover, the cohomology of the cochain complex induced by α via the twisting procedure described in Proposition 2.8 coincides with the cohomology of the shifted cochain complex $s\mathrm{C}^\bullet_{\mathrm{mRB3\text{-}Lie}^\lambda}(\mathfrak{g})$ defined in Definition 5.12.

6.3 Comparison of the Controlling $L_\infty[1]$-structures for relative Modified Rota–Baxter and relative Rota–Baxter 3-Lie Algebras

Hou, Sheng, and Zhou [55] constructed the controlling $L_\infty[1]$-algebra structure for relative Rota–Baxter operators of weight λ on 3-Lie algebras. In this subsection, we will construct a controlling L_∞-subalgebra of the $L_\infty[1]$-algebra introduced in Section 6.1, which controls the deformation of relative modified Rota–Baxter 3-Lie algebras. This subalgebra simultaneously controls the deformations of both the relative Rota–Baxter operators and the underlying 3-Lie algebra structures.

Proposition 6.6. *Let $V_{\mathfrak{h}}$ be a subspace of $V_{r\mathrm{Pair}}$, and let $\iota : V_{\mathfrak{h}} \hookrightarrow V_{r\mathrm{Pair}}$ be the natural embedding. Then ι induces an L_∞-subalgebra structure $\{l_k^{\mathrm{RB}}\}_{k\geq 1}$ on $V_{\mathfrak{h}}[1] \oplus \mathfrak{a}$, as well as an L_∞-morphism*

$$\begin{aligned}\tilde{\iota} : \left(sV_{\mathfrak{h}} \oplus \mathfrak{a}, \{l_k^{\mathrm{RB}}\}_{k\geq 1}\right) &\longrightarrow \left(sV_{r\mathrm{Pair}} \oplus \mathfrak{a}, \{l_k^{m\mathrm{RB}}\}_{k\geq 1}\right) \\ (sf, \theta) &\longrightarrow (s\iota(f), \theta),\end{aligned}$$

where

$$l_2^{\mathrm{RB}}(sf \otimes sg) = s\iota^{-1}[\iota(f), \iota(g)]_{\mathrm{R}}$$
$$l_k^{\mathrm{RB}}\left(f[1] \otimes \theta_1 \otimes \ldots \otimes \theta_{k-1}\right) = \mathscr{P}\left[\cdots[[\iota(f), \theta_1]_{\mathrm{R}}, \theta_2]_{\mathrm{R}}, \ldots, \theta_{k-1}\right]_{\mathrm{R}}, \text{ for } k \geq 2$$

for all $\theta_1, \ldots, \theta_{k-1} \in \mathfrak{a}$ and $f, g \in V_{\mathfrak{h}}$.

Proof. By the proof of Proposition 6.1, we have that $V_{\mathfrak{h}}$ is also a subalgebra of the graded Lie algebra $V_{r\mathrm{Pair}}$. Thus, by Theorem 2.1, the bracket $\iota^{-1}[\iota(-),\ \iota(-)]$ is well-defined, and the map

$$\tilde{\iota}:\ \left(sV_{\mathfrak{h}}\oplus\mathfrak{a},\ \{l_k^{\mathrm{RB}}\}_{k\geq 1}\right)\longrightarrow\left(sV_{r\mathrm{Pair}}\oplus\mathfrak{a},\ \{l_k^{m\mathrm{RB}}\}_{k\geq 1}\right)$$

is an L_∞-morphism. □

Theorem 6.4. *With all the above notations. Suppose there are maps*

$$\pi\in\mathrm{Hom}_{\mathbf{k}}(\wedge^3\mathfrak{g},\mathfrak{g}),$$

$$\mu\in\mathrm{Hom}_{\mathbf{k}}(\wedge^3\mathfrak{h},\mathfrak{h}),$$

$$\rho\in\mathrm{Hom}_{\mathbf{k}}(\wedge^2\mathfrak{g}\otimes\mathfrak{h},\mathfrak{h}),$$

$$T\in\mathrm{Hom}_{\mathbf{k}}(\mathfrak{h},\mathfrak{g})$$

and $\delta=\pi+\rho+\lambda\mu$ with nonzero weight λ.

Then

$$(s\delta,T)\in(sV_{\mathfrak{h}}\oplus\mathfrak{a})^0$$

is a Maurer-Cartan element in $L_\infty[1]$-algebra

$$(sV_{\mathfrak{h}}\oplus\mathfrak{a},\{l_k^{\mathrm{RB}}\}_{k\geq 1})$$

if and only

$$((\mathfrak{h},\pi),(\mathfrak{g},\mu),\rho,T)$$

is a relative Rota–Baxter 3-Lie algebra structure.

Proof. As in the proof of Theorem 6.2, we omit the details here. □

Remark 6.7. *Let $\alpha=((\mathfrak{h},\pi),(\mathfrak{g},\mu),\rho,T)$ be a relative Rota–Baxter 3-Lie algebra with trivial T. By Theorem 6.4, α is also a Maurer-Cartan element in the $L_\infty[1]$-algebra $(sV_{\mathfrak{h}}\oplus\mathfrak{a},\{l_k^{\mathrm{RB}}\}_{k\geq 1})$.*

By Proposition 2.8, we obtain the twisted $L_\infty[1]$-algebra structure $\{(l^{\mathrm{RB}})_k^{\alpha}\}_{k\geq 1}$ on $sV_{\mathfrak{h}}\oplus\mathfrak{a}$ induced by the Maurer-Cartan element α. In fact, the subalgebra $\mathfrak{a}$ of the twisted $L_\infty[1]$-algebra

$$(sV_{\mathfrak{h}}\oplus\mathfrak{a},\{(l^{\mathrm{RB}})_k^{\alpha}\}_{k\geq 1})$$

coincides with the $L_\infty[1]$-algebra introduced in Section 4 of [55] by Hou, Sheng, and Zhou. This $L_\infty[1]$-algebra controls the deformation theory of relative Rota–Baxter operators of weight λ on 3-Lie algebras $(\mathfrak{h},\pi)$, $(\mathfrak{g},\mu)$ together with the action ρ.

Part II

From homotopy Rota–Baxter algebras to pre-Calabi-Yau and homotopy double Poisson algebras

Chapter 7

Preliminaries

7.1 Notations

Let $\mathbf{k}$ be a field of characteristic 0. A (homologically) *graded space* is a $\mathbb{Z}$-indexed family of $\mathbf{k}$-vector spaces $V = \{V_n\}_{n\in\mathbb{Z}}$. Elements of $\bigcup_{n\in\mathbb{Z}} V_n$ are called homogeneous and have a degree $|v| = n$ if $v \in V_n$.

Given two graded spaces V and W, a graded map of degree r is a linear map $f : V \to W$ such that $f(V_n) \subseteq W_{n+r}$ for all n, and we denote the degree of f by $|f| = r$. Define

$$\mathrm{Hom}(V,W)_r = \prod_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathbf{k}}(V_p, W_{p+r})$$

as the space of graded maps of degree r. The graded space $\mathrm{Hom}(V,W)$ is then given by $\{\mathrm{Hom}(V,W)_r\}_{r\in\mathbb{Z}}$.

The tensor product $V \otimes W$ of two graded spaces V and W is defined by

$$(V\otimes W)_n = \bigoplus_{p+q=n} V_p \otimes W_q.$$

We adopt Sweedler's notation for elements in tensor products of graded spaces. Let $V^1 \otimes \cdots \otimes V^n$ be the tensor product of graded spaces $V^1, \cdots, V^n$. An element r in this tensor product can be expressed as

$$r = \sum_{i_1,\cdots,i_n} r^{[1]}_{i_1} \otimes \cdots \otimes r^{[n]}_{i_n},$$

where $r^{[k]}_{i_k} \in V^k$. For simplicity, we omit the subscripts i_k and write:

$$r = \sum r^{[1]} \otimes \cdots \otimes r^{[n]}.$$

If V is a finite-dimensional graded space, there is an isomorphism of graded spaces:

$$\mathrm{Hom}(V,W) \cong W \otimes V^\vee.$$

Moreover, if both V and W are finite-dimensional graded spaces, we have the isomorphism:

$$\mathrm{End}_{\mathrm{gr}}(V\otimes W) \cong V \otimes W \otimes W^\vee \otimes V^\vee.$$

The *suspension* of a graded space V is the graded space sV, defined by $(sV)_n = V_{n-1}$ for all $n \in \mathbb{Z}$. For any $v \in V_{n-1}$, we denote the corresponding element in $(sV)_n$ by sv. The map $s : V \to sV$, defined by $v \mapsto sv$, is a graded map of degree 1.

Similarly, the *desuspension* of V, denoted $s^{-1}V$, is defined by $(s^{-1}V)_n = V_{n+1}$. For $v \in V_{n+1}$, the corresponding element in $(s^{-1}V)_n$ is written as $s^{-1}v$. The map $s^{-1} : V \to s^{-1}V$, given by $v \mapsto s^{-1}v$, is a graded map of degree -1.

Let V be a graded vector space. Define the graded symmetric algebra $\mathrm{Sym}(V)$ of V to be $\mathrm{T}(V)/I$ where the two-sided ideal I is generated by $x \otimes y - (-1)^{|x||y|} y \otimes x$ for all homogeneous elements $x, y \in V$. For homogeneous elements $x_1, \ldots, x_n \in V$ and $\sigma \in \mathfrak{S}_n$, the *Koszul sign* $\varepsilon(\sigma; x_1, \ldots, x_n)$ is defined by

$$x_1 \odot x_2 \odot \cdots \odot x_n = \varepsilon(\sigma; x_1, \ldots, x_n) x_{\sigma(1)} \odot x_{\sigma(2)} \odot \cdots \odot x_{\sigma(n)} \in \mathrm{Sym}(V), \tag{1.1}$$

where $\odot$ is the multiplication in $\mathrm{Sym}(V)$; For instance, $x \cdot y = (-1)^{|x||y|} y \cdot x$, so $\varepsilon((1\ 2), x, y) = (-1)^{|x||y|}$. we also define

$$\chi(\sigma; x_1, \ldots, x_n) = \mathrm{sgn}(\sigma) \varepsilon(\sigma; x_1, \ldots, x_n), \tag{1.2}$$

where $\mathrm{sgn}(\sigma)$ is the sign of the permutation σ.

Let $\mathfrak{S}_n$ denote the symmetric group on n elements, and let V be a graded space. The left action of $\mathfrak{S}_n$ on $V^{\otimes n}$ is defined as follows: for $\sigma \in \mathfrak{S}_n$ and any $r = \sum r^{[1]} \otimes \cdots \otimes r^{[n]} \in V^{\otimes n}$,

$$\sigma \cdot r = \sum \varepsilon(\sigma; r^{[1]}, \ldots, r^{[n]}) r^{[\sigma^{-1}(1)]} \otimes \cdots \otimes r^{[\sigma^{-1}(n)]},$$

where $\varepsilon(\sigma; r^{[1]}, \ldots, r^{[n]})$ is the Koszul sign obtained from permuting the graded elements $r^{[1]}, \ldots, r^{[n]}$. We write $\sigma^{-1} \cdot r$ as $r^{\sigma(1), \ldots, \sigma(n)}$.

For $0 \leqslant i_1, \ldots, i_r \leqslant n$ with $i_1 + \cdots + i_r = n$, let $\mathrm{Sh}(i_1, i_2, \ldots, i_r)$ denote the set of $(i_1, \ldots, i_r)$-shuffles, i.e., permutations $\sigma \in \mathfrak{S}_n$ such that:

$$\sigma(1) < \cdots < \sigma(i_1), \sigma(i_1+1) < \cdots < \sigma(i_1+i_2), \ldots, \sigma(i_1+\cdots+i_{r-1}+1) < \cdots < \sigma(n).$$

The following fact shows that shuffles can be viewed as representatives of left cosets of symmetric groups by Young subgroups:

Lemma 7.1. *Let $n \geqslant 1, 1 \leqslant i \leqslant n-1$. Then for any $\delta \in \mathfrak{S}_n$, there exists a unique triple (τ, σ, π) with $\sigma \in \mathrm{Sh}(i, n-i), \tau \in \mathfrak{S}_i, \pi \in \mathfrak{S}_{n-i}$ such that $\delta(l) = \sigma\tau(l)$ for $1 \leqslant l \leqslant i$, and $\delta(i+m) = \sigma(i+\pi(m))$ for $1 \leqslant m \leqslant n-i$.*

7.2 L_∞-algebras and homotopy Possion algebras

7.2.1 Differential graded Lie algebras

Now, let's recall some basics on differential graded Lie algebras.

Definition 7.2. Let $L = \oplus_{i \in \mathbb{Z}} L_i$ be a graded space. A *differential graded (dg) Lie algebra* structure on L consists of two operations

$$\tau_1 : L \to L \quad \tau_2 : L \otimes L \to L$$

with $|\tau_1| = -1, |\tau_2| = 0$ satisfying the following conditions.

(i) $\tau_1 \circ \tau_1 = 0$.

(ii) $\tau_1 \circ \tau_2 = \tau_2(\tau_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes \tau_1)$.

(iii) $\tau_2(x \otimes y) + (-1)^{|x||y|} \tau_2(y \otimes x) = 0$.

(iv) $\tau_2(\tau_2(x \otimes y) \otimes z) + (-1)^{|x|(|y|+|z|)} \tau_2(\tau_2(y \otimes z) \otimes x) + (-1)^{|z|(|x|+|y|)} \tau_2(\tau_2(z \otimes x) \otimes y)$.

Definition 7.3. Let (L, τ_1, τ_2) be a dg Lie algebra. An element $\alpha \in L_{-1}$ is called a *Maurer-Cartan element* if α satisfies

$$\tau_1(\alpha) - \frac{1}{2}\tau_2(\alpha \otimes \alpha) = 0.$$

Lemma 7.4. *Let (L, τ_1, τ_2) be a dg Lie algebra and α be a Maurer-Cartan element. Define operations*

$$\tau_1^\alpha(x) = \tau_1(x) - \tau_2(\alpha \otimes x),\ \tau_2^\alpha := -\tau_2.$$

Then $(L, \tau_1^\alpha, \tau_2^\alpha)$ is a dg Lie algebra.

7.2.2 L_∞-algebras and Maurer-Cartan elements

Now, let's recall some basics on L_∞-algebras.

Definition 7.5. Let $L = \bigoplus_{i \in \mathbb{Z}} L_i$ be a graded space over $\mathbf{k}$. Assume that L is endowed with a family of graded linear operators $l_n : L^{\otimes n} \to L, n \geqslant 1$ with $|l_n| = n - 2$ subject to the following conditions: for arbitrary $n \geqslant 1$, $\sigma \in \mathfrak{S}_n$ and $x_1, \ldots, x_n \in L$,

- (skew-symmetry)

$$l_n(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}) = \chi(\sigma; x_1, \ldots, x_n)\ l_n(x_1 \otimes \cdots \otimes x_n);$$

- (generalised Jacobi identity)

$$\sum_{i=1}^{n} \sum_{\sigma \in \mathrm{Sh}(i,n-i)} \chi(\sigma; x_1, \ldots, x_n)(-1)^{i(n-i)} l_{n-i+1}(l_i(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(i)}) \otimes x_{\sigma(i+1)} \otimes \cdots \otimes x_{\sigma(n)}) = 0,$$

  where $\mathrm{Sh}(i, n-i)$ is the set of $(i, n-i)$ shuffles, that is, permutations $\sigma \in \mathfrak{S}_n$ such that

$$\sigma(1) < \cdots < \sigma(i) \text{ and } \sigma(i+1) < \cdots < \sigma(n).$$

Then $(L, \{l_n\}_{n \geqslant 1})$ is called an L_∞*-algebra*.

Remark 7.6. *Let us consider the generalised Jacobi identity for $n \leqslant 3$ with the assumption of generalised skew-symmetry.*

(i) $n = 1$, $l_1 \circ l_1 = 0$*, that is,* l_1 *is a differential,*

(ii) $n = 2$, $l_1 \circ l_2 = l_2 \circ (l_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes l_1)$*, that is* l_1 *is a derivation for* l_2*,*

(iii) $n = 3$*, for homogeneous elements* $x_1, x_2, x_3 \in L$

$$\begin{aligned} & l_2(l_2(x_1 \otimes x_2) \otimes x_3) + (-1)^{|x_1|(|x_2|+|x_3|)} l_2(l_2(x_2 \otimes x_3) \otimes x_1) \\ & +(-1)^{|x_3|(|x_1|+|x_2|)} l_2(l_2(x_3 \otimes x_1) \otimes x_2) \\ = \ & -\Big(l_1(l_3(x_1 \otimes x_2 \otimes x_3)) + l_3(l_1(x_1) \otimes x_2 \otimes x_3) + (-1)^{|x_1|} l_3(x_1 \otimes l_1(x_2) \otimes x_3) \\ & +(-1)^{|x_1|+|x_2|} l_3(x_1 \otimes x_2 \otimes l_1(x_3))\Big), \end{aligned}$$

that is, l_2 *satisfies the Jacobi identity up to homotopy.*

In particular, if all $l_n = 0$ *with* $n \geqslant 3$*, then* (L, l_1, l_2) *is just a dg Lie algebra.*

One can also define Maurer-Cartan elements in L_∞-algebras.

Definition 7.7. Let $(L,\{l_n\}_{n\geqslant 1})$ be an L_∞-algebra. An element $\alpha\in L_{-1}$ is called a *Maurer-Cartan element* if it satisfies the Maurer-Cartan equation:

$$\sum_{n=1}^{\infty}\frac{1}{n!}(-1)^{\frac{n(n-1)}{2}}l_n(\alpha^{\otimes n})=0, \tag{2.3}$$

whenever this infinite sum exists.

Proposition 7.8 (Twisting procedure)**.** *Given a Maurer-Cartan element α in L_∞-algebra L, one can introduce a new L_∞-structure $\{l_n^\alpha\}_{n\geqslant 1}$ on graded space L, where $l_n^\alpha: L^{\otimes n}\to L$ is defined as :*

$$l_n^\alpha(x_1\otimes\cdots\otimes x_n)=\sum_{i=0}^{\infty}\frac{1}{i!}(-1)^{in+\frac{i(i-1)}{2}}l_{n+i}(\alpha^{\otimes i}\otimes x_1\otimes\cdots\otimes x_n),\ \forall x_1,\dots,x_n\in L, \tag{2.4}$$

whenever these infinite sums exist. The new L_∞-algebra $(L,\{l_n^\alpha\}_{n\geqslant 1})$ is called the twisted L_∞-algebra (by the Maurer-Cartan element α).

Definition 7.9. A *homotopy Poisson algebra* (also called a *derived Poisson algebra*) is a graded vector space L equipped with both an L_∞-algebra structure $\{l_n\}_{n\geqslant 1}$ and a graded commutative associative algebra structure, such that the following Leibniz$_\infty$ rule holds: for all $n\geqslant 1$ and $x_1,\dots,x_{n-1},x'_n,x''_n\in L$,

$$l_n(x_1\otimes\cdots\otimes x'_n\cdot x''_n)=l_n(x_1\otimes\cdots\otimes x'_n)\cdot x''_n+(-1)^{|x'_n|(\sum_{i=1}^{n-1}|x_i|+n-2)}x'_n\cdot l_n(x_1\otimes\cdots\otimes x_{n-1}\otimes x''_n).$$

Proposition 7.10. *Let $(L,\{l_n\}_{n\geqslant 1})$ be an L_∞-algebra. Define a family of operations $\{\tilde{l}_n\}_{n\geqslant 1}$on the graded symmetric algebra* Sym(L) *as follows: for all homogeneous elements $x_1^1,\cdots,x_{k_1}^1,\cdots,x_1^n,\cdots,x_{k_n}^n\in L$*

$$\tilde{l}_n(x_1^1\cdots x_{k_1}^1\otimes\cdots\otimes x_1^n\cdots x_{k_n}^n):=\sum_{1\leqslant q_1\leqslant k_1,\cdots 1\leqslant q_n\leqslant k_n}(-1)^{\sum_{s=1}^{n}\left(\sum_{t=1}^{s-1}(\sum_{j=1}^{q_t-1}|x_j|+\sum_{j=q_t+1}^{k_t}|x_j|)+\sum_{j=1}^{q_s-1}|x_j|\right)|x_{q_s}|}$$
$$l_n(x_{q_1}^1\otimes\cdots\otimes u_{q_n}^n)\cdot x_1^1\cdots\widehat{x_{q_1}^1}\cdots x_{k_1}^1\cdots x_1^n\cdots\widehat{x_{q_n}^n}\cdots x_{k_n}^n.$$

Then $\left(\mathrm{Sym}(L),\{l_n\}_{n\geqslant 1}\right)$ defines a homotopy Poisson algebra.

7.3 A_∞-algebras and Pre-Calabi-Yau algebras

7.3.1 A_∞-algebras and A_∞-bimodules

We first recall some basics on A_∞-algebras and A_∞-bimodules, following [59, 61].

Definition 7.11. Let $A=\oplus_{n\in\mathbb{Z}}A_n$ be a graded vector space. If A is equipped with a family of homogeneous linear maps $\{m_n: A^{\otimes n}\to A\}_{n\geqslant 1}$, with $|m_n|=n-2$ satisfying the Stasheff identity: for all $n\geqslant 1$,

$$\sum_{\substack{i+j+k=n,\\ i,k\geqslant 0,j\geqslant 1}}(-1)^{i+jk}m_{i+1+k}\circ\left(\mathrm{Id}^{\otimes i}\otimes m_j\otimes\mathrm{Id}^{\otimes k}\right)=0, \tag{3.5}$$

then $(A,\{m_n\}_{n\geqslant 1})$ is called an *A_∞-algebra*.

Remark 7.12. *For small n, one can write down Equation* (3.5) *explicitly as follows:*

(i) $n = 1$, $m_1 \circ m_1 = 0$, *that is,* m_1 *is a differential,;*

(ii) $n = 2$, $m_1 \circ m_2 = m_2 \circ (m_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes m_1)$, *that is* m_1 *is a derivation for* l_2,

(iii) $n = 3$

$$m_2 \circ (\mathrm{Id} \otimes m_2 - m_2 \otimes \mathrm{Id}) = m_1 \circ m_3 + m_3 \circ (m_1 \otimes \mathrm{Id} \otimes \mathrm{Id} + \mathrm{Id} \otimes m_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathrm{Id} \otimes m_1),$$

that is, operation m_2 *is associative up to homotopy and the obstruction of associativity is provided by* m_3.

In particular, if all $m_n = 0$ *with* $n \geqslant 3$*, then* (A, m_1, m_2) *is just a dg algebra.*

Definition 7.13. Let $(A, \{m_n\}_{n\geqslant 1})$ be an A_∞-algebra. An A_∞*-bimodule* over A is a graded space $M = \bigoplus_{n\in\mathbb{Z}} M_n$ equipped with a family of homogeneous maps $\{m_{p,q} : A^{\otimes p} \otimes M \otimes A^{\otimes q} \to M\}_{p,q\geq 0}$ with $|m_{p,q}| = p + q - 1$ satisfying: for all $p, q \geqslant 0$,

$$\begin{aligned}
&\sum_{\substack{1\leqslant j\leqslant p\\ 0\leqslant i\leqslant p-j}} (-1)^{i+j(p-i-j+1+q)} m_{p-j+1,q} \circ \left(\mathrm{Id}^{\otimes i} \otimes m_j \otimes \mathrm{Id}^{\otimes p-i-j} \otimes \mathrm{Id}_M \otimes \mathrm{Id}^{\otimes q}\right) \\
&+ \sum_{\substack{i+r=p,\\ s+k=q,\\ i,r,s,k\geqslant 0}} (-1)^{i+(r+s-1)k+1} m_{i+1,k+1} \circ \left(\mathrm{Id}^{\otimes i} \otimes m_{r,s} \otimes \mathrm{Id}^{\otimes k}\right) \\
&+ \sum_{\substack{1\leqslant j\leqslant q\\ 0\leqslant i\leqslant q-j}} (-1)^{p+i+1+j(q-i-j)} m_{p,q-j+1} \circ \left(\mathrm{Id}^{\otimes p} \otimes \mathrm{Id}_M \otimes \mathrm{Id}^{\otimes i} \otimes m_j \otimes \mathrm{Id}^{\otimes q-i-j}\right) \\
&= 0.
\end{aligned} \tag{3.6}$$

7.3.2 Cyclic A_∞-algebras and their cyclic complements

Definition 7.14. Let d be an integer and let A be a graded space. Suppose that A is endowed with a graded symmetric bilinear form of degree $-d$: $\gamma : A \times A \to \mathbf{k}$, that is, γ is a homogeneous bilinear map of degree $-d$ satisfying $\gamma(a,b) = (-1)^{|a||b|}\gamma(b,a)$ for all homogeneous elements $a, b \in A$. An operation $m_n : A^{\otimes n} \to A$ is called d*-cyclic* with respect to γ if it satisfies

$$\gamma(m_n(a_1 \otimes \cdots \otimes a_n), a_0) = (-1)^{n+|a_0|(\sum_{i=1}^{n}|a_i|)} \gamma(m_n(a_0 \otimes \cdots \otimes a_{n-1}), a_n),$$

for all homogeneous elements $a_0, \ldots, a_n \in A$.

Definition 7.15. Let $d \in \mathbb{Z}$. A d*-cyclic* A_∞*-algebra* is an A_∞-algebra $(A, \{m_n\}_{n\geqslant 1})$ equipped with a graded symmetric, nondegenerate bilinear form $\gamma : A \times A \to \mathbf{k}$, such that each m_n is d-cyclic with respect to γ.

Let $d \in \mathbb{Z}$. Set

$$\partial_d A = A \oplus s^d A^\vee.$$

There is a natural bilinear form

$$\zeta_A : \partial_d A \times \partial_d A \to \mathbf{k}$$

of degree $-d$ on $\partial_d A$ defined as follows:

$$\zeta_A(s^d f, a) = (-1)^{|a|(|f|+d)}\zeta_A(a, s^d f) = f(a), \;\; \zeta_A(a,b) = \zeta_A(s^d f, s^d g) = 0,$$

for all homogeneous $a, b \in A$ and $f, g \in A^\vee$. Note that ζ_A has degree $-d$.

A 0-cyclic A_∞-algebra will be simply called a *cyclic* A_∞*-algebra*.

Remark 7.16. *Let A be a d-cyclic A_∞-algebra. Then the canonical map*

$$\varphi : A \longrightarrow s^d A^\vee, \quad a \mapsto s^d \gamma(a,-) \in s^d A^\vee,$$

is an isomorphism of A_∞-bimodules over A. For further details, see [52, Lemma 3.1].

If A is an A_∞-algebra, then $\partial_0 A$ carries a natural A_∞-algebra structure, namely the trivial extension A_∞-algebra. We refer to $\partial_0 A$ as the *cyclic completion* of the A_∞-algebra A:

Proposition 7.17. *Let A be a locally finite-dimensional A_∞-algebra. Then $\partial_0 A$ is a cyclic A_∞-algebra with respect to the bilinear form ζ_A. Precisely, the A_∞-operator $\{\widetilde{m}_n : \partial_d A^{\otimes n} \to \partial_d A\}_{n\geqslant 1}$ is given by the following formulas: for homogeneous elements $(a_1, f_1), \cdots, (a_n, f_n) \in \partial_0 A = A \oplus A^\vee$,*

$$\widetilde{m}_n : (\partial_0 A)^{\otimes n} \longrightarrow \partial_0 A$$

$$\widetilde{m}_n((a_1,f_1)\otimes\cdots\otimes(a_n,f_n)) = \left(m_n(a_1\otimes\cdots\otimes a_n), \sum_{j=1}^{n} (-1)^{\xi_j^{m_n}} f_j \circ m_n(a_{j+1}\otimes\cdots\otimes a_n \otimes - \otimes a_1 \otimes \cdots \otimes a_{j-1}) \right),$$

where

$$\xi_j^{m_n} = jn + |m_n||f_j| + (\sum_{k=1}^{j-1}(|a_k|))(|f_j| + \sum_{k=j+1}^{n}(|a_k|)).$$

7.3.3 Pre-Calabi–Yau algebras

Definition 7.18. [62] Let $d \in \mathbb{Z}$. A *d-pre-Calabi–Yau structure* on a graded space $A = \oplus_{n\in\mathbb{Z}} A_n$ consists of a $(d-1)$-cyclic A_∞-algebra structure $\{m_n\}_{n\geqslant 1}$ on $\partial_{d-1} A = A \oplus s^{d-1} A^\vee$ with respect to the natural bilinear form $\zeta_A : \partial_{d-1} A \otimes \partial_{d-1} A \to \mathbf{k}$ such that $m_n(A^{\otimes n}) \subset A$ for all $n \geqslant 1$; that is, $\partial_{d-1} A$ contains A as an A_∞-subalgebra.

A 0-pre-Calabi-Yau algebra will be simply called a *pre-Calabi–Yau algebra*.

Remark 7.19. *In the original work of Kontsevich and Vlassopoulos [63] (see also [62]), a pre-Calabi-Yau algebra is defined as a vector space endowed with a family of operations with multiple inputs and outputs, subject to certain compatibility conditions. They showed that, on a finite-dimensional vector space, a pre-Calabi-Yau structure in this sense is equivalent to the one described above.*

Example 7.20. *[62] Let M be a compact oriented manifold of dimension d with compact boundary ∂M then the cohomology $\mathrm{H}^\bullet(M)$ of M has the structure of a pre-Calabi-Yau algebra of dimension d.*

We now introduce certain pre-Calabi-–Yau algebras satisfying specific desirable properties, following primarily [36].

Definition 7.21. Let A be a pre-Calabi-Yau algebra. We say that A is

- *good* if the A_∞-algebra structure $\{m_n\}_{n\geqslant 1}$ on $\partial_{-1} A = A \oplus s^{-1} A^\vee$ satisfies:
  - for all $i > 1$, $m_{2i} = 0$;
  - for all $i \geqslant 1$,

$$m_{2i-1}(A \otimes s^{-1}A^\vee \otimes \cdots \otimes s^{-1}A^\vee \otimes A) \subseteq A,$$
$$m_{2i-1}(s^{-1}A^\vee \otimes A \otimes \cdots \otimes A \otimes s^{-1}A^\vee) \subseteq s^{-1}A^\vee,$$

  and m_{2i-1} vanishes in all other cases;

- *fine* if it is good and m_2 also vanishes.
- *manageable* if m_2 restricted to A is an associative multiplication, denoted by "$\cdot$", and for all

$$(a, s^{-1}f), (b, s^{-1}g) \in \partial_{-1}A,$$

we have:

$$m_2((a, s^{-1}f) \otimes (b, s^{-1}g) = (a \cdot b, (-1)^{|a|} s^{-1} a \triangleright g + s^{-1} f \triangleleft b);$$

where the symbols "$\triangleright$" and "$\triangleleft$" denote the natural left and right actions of A on $A^\vee$ respectively, induce by the multiplication on A.

- *special* if the A_∞-algebra structure $\{m_n\}_{n\geqslant 1}$ on $\partial_{-1}A = A \oplus s^{-1}A^\vee$ satisfies: for all $n > 1$ m_{2n-1} is ultracyclic, that is, $v_1, \cdots v_{2n} \in \partial_{-1}A$, and $\sigma \in \mathfrak{S}_n$,

$$\begin{aligned}&\zeta_A(m_{2n-1}(v_1 \otimes \cdots \otimes v_{2n-1}), v_{2n})\\ =&\varepsilon(\widetilde{\sigma}; v_1, v_2, \ldots, v_{2n-1}, v_{2n})\zeta_A(m_{2n-1}(v_{\widetilde{\sigma}(1)} \otimes \cdots \otimes v_{\widetilde{\sigma}(2n-1)}), v_{\widetilde{\sigma}(2n)}),\end{aligned}$$

where $\widetilde{\sigma} \in \mathfrak{S}_{2n}$ is defined as $\widetilde{\sigma}(2i) = 2\sigma(i)$, $\widetilde{\sigma}(2i-1) = 2\sigma(i) - 1$ for all $1 \leqslant i \leqslant n$.

Chapter 8

Rota–Baxter algebras and homotopy Rota–Baxter algebras

8.1 Rota–Baxter algebras, double Lie algebras, and Yang-Baxter equations

In this section, we will first recall some basic notions on Rota–Baxter algebras, double Lie algebras and associative Yang-Baxter equations. Then we will recall the connections among these three objects introduced by Schedler [88], Goncharov and Kolesnikov [42].

Definition 8.1. Let $(A, \mu = \cdot)$ be an associative algebra over a field $\mathbf{k}$, and let M be a bimodule over A. A linear operator $T : M \to A$ is called a *relative Rota–Baxter operator on M* if it satisfies the following relation:

$$T(a) \cdot T(b) = T\big(a \cdot T(b) + T(a) \cdot b\big), \tag{1.1}$$

for all $a, b \in A$. In this case, the triple (A, M, T) is called a *relative Rota–Baxter algebra*.

In particular, if we take $M = A$, then T is simply called a *Rota–Baxter operator*, and $(A, \cdot, T)$ is called a *Rota–Baxter algebra*.

Definition 8.2. [88, 92] A *double Lie algebra* is a linear space V equipped with a linear map

$$\{\{-,-\}\} : V \otimes V \to V \otimes V$$

satisfying the following identities for all $a, b, c \in V$

- Skew-symmetry:
$$\{\{a,b\}\} = -\sigma_{(12)}\{\{b,a\}\}; \tag{1.2}$$
- Double Jacobi identity:
$$\{\{-,\{\{-,-\}\}\}\}_L + \sigma_{(123)}\{\{-,\{\{-,-\}\}\}\}_L\sigma_{(123)}^{-1} + \sigma_{(123)}^2\{\{-,\{\{-,-\}\}\}\}_L\sigma_{(123)}^{-2} = 0. \tag{1.3}$$

  where $\{\{-,-\}\}_L(x_1 \otimes x_2 \otimes x_3) := \{\{x_1, x_2\}\} \otimes x_3$.

Definition 8.3. [92] A *double Poisson algebra* is an associative algebra $(A,\cdot)$ equipped with a double Lie algebra structure $\{\{-,-\}\}$ satisfying the Leibniz rule: for all $a,b,c\in A$

$$\{\{a,b\cdot c\}\} = \{\{a,b\}\}\cdot c + b\cdot\{\{a,c\}\}, \tag{1.4}$$

where

$$\{\{a,b\}\}\cdot c = \{\{a,b\}\}^{[1]}\otimes(\{\{a,b\}\}^{[2]}\cdot c),$$
$$b\cdot\{\{a,c\}\} = (b\cdot\{\{a,c\}\}^{[1]})\otimes\{\{a,c\}\}^{[2]}.$$

Goncharov and Kolesnikov [42] proved that double Lie algebra structures on a finite-dimensional vector space V are equivalent to cyclic Rota–Baxter operators (referred to as a skew-symmetric Rota–Baxter operators in their work) on the associative algebra $\mathrm{End}(V)$. We briefly recall this correspondence below.

For a finite-dimensional vector space V, there is a natural nondegenerate bilinear form $\langle -,-\rangle$ on $\mathrm{End}_{\mathbf{k}}(V)$ which is given as:

$$\langle f,g\rangle := \mathrm{tr}(f\circ g), \forall f,g\in \mathrm{End}_{\mathbf{k}}(V).$$

Thus we have an isomorphism

$$\mathrm{End}_{\mathbf{k}}(V)\cong \mathrm{End}_{\mathbf{k}}(V)^{\vee},$$

which induces the following isomorphisms:

$$\mathrm{End}_{\mathbf{k}}(V\otimes V)\cong \mathrm{End}_{\mathbf{k}}(V)\otimes\mathrm{End}_{\mathbf{k}}(V)\cong \mathrm{End}_{\mathbf{k}}(V)\otimes\mathrm{End}_{\mathbf{k}}(V)^{\vee}\cong \mathrm{End}_{\mathbf{k}}(\mathrm{End}_{\mathbf{k}}(V)).$$

In this way, any double bracket

$$\{\{-,-\}\}: V\otimes V\to V\otimes V$$

can be uniquely determined by a linear operator

$$T:\mathrm{End}_{\mathbf{k}}(V)\to \mathrm{End}_{\mathbf{k}}(V).$$

Conversely, given a linear operator T on $\mathrm{End}_{\mathbf{k}}(V)$, the corresponding bracket $\{\{-,-\}\}: V\otimes V\to V\otimes V$ can be expressed in terms of T as follows:

$$\{\{a,b\}\} = \sum_{i=1}^{N} T^{\vee}(e^i)(a)\otimes e_i(b) = \sum_{i=1}^{N} e^i(a)\otimes T(e_i)(b), \quad a,b\in V, \tag{1.5}$$

where $\{e_1,\cdots,e_N\}$ is a basis of $\mathrm{End}_{\mathbf{k}}(V)$, and $\{e^1,\cdots,e^N\}$ is the corresponding dual basis with respect to the trace form, i.e., $\langle e^i,e_j\rangle=\delta^i_j$. Here, $T^{\vee}$ denotes the adjoint (or conjugate) operator of T on $\mathrm{End}_{\mathbf{k}}(V)$ with respect to the trace form.

Goncharov and Kolesnikov proved that the bracket $\{\{-,-\}\}$ defines a double Lie algebra structure if and only if the operator T is a cyclic Rota–Baxter operator on $\mathrm{End}_{\mathbf{k}}(V)$, that is, T is a Rota–Baxter operator satisfying $T=-T^{\vee}$.

On the other hand, Schedler [88] established a correspondence between skew-symmetric solutions of the associative Yang-Baxter equation (AYBE) in $\mathrm{End}_{\mathbf{k}}(V)$ and double Lie algebra structures on V.

Now let $A=\mathrm{End}_{\mathbf{k}}(V)$ for a vector space V. Then there is a canonical isomorphism

$$\mathrm{End}_{\mathbf{k}}(V)\otimes\mathrm{End}_{\mathbf{k}}(V)\cong\mathrm{End}_{\mathbf{k}}(V\otimes V),$$

under which each element $r=\sum_i a_i\otimes b_i$ corresponds to a unique bilinear operation

$$\{\{-,-\}\}: V\otimes V\to V\otimes V.$$

Schedler proved that an element $r\in\mathrm{End}_{\mathbf{k}}(V)\otimes\mathrm{End}_{\mathbf{k}}(V)$ is a skew-symmetric solution of the associative Yang-Baxter equation in $\mathrm{End}_{\mathbf{k}}(V)$ if and only if the associated double bracket $\{\{-,-\}\}$ defines a double Lie algebra structure on V.

In summary, we have the following equivalence:

Theorem 8.4. *[42, 88] Let V be a finite-dimensional vector space over $\boldsymbol{k}$. The following data are equivalent:*

(i) A double Lie algebra structure $\{\{-,-\}\}$ on V.

(ii) A linear operator $T : \mathrm{End}_{\boldsymbol{k}}(V) \to \mathrm{End}_{\boldsymbol{k}}(V)$ that is a Rota–Baxter operator with respect to composition (i.e., on $(\mathrm{End}_{\boldsymbol{k}}(V), \circ)$), and is cyclic, meaning $T^{\vee} = -T$.

(iii) A skew-symmetric solution $r \in \mathrm{End}_{\boldsymbol{k}}(V \otimes V)$ of the associative Yang-Baxter equation, i.e., $r = -r^{21}$ and $\mathrm{AYBE}(r) = 0$.

In [92], Van den Bergh proved that for a double Poisson algebra $(A, \cdot, \{\{-,-\}\})$, there exists a Lie algebra structure $\{-,-\}$ on the 0-th Hochschild homology of A: $\mathrm{HH}_0(A) = A/[A,A]$. The Lie bracket $\{-,-\}$ is defined as follows: for $a, b \in A$,

$$\{a,b\} := \{\{a,b\}\}^{[1]} \cdot \{\{a,b\}\}^{[2]}.$$

Inspired by this construction, we have the following result.

Lemma 8.5. *Let $(A, \cdot, \{\{-,-\}\})$ be a double Poisson algebra. Define a bracket $\{-,-\}$ on the quotient algebra $A/\langle [A,A] \rangle$ as follows:*

$$\{\bar{a}, \bar{b}\} = \overline{\{\{a,b\}\}^{[1]} \cdot \{\{a,b\}\}^{[2]}}.$$

Then $(A/\langle [A,A] \rangle, \{-,-\})$ is a Poisson algebra.

Proof. Since $\{\{-,-\}\}$ is skew-symmetric, so is $\{-,-\}$. For $a,b,c,d,e \in A$,

$$\begin{aligned}
&\{\bar{a}, \overline{bcde} - \overline{bdce}\} \\
=&\overline{\{\{a,b\}\}^{[1]} \cdot \{\{a,b\}\}^{[2]} cde} + \overline{\{\{a,c\}\}^{[1]} \cdot \{\{a,c\}\}^{[2]} bde} + \overline{\{\{a,d\}\}^{[1]} \cdot \{\{a,d\}\}^{[2]} bce} \\
&+ \overline{\{\{a,e\}\}^{[1]} \cdot \{\{a,e\}\}^{[2]} bcd} - \overline{\{\{a,b\}\}^{[1]} \cdot \{\{a,b\}\}^{[2]} cde} + \overline{\{\{a,d\}\}^{[1]} \cdot \{\{a,d\}\}^{[2]} bce} \\
&+ \overline{\{\{a,c\}\}^{[1]} \cdot \{\{a,c\}\}^{[2]} bde} + \overline{\{\{a,e\}\}^{[1]} \cdot \{\{a,e\}\}^{[2]} bcd} \\
=&0.
\end{aligned}$$

Thus, $\{-,-\}$ is well-defined on $A/\langle [A,A] \rangle$.

For the Jacobi identity,

$$\begin{aligned}
&\{-,\{-,-\}\} + \{-,\{-,-\}\}\sigma_{(123)}^{-1} + \{-,\{-,-\}\}\sigma_{(123)}^{-2} \\
=&\mu \circ \Big(\{\{-,\{\{-,-\}\}\}\}_L + \sigma_{(123)} \{\{-,\{\{-,-\}\}\}\}_L \sigma_{(123)}^{-1} + \sigma_{(123)}^{2} \{\{-,\{\{-,-\}\}^r\}\}_L \sigma_{(123)}^{-2} \Big) \\
&+ \mu \circ \Big(\{\{-,\{\{-,-\}\}\}\}_L + \sigma_{(123)} \{\{-,\{\{-,-\}\}\}\}_L \sigma_{(123)}^{-1} + \sigma_{(123)}^{2} \{\{-,\{\{-,-\}\}^r\}\}_L \sigma_{(123)}^{-2} \Big) \sigma_{(23)}^{-1} \\
=&0,
\end{aligned}$$

where μ is the multiplication of $A/\langle [A,A] \rangle$.

Finally, it is easy to verify that $\{-,-\}$ satisfy the Leibniz rule. □

It is well-known that the symmetric algebra of a Lie algebra has a Poisson algebra structure. For double Lie algebra, we have the following adapted construction.

Proposition 8.6. *Let $(V, \{\{-,-\}\})$ be a double Lie algebra. Define a double bracket operation $\{\{-,-\}\}$: $\mathrm{T}(V)\otimes\mathrm{T}(V)\to\mathrm{T}(V)\otimes\mathrm{T}(V)$, where $\mathrm{T}(V)$ is the tensor algebra generated by V, as follows. For $m,n\geq 1$ and $u_1,\dots,u_m,v_1,\dots,v_n\in V$, set*

$$\{\{u_1\cdots u_m, v_1\cdots v_n\}\} = \sum_{p,q} v_1\cdots v_{q-1}\,\{\{u_p,v_q\}\}^{[1]}\,u_{p+1}\cdots u_m\otimes u_1\cdots u_{p-1}\,\{\{u_p,v_q\}\}^{[2]}\,v_{q+1}\cdots v_n. \quad (1.6)$$

Then $\mathrm{T}(V)$ is a double Poisson algebra with respect to this double bracket.

Proof. We denote the multiplication of $\mathrm{T}(V)$ by $\cdot$. Define a double bracket

$$\{\{-,-\}\}:\ \mathrm{T}(V)\otimes\mathrm{T}(V)\longrightarrow\mathrm{T}(V)\otimes\mathrm{T}(V)$$

as follows. For $m,n\geq 1$ and $u_1,\dots,u_m,v_1,\dots,v_n\in V$, set

$$\{\{u_1\cdots u_m,\ v_1\cdots v_n\}\} := \sum_{p,q} v_1\cdots v_{q-1}\cdot\{\{u_p,v_q\}\}^{[1]}\cdot u_{p+1}\cdots u_m\otimes u_1\cdots u_{p-1}\cdot\{\{u_p,v_q\}\}^{[2]}\cdot v_{q+1}\cdots v_n.$$

By construction, $\{\{-,-\}\}$ satisfies the double Leibniz rule. Since the double bracket is double skew-symmetric on V, it is also double skew-symmetric on $\mathrm{T}(V)$.

It remains to verify the double Jacobi identity. Let $m,n,k\geq 1$ and $u_i,v_j,z_t\in V$. Consider the Jacobiator

$$\begin{aligned}&\{\{u_1\cdots u_m,\{\{v_1\cdots v_n,z_1\cdots z_k\}\}\}\}_L\\ &\quad+\sigma_{(123)}\{\{z_1\cdots z_k,\{\{u_1\cdots u_m,v_1\cdots v_n\}\}\}\}_L\\ &\quad+\sigma^2_{(123)}\{\{v_1\cdots v_n,\{\{z_1\cdots z_k,u_1\cdots u_m\}\}\}\}_L.\end{aligned}$$

Expanding each term using the defining formula and the double Leibniz rule, one obtains a finite sum indexed by the positions of the generators. After regrouping terms, all summands cancel pairwise by the double Jacobi identity on V. Hence the above expression vanishes.

Therefore, $\{\{-,-\}\}$ defines a double Poisson algebra structure on $\mathrm{T}(V)$. It is straightforward to check that the induced bracket $\{-,-\}$ is well defined on $\mathrm{Sym}(V)$, and that $(\mathrm{Sym}(V),\{-,-\})$ is a Poisson algebra. ☐

Corollary 8.7. *Let $(V,\{\{-,-\}\})$ be a double Lie algebra. Then $\mathrm{Sym}(V)$ is a Poisson algebra.*

Remark 8.8. *The above corollary suggests that the dual space of a double Lie algebra can be regarded as a formal Poisson manifold.*

8.2 Homotopy Rota–Baxter algebras and homotopy Rota–Baxter modules

In this section, we review the notions of homotopy Rota–Baxter algebras and relative Rota–Baxter algebras. We then introduce the concepts of homotopy Rota–Baxter modules, along with a trivial extension construction that produces homotopy Rota–Baxter algebras from such modules.

Definition 8.9. [95] Let $(A,\{m_n\}_{n\geqslant 1})$ be an A_∞-algebra. A homotopy Rota–Baxter operator consists of a family of operators $\{T_n : A^{\otimes n}\to A\}_{n\geqslant 1}$ with $|T_n| = n-1$ subjecting to the following identities for all $n\geqslant 1$:

$$\begin{aligned}&\sum_{\substack{l_1+\cdots+l_k=n,\\ l_1,\dots,l_k\geqslant 1}}(-1)^{\delta} m_k\circ\Big(T_{l_1}\otimes\cdots\otimes T_{l_k}\Big) \quad (2.7)\\ =&\sum_{1\leqslant j\leqslant p}\ \sum_{\substack{r_1+\cdots+r_p=n,\\ r_1,\dots,r_p\geqslant 1}}(-1)^{\eta}T_{r_1}\circ\Big(\mathrm{Id}^{\otimes i}\otimes m_p\circ(T_{r_2}\otimes\cdots\otimes T_{r_j}\otimes\mathrm{Id}\otimes T_{r_{j+1}}\otimes\cdots\otimes T_{r_p})\otimes\mathrm{Id}^{\otimes k}\Big)\end{aligned}$$

where

$$\delta = \frac{k(k-1)}{2} + \sum_{j=1}^{k}(k-j)l_j, \ \ \eta = i + (p + \sum_{j=2}^{p}(r_j-1))k + \sum_{t=2}^{j}(r_t-1) + \sum_{t=2}^{p}(r_t-1)(p-t).$$

The triple $(A, \{m_n\}_{n\geqslant 1}, \{T_n\}_{\geqslant 1})$ is called a *homotopy Rota–Baxter algebra*.

We also need the concepts of modules over homotopy Rota–Baxter algebras.

Definition 8.10. Let $(A, \{m_n\}_{n\geqslant 1}, \{T_n\}_{n\geqslant 1})$ be a homotopy Rota–Baxter algebras. A *Rota–Baxter module over A* is an A_∞-bimodule $(M, \{m_{i,j}\}_{i,j\geqslant 0})$ over A which is endowed with a family of graded maps $\{T^M_{i,j} : A^{\otimes i} \otimes M \otimes A^{\otimes j} \to M\}_{i,j\geqslant 0}$, with $|T^M_{i,j}| = i + j$, such that the following identities hold for any $m,n \geqslant 0$:

$$\begin{aligned}
&\sum_{\substack{i_1+\cdots+i_p+l=m\\ j_1+\cdots+j_q+k=n\\ p,q,l,k\geqslant 0}} (-1)^{\alpha} m_{p,q} \circ \Big(T_{i_1} \otimes \cdots \otimes T_{i_p} \otimes T^M_{l,k} \otimes T_{j_1} \otimes \cdots \otimes T_{j_q}\Big) \qquad (2.8)\\
= &\sum_{\substack{i_1+\cdots+i_p+l=m,\\ j_1+\cdots+j_q+k=n,\\ i_1,\cdots,i_p,j_1,\cdots,j_q\geqslant 1\\ p,\ q,\ l,\ k\geqslant 0;}} (-1)^{\beta_1} T^M_{l,k} \circ \Big(\mathrm{Id}_A^{\otimes l} \otimes m_{p,q} \circ (T_{i_1} \otimes \cdots \otimes T_{i_p} \otimes \mathrm{Id}_M \otimes T_{j_1} \otimes \cdots \otimes T_{j_q}) \otimes \mathrm{Id}_A^{\otimes k}\Big)\\
&+ \sum_{\substack{i_1+\cdots+i_p+l+r+1=m\\ j_1+\cdots+j_q+k+t=n\\ i_1,\cdots,i_p,j_1,\cdots,j_q\geqslant 0\\ v,l,k,p,q\geqslant 0}} (-1)^{\beta_2} T^M_{l,k} \circ \Big(\mathrm{Id}_A^{\otimes l} \otimes m_{p+1,q} \circ (T_{i_1} \otimes \cdots \otimes T_{i_v} \otimes \mathrm{Id}_A \otimes T_{i_{v+1}} \cdots\\
&\qquad \cdots \otimes T_{i_p} \otimes T^M_{r,t} \otimes T_{j_1} \otimes \cdots \otimes T_{j_q}) \otimes \mathrm{Id}_A^{\otimes k}\Big)\\
&+ \sum_{\substack{i_1+\cdots+i_p+l+r=m\\ j_1+\cdots+j_q+k+t+1=n\\ i_1,\cdots,i_p,j_1,\cdots,j_q\geqslant 0\\ v,l,k,p,q\geqslant 0}} (-1)^{\beta_3} T^M_{l,k} \circ \Big(\mathrm{Id}_A^{\otimes l} \otimes m_{p,q+1} \circ (T_{i_1} \otimes \cdots \otimes T_{i_p} \otimes T^M_{r,t} \otimes T_{j_1} \otimes \cdots\\
&\qquad \cdots \otimes T_{j_v} \otimes \mathrm{Id}_A \otimes T_{j_{v+1}} \otimes \cdots T_{j_q}) \otimes \mathrm{Id}_A^{\otimes k}\Big),
\end{aligned}$$

where

$$\begin{aligned}
\alpha =& \frac{(p+q)(p+q+1)}{2} + q(l+k) + \sum_{t=1}^{q}(q-t)j_t + \sum_{t=1}^{p}(p+q+1-t)i_t,\\
\beta_1 =& l + k(m+n-l) + \sum_{t=1}^{p}(i_t-1) + \sum_{t=1}^{p}(i_t-1)(p+q-t) + \sum_{t=1}^{q}(j_t-1)(q-t),\\
\beta_2 =& l + k(m+n-l) + \sum_{s=1}^{v}(i_s-1) + (r+t)q + \sum_{s=1}^{p}(i_s-1)(p+q+1-t) + \sum_{s=1}^{q}(j_s-1)(q-t),\\
\beta_3 =& l + k(m+n-l) + \sum_{s=1}^{p}(i_s-1) + \sum_{s=1}^{v}(j_s-1) + (r+t)(q-1) + \sum_{s=1}^{p}(i_s-1)(p+q+1-t) + \sum_{s=1}^{q}(j_s-1)(q-t).
\end{aligned}$$

Definition 8.11. [23] Let $(A, \{m_i\}_{i\geqslant 1})$ be an A_∞-algebra and $(M, \{m_{p,q}\}_{p,q\geq 0})$ an A_∞-bimodule over A. A *homotopy relative Rota–Baxter operator* on M is a family of operators $\{T_n : M^{\otimes n} \to A\}_{n\geqslant 1}$ of degree

$|T_n| = n-1$ satisfying the following identity for all $n \geqslant 1$:

$$\sum_{\substack{l_1+\cdots+l_k=n,\\ l_1,\dots,l_k\geqslant 1}} (-1)^{\delta} m_k \circ \left(T_{l_1} \otimes \cdots \otimes T_{l_k}\right) \tag{2.9}$$
$$= \sum_{1\leqslant j\leqslant p} \sum_{\substack{r_1+\cdots+r_p=n,\\ r_1,\dots,r_p\geqslant 1}} (-1)^{\eta} T_{r_1} \circ \left(\mathrm{Id}^{\otimes i} \otimes m_{j-1,p-j} \circ (T_{r_2} \otimes \cdots \otimes T_{r_j} \otimes \mathrm{Id} \otimes T_{r_{j+1}} \otimes \cdots \otimes T_{r_p}) \otimes \mathrm{Id}^{\otimes k}\right),$$

where the signs δ and η are as defined in Definition 8.9. The triple $(A, M, \{T_i\}_{i\geqslant 1})$ is called a *homotopy relative Rota–Baxter algebra*.

In particular, when the underlying A_∞-algebra and A_∞-bimodule of a homotopy relative Rota–Baxter algebra are simply a differential graded (dg) algebra and a dg bimodule over the dg algebra, respectively, the notion simplifies as follows. This special case will be used in later sections.

Definition 8.12. Let (A, d, m) be a dg algebra, and let (M, d_M, m^l, m^r) be a dg A-bimodule, where m^l and m^r denote the left and right actions of A, respectively. A *homotopy relative Rota–Baxter operator* on M is a family of operations $\{T_n : M^{\otimes n} \to A\}_{n\geqslant 1}$ with $|T_n| = n-1$ satisfying the identity

$$d \circ T_n - \sum_{s+k+1=n} (-1)^{n-1} T_n \circ (\mathrm{Id}^{\otimes s} \otimes d_M \otimes \mathrm{Id}^{\otimes k}) \tag{2.10}$$
$$= -\sum_{i+j=n} (-1)^{1+i} m \circ (T_i \otimes T_j)$$
$$+ \sum_{i+j+k+1=n} (-1)^{i+(j-1)(k+1)} T_{i+k+1} \circ (\mathrm{Id}^{\otimes i} \otimes m^l \circ (T_j \otimes \mathrm{Id}) \otimes \mathrm{Id}^{\otimes k})$$
$$+ \sum_{i+j+k+1=n} (-1)^{i+(j-1)k} T_{i+k+1} \circ (\mathrm{Id}^{\otimes i} \otimes m^r \circ (\mathrm{Id} \otimes T_j) \otimes \mathrm{Id}^{\otimes k})$$

for all $n \geqslant 1$. In this case, the triple $(A, M, \{T_n\}_{n\geq 1})$ is called a *dg homotopy relative Rota–Baxter algebra*.

Remark 8.13. *(i) Given a homotopy Rota–Baxter algebra $(A, \{m_n\}_{n\geq 1}, \{T_n\}_{n\geq 1})$, the space A itself naturally forms a homotopy Rota–Baxter module over A. Explicitly, the structure maps are given by $m^A_{p,q} = m_{p+q+1}$ and $T^A_{p,q} = T_{p+q+1}$.*

(ii) In Equation (2.7) *of Definition 8.9, if we replace one instance of A in the inputs with a module M, and correspondingly replace the operations m_n and T_n with $m_{p,q}$ and $T_{p,q}$ to reflect the presence of M, we recover Equation* (2.8)*. Furthermore, if all instances of A in the inputs are replaced by M, we obtain Equation* (2.9) *from the definition of homotopy relative Rota–Baxter algebras.*

The above two notions, homotopy Rota–Baxter algebras and homotopy relative Rota–Baxter algebras, are related by the following construction.

Proposition 8.14. *Let A be an A_∞-algebra and M an A_∞-bimodule over A. Let*

$$\{T_n : M^{\otimes n} \longrightarrow A\}_{n\geqslant 1}$$

be a family of operators. Let $A \ltimes M$ denote the trivial extension of the A_∞-algebra A by the A_∞-bimodule M. Define

$$\overline{T}_n : (A \ltimes M)^{\otimes n} \twoheadrightarrow M^{\otimes n} \xrightarrow{T_n} A \hookrightarrow A \ltimes M.$$

Then

$$(A, M, \{T_n\}_{n\geqslant 1})$$

is an homotopy relative Rota–Baxter algebra if and only if

$$(A \ltimes M, \{\overline{T}_n\}_{n\geqslant 1})$$

is a homotopy Rota–Baxter algebra.

Proof. The claim follows from a direct verification of the defining equations, so we omit the details. $\square$

Proposition 8.15. *Let*

$$(A, \{m_n\}_{n\geqslant 1}, \{T_n\}_{n\geqslant 1})$$

be a homotopy Rota–Baxter algebra, and let

$$(M, \{m_{i,j}\}_{i,j\geqslant 0}, \{T^M_{i,j}\}_{i,j\geqslant 0})$$

be a homotopy Rota–Baxter module over A. Then there exists a canonical homotopy Rota–Baxter algebra structure

$$\left(\{\widetilde{m}_n\}_{n\geqslant 1}, \{\widetilde{T}_n\}_{n\geqslant 1}\right)$$

on the graded space $A\oplus M$, where the structure maps

$$\widetilde{m}_n : (A\oplus M)^{\otimes n} \to A\oplus M \quad \text{and} \quad \widetilde{T}_n : (A\oplus M)^{\otimes n} \to A\oplus M$$

are defined as follows:

(i) *On the summand $A^{\otimes n} \subset (A\oplus M)^{\otimes n}$, the maps $\widetilde{m}_n, \widetilde{T}_n$ are given by*

$$\widetilde{m}_n|_{A^{\otimes n}} = m_n, \ \ \widetilde{T}_n|_{A^{\otimes n}} = T_n;$$

(ii) *On the summands of the form $A^{\otimes i}\otimes M\otimes A^{\otimes j} \subset (A\oplus M)^{\otimes n}$ with $i+j=n-1$ and $i,j\geqslant 0$, the maps $\widetilde{m}_n, \widetilde{T}_n$ on the direct summand $A^{\otimes i}\otimes M\otimes A^{\otimes j}$ are given as*

$$\widetilde{m}_n|_{A^{\otimes i}\otimes M\otimes A^{\otimes j}} = m_{i,j}, \ \ \widetilde{T}_n|_{A^{\otimes i}\otimes M\otimes A^{\otimes j}} = T^M_{i,j};$$

(iii) *On all other summands of $(A\oplus M)^{\otimes n}$, the maps $\widetilde{m}_n$ and $\widetilde{T}_n$ vanish.*

The resulting homotopy Rota–Baxter algebra, denoted by $A\ltimes M$, is called the trivial extension of A by M.

Proof. This is just the analog of classical trivial extension of A_∞-algebras by A_∞-bimodules. It can be checked by direct computations, so we omit the details here. $\square$

Proposition 8.16. *Let $(M, \{m_{i,j}\}_{i,j\geqslant 0}, \{T_{i,j}\}_{i,j\geqslant 0})$ be a homotopy Rota–Baxter module over homotopy Rota–Baxter algebras $(A, \{m_n\}_{n\geqslant 1}, \{T_n\}_{n\geqslant 1})$. Then $M^\vee$ has a canonical homotopy Rota–Baxter module structure, in which $T^{M^\vee}_{i,j}, m^{M^\vee}_{i,j} : A^{\otimes i}\otimes M^\vee\otimes A^{\otimes j} \to M^\vee$ are defined as follows:*

$$\begin{aligned} & m^{M^\vee}_{i,j}(a_1\otimes\cdots a_i\otimes f\otimes b_1\otimes\cdots\otimes b_j)(x) \\ =&(-1)^{(j+1)(i+j+1)+(\sum\limits_{k=1}^{i}|a_k|)(|f|+|x|+\sum\limits_{k=1}^{j}|b_k|)+|f|(i+j-1)} f\big(m^M_{j,i}(b_1\otimes\cdots\otimes b_j\otimes x\otimes a_1\otimes\cdots\otimes a_i)\big) \end{aligned}$$

$$\begin{aligned} & T^{M^\vee}_{i,j}(a_1\otimes\cdots a_i\otimes f\otimes b_1\otimes\cdots\otimes b_j)(x) \\ =&(-1)^{(j+1)(i+j+1)+(\sum\limits_{k=1}^{i}|a_k|)(|f|+|x|+\sum\limits_{k=1}^{j}|b_k|)+|f|(i+j)} f\big(T^M_{j,i}(b_1\otimes\cdots\otimes b_j\otimes x\otimes a_1\otimes\cdots\otimes a_i)\big). \end{aligned}$$

In particular, $A^\vee$ is a homotopy Rota–Baxter module over A.

Proof. The proof of this proposition involves extensive computations. For the sake of readability, we have placed the proof in the Appendix B. $\square$

8.3 Cylic homotopy Rota–Baxter algebras and their cyclic complements

In this subsection, we present the concept of cyclic homotopy Rota–Baxter algebras—a distinguished class of homotopy Rota–Baxter algebras endowed with a desirable cyclic invariance property—and provide a canonical construction of these structures.

Definition 8.17. Let A be a cyclic A_∞-algebra with respect to a nondegenerate bilinear form $\gamma : A\otimes A\to \mathbf{k}$. A homotopy Rota–Baxter operator $\{T_n\}_{n\geqslant 1}$ on A is said to be *cyclic* if each operator $T_n : A^{\otimes n}\to A$ is cyclic with respect to the bilinear form γ. Then $(A,\{T_n\}_{n\geqslant 1})$ is called a *cyclic homotopy (absolute) Rota–Baxter algebras*.

Moreover, we call $\{T_n\}_{n\geqslant 1}$ an *ultracyclic homotopy Rota–Baxter operator* if each operator T_n is both cyclic and skew-symmetric, i.e., for all $\sigma\in\mathfrak{S}_n$, the identity

$$T_n\circ\sigma = \operatorname{sgn}(\sigma)T_n$$

holds. In this case, the pair $(A,\{T_n\}_{n\geqslant 1})$ is called an *ultracyclic homotopy Rota–Baxter algebra.*

We give a method to construct the cyclic homotopy Rota–Baxter algebras from homotopy Rota–Baxter algebras, called the *cyclic completion for homotopy Rota–Baxter algebras*.

Proposition 8.18. *Let $(A,\{m_n\}_{n\geqslant 1},\{T_n\}_{n\geqslant 1})$ be a locally finite-dimensional homotopy Rota–Baxter algebra. Then $\partial_0 A := A\ltimes A^\vee$ is a cyclic homotopy Rota–Baxter algebra. Precisely, the homotopy Rota–Baxter operator $\{\widetilde{T}_n\}_{n\geqslant 1}$ is given by the following formulas: for homogeneous elements $(a_1,f_1),\cdots,(a_n,f_n)\in \partial_0 A = A\oplus A^\vee$,*

$$\widetilde{T}_n : (\partial_0 A)^{\otimes n}\longrightarrow \partial_0 A$$

$$\begin{aligned}&\widetilde{T}_n\left((a_1,f_1)\otimes\cdots\otimes(a_n,f_n)\right)\\ =&\left(T_n(a_1\otimes\cdots\otimes a_n),\sum_{j=1}^{n}(-1)^{\xi_j^{T_n}} f_j\circ T_n(a_{j+1}\otimes\cdots\otimes a_n\otimes - \otimes a_1\otimes\cdots\otimes a_{j-1})\right),\end{aligned}$$

where

$$\xi_j^{T_n} = jn+|T_n||f_j|+(\sum_{k=1}^{j-1}(|a_k|))(|f_j|+\sum_{k=j+1}^{n}(|a_k|)).$$

Moreover, if $\{T_n\}_{n\geqslant 1}$ is skew-symmetric, then $\partial_0 A$ is an ultracyclic homotopy Rota–Baxter algebra.

Proof. According to Proposition 8.15 and Proposition 8.16, we have that $\partial_0 A$ is a homotopy Rota–Baxter algebra, and it can be seen that this homotopy Rota–Baxter structure on $\partial_0 A$ is cyclic with respect to the natural bilinear form on $\partial_0 A$. According to the formulas of $\widetilde{T}_n$ presented above, one can see that $\widetilde{T}_n$ is skew-symmetric if each T_n is skew-symmetric. □

We also have the notion of relative cyclic homotopy Rota–Baxter operators.

Definition 8.19. Let A be an A_∞-algebra, and let $\{T_n : (A^\vee)^{\otimes n}\to A\}_{n\geqslant 1}$ be a homotopy relative Rota–Baxter operator on the dual bimodule $A^\vee$. We say that $\{T_n\}_{n\geqslant 1}$ is a *cyclic homotopy relative Rota–Baxter operator* if, for all $n\geqslant 1$ and homogeneous elements $f_0,\dots,f_n\in A^\vee$, the following identity holds:

$$\langle T_n(f_0\otimes\cdots\otimes f_{n-1}),f_n\rangle = (-1)^{n+|f_n|(\Sigma_{j=0}^{n-1}|f_j|)}\langle T_n(f_n\otimes f_0\otimes\cdots\otimes f_{n-2}),f_{n-1}\rangle, \tag{3.11}$$

where $\langle -,-\rangle : A\times A^\vee\to\mathbf{k}$ denotes the natural pairing. Then $(A,A^\vee,\{T_n\}_{n\geqslant 1})$ is called a *cyclic homotopy relative Rota–Baxter algebra*.

Moreover, we call $\{T_n\}_{n\geqslant 1}$ an *ultracyclic homotopy relative Rota–Baxter operator* if each operator T_n is cyclic and skew-symmetric, that is , each T_n satisfies

$$T_n \circ \sigma = \mathrm{sgn}(\sigma) T_n,$$

for all $\sigma \in \mathfrak{S}_n$. In this case, $(A, A^\vee, \{T_n\}_{n\geqslant 1})$ is called an *ultracyclic homotopy relative Rota–Baxter algebra*.

The above two notions, cyclic absolute homotopy Rota–Baxter algebras and cyclic homotopy relative Rota–Baxter algebras are related by the following construction.

Proposition 8.20. *Let A be a locally finite-dimensional A_∞-algebra, and let*

$$\{T_n : (A^\vee)^{\otimes n} \longrightarrow A\}_{n\geqslant 1}$$

be a family of operators. Define

$$\overline{T}_n : (\partial_0 A)^{\otimes n} \twoheadrightarrow (A^\vee)^{\otimes n} \xrightarrow{T_n} A \hookrightarrow \partial_0 A.$$

Then

$$(\partial_0 A, \{\overline{T}_n\}_{n\geqslant 1})$$

is a cyclic (resp. ultracyclic) absolute homotopy Rota–Baxter algebra if and only if

$$(A, A^\vee, \{T_n\}_{n\geqslant 1})$$

is a cyclic (resp. ultracyclic) homotopy relative Rota–Baxter algebra.

Proof. This can be proved by direct computations, so we omit the details. □

Remark 8.21. *Every cyclic homotopy absolute Rota–Baxter algebra $(A, \{T_n\}_{n\geqslant 1})$ can naturally be regarded as a cyclic homotopy relative Rota–Baxter algebra $(A, A^\vee, \{T'_n\}_{n\geqslant 1})$, where each T'_n is defined as the composition:*

$$T'_n : (A^\vee)^{\otimes n} \xrightarrow{(\varphi^{-1})^{\otimes n}} A^{\otimes n} \xrightarrow{T_n} A,$$

and $\varphi^{-1} : A^\vee \to A$ denotes the inverse of the A_∞-bimodule isomorphism $\varphi : A \to A^\vee$ (see Remark 7.16) induced by the non-degenerate bilinear form γ defining the cyclic A_∞-structure on A.

Chapter 9

Pre-Calabi–Yau structures arising from cyclic homotopy Rota–Baxter algebras

In this chapter, we construct pre-Calabi–Yau algebras from homotopy Rota–Baxter algebras. We begin by introducing the notion of interactive pairs, consisting of two dg algebras–referred to as the acting algebra and base algebra–equipped with mutually interacting module structures that satisfy a key compatibility condition. We then demonstrate that if the acting algebra of an interactive pair is endowed with a cyclic homotopy relative Rota–Baxter algebra satisfying certain additional conditions, then the base algebra naturally inherits a pre-Calabi–Yau algebra structure. In particular, a dg module over a dg algebra which is endowed with a homotopy relative Rota–Baxter algebra structure naturally inherits a pre-Calabi–Yau algebra structure.

9.1 Interactive pairs and relative derivatives

Definition 9.1. An *interactive pair* (A,B) consists of the following data:

- A pair of dg algebras $(A,d_A,\cdot)$ and $(B,d_B,*)$.
- A left dg B-module structure on the complex (A,d_A) and a left dg A-module structure on the complex (B,d_B). To distinguish between them, the left action of A on B is denoted by $\triangleright$, while the left action of B on A is denoted by $\blacktriangleright$.
- A compatibility condition ensuring that for all $a\in A, b_1,b_2\in B$, the following identity holds:

$$(b_1\blacktriangleright a)\triangleright b_2 = b_1 * (a\triangleright b_2).$$

We call A the acting algebra of the interactive pair and B the base algebra of the interactive pair.

Example 9.2.

(i) Let A be a dg algebra. Then (A,A) is a interactive pair.

(ii) Let A be a dg algebra and B a dg A-module. By viewing B as a dg algebra with trivial multiplication and A as a B-module with trivial action, the pair (A,B) forms an interactive pair.

(iii) Let $(B,\cdot)$ be a dg algebra. The graded vector space $\operatorname{End}_{\mathrm{gr}}(B)$ carries a natural dg algebra structure, with multiplication given by composition. The algebra B becomes a left dg $\operatorname{End}_{\mathrm{gr}}(B)$-module in the canonical way. For each element $b \in B$, define $l_b \in \operatorname{End}_{\mathrm{gr}}(B)$ by $l_b(x) := b \cdot x$ for all $x \in B$. This gives rise to a left action of B on $\operatorname{End}_{\mathrm{gr}}(B)$ defined by

$$b \blacktriangleright f := l_b \circ f,$$

which equips $\operatorname{End}_{\mathrm{gr}}(B)$ with the structure of a left dg B-module. Moreover, for all $b_1, b_2 \in B$ and $f \in \operatorname{End}_{\mathrm{gr}}(B)$, we have

$$(l_{b_1} \circ f)(b_2) = b_1 \cdot f(b_2).$$

Hence, $(\operatorname{End}_{\mathrm{gr}}(B), B)$ forms an interactive pair.

Definition 9.3. Let (A,B) be an interactive pair. An operator $T_n : (A^\vee)^{\otimes n} \to A$ is called

- an *n-derivation relative to B*, if for all $b_1, b_2 \in B$, and $f_1 \cdots, f_n \in A^\vee$:

$$\begin{aligned} & T_n(f_1 \otimes \cdots \otimes f_n) \rhd (b_1 * b_2) \\ = \quad & T_n(f_1 \otimes \cdots \otimes f_n \blacktriangleleft b_1) \rhd b_2 \\ & + (T_n(f_1 \otimes \cdots \otimes f_n) \rhd b_1) * b_2; \end{aligned} \tag{1.1}$$

- a *strong n-derivation relative to B*, if T_n is an n-derivation relative to B and for all $b_1, b_2 \in B$, $g \in B^\vee$, and $f_1, \cdots, f_{n-1} \in A^\vee$, the following identities hold:

$$\begin{aligned} & T_n\big(\kappa(b_1 * b_2 \otimes f_1) \otimes f_2 \otimes \cdots \otimes f_n\big) \\ & = (-1)^{|T_n||b_1|}\Big(b_1 \blacktriangleright \big(T_n\big(\kappa(b_2 \otimes g) \otimes f_1 \otimes \cdots \otimes f_{n-1}\big)\big)\Big) \\ & \quad + T_n\big(\kappa\big(b_1 \otimes b_2 \blacktriangleright g\big) \otimes f_1 \otimes \cdots \otimes f_{n-1}\big); \end{aligned} \tag{1.2}$$

$$\begin{aligned} & T_n\big(f_1 \otimes \cdots \otimes f_{l-1} \otimes \kappa(b_1 * b_2 \otimes g) \otimes f_l \otimes \cdots \otimes f_{n-1}\big) \\ = & T_n(f_1 \otimes \cdots \otimes f_{l-1} \blacktriangleleft b_1 \otimes \kappa(b_2 \otimes g) \otimes f_l \otimes \cdots \otimes f_{n-1}) \\ & + T_n(f_1 \otimes \cdots \otimes f_{l-1} \otimes \kappa(b_1 \otimes b_2 \blacktriangleright g) \otimes f_l \otimes \cdots \otimes f_{n-1}), \end{aligned} \tag{1.3}$$

for all $1 < l \leqslant n$.

Here " $\blacktriangleleft$ " is the right action of B on $A^\vee$ induced by " $\blacktriangleright$ " and $\kappa : B \otimes B^\vee \to A^\vee$ is defined as $\kappa(b \otimes f)(a) = (-1)^{|b|(|f|+|a|)} f(a \rhd b)$, for any $b \in B$, $f \in B^\vee$ and $a \in A$.

Remark 9.4. *Given an interactive pair (A,B), there is an isomorphism:*

$$\begin{aligned} \iota : A^{\otimes n} \otimes B &\cong \operatorname{Hom}((A^\vee)^{\otimes n}, B) \\ a_n \otimes \cdots \otimes a_1 \otimes b &\to Q \end{aligned}$$

where $Q(f_1 \otimes \cdots \otimes f_n) = (-1)^{(\Sigma_{j=1}^n |f_j|)|b| + (\Sigma_{j=1}^n |f_j||a_j|)} f_1(a_1) \cdots f_n(a_n) b$, for all $f_1, \ldots, f_n \in A^\vee$. Since A is a left B-module and B is a right B-module, then $A^{\otimes n} \otimes B$ is a B-bimodule. Therefore, each n-derivation T_n relative to B gives rise to a usual derivation of B into the B-bimodule $A^{\otimes n} \otimes B$: for all $b_1, b_2 \in B$

$$\begin{aligned} & \iota^{-1}\big(T_n(- \otimes \cdots \otimes -) \rhd (b_1 * b_2)\big) \\ = & (-1)^{|b_1||T_n|} b_1 \blacktriangleright \iota^{-1}(T_n(- \otimes \cdots \otimes -) \rhd b_2) \\ & + \iota^{-1}(T_n(- \otimes \cdots \otimes -) \rhd b_1) * b_2. \end{aligned}$$

In particular, if one takes $(B,*)$ to be a finite dimensional algebra and $A=\mathrm{End}_{\mathrm{gr}}(B)$, the above construction yields a bijection between the space of n-derivation relative to B on A and the space of derivations from B to $A^{\otimes n}\otimes B$.

Proposition 9.5. *Let (A,B) be an interactive pair with the acting algebra A being locally finite-dimensional. Let $T_n\colon (A^\vee)^{\otimes n}\to A$ be an n-derivation relative to B satisfying Equation* (3.11)*. Then T_n is also a strong n-derivation relative to B.*

Proof. We will check that T_n satisfies Equations (1.2)(1.3) in Definition 9.3. For all $b_1,b_2,b_3\in B$, $g\in B^\vee$, and $f_1,\cdots,f_n\in A^\vee$

$$
\begin{aligned}
&\langle T_n(\kappa(b_1*b_2\otimes g)\otimes f_1\otimes\cdots\otimes f_{n-1}),f_n\rangle\\
&-(-1)^{|b_1|(\sum\limits_{i=1}^{n}|f_i|+|g|+|b_2|)}\langle T_n(\kappa(b_2\otimes g)\otimes f_1\otimes\cdots\otimes f_{n-1}),f_n\blacktriangleleft b_1\rangle\\
&-\langle T_n(\kappa(b_1\otimes b_2\blacktriangleright g),f_1,\cdots,f_{n-1}),f_n\rangle\\
=&(-1)^{n+(|b_1|+|b_2|+|g|)(\sum\limits_{i=1}^{n}|f_i|)}\langle T_n(f_1\otimes\cdots\otimes f_n)\rhd(b_1*b_2),g\rangle\\
&-(-1)^{n+(|b_1|+|b_2|+|g|)(\sum\limits_{i=1}^{n}|f_i|)}\langle T_n(f_1\otimes\cdots\otimes f_n\blacktriangleleft b_1)\rhd b_2,g\rangle\\
&-(-1)^{n+(|b_1|+|b_2|+|g|)(\sum\limits_{i=1}^{n}|f_i|)}\langle (T_n(f_1\otimes\cdots\otimes f_n)\rhd b_1)*b_2,g\rangle\\
=&0
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
T_n\big(\kappa(b_1*b_2\otimes g)\otimes f_1\otimes\cdots\otimes f_{n-1}\big)=&(-1)^{|T_n||b_1|}b_1*(T_n(\kappa(b_2\otimes g)\otimes f_1\otimes\cdots\otimes f_{n-1})\\
&+T_n(\kappa(b_1\otimes b_2\blacktriangleright g)\otimes f_1\otimes\cdots\otimes f_{n-1}),
\end{aligned}
$$

that is, T_n fulfills Equation (1.2).

Similarly, for any $1<l\leqslant n$, $f_1,\cdots,f_n\in A^\vee$, $g\in B^\vee$ and $b_1,b_2\in B$,

$$
\begin{aligned}
&\langle T_n(f_1\otimes\cdots\otimes f_{l-1}\blacktriangleleft b_1\otimes\kappa(b_2\otimes g)\otimes f_l\otimes\cdots\otimes f_{n-1}),f_n\rangle\\
&+\langle T_n(f_1\otimes\cdots\otimes f_{l-1}\otimes\kappa(b_1\otimes b_2\blacktriangleright g)\otimes f_l\otimes\cdots\otimes f_{n-1}),f_n\rangle\\
&-\langle T_n(f_1\otimes\cdots\otimes f_{l-1}\otimes\kappa(b_1*b_2\otimes g)\otimes f_l\otimes\cdots\otimes f_{n-1}),f_n\rangle\\
=&(-1)^{\varepsilon}\langle T_n(f_l\otimes\cdots\otimes f_n\otimes f_1\cdots\otimes f_{l-1}\blacktriangleleft b_1),\kappa(b_2\otimes g)\rangle\\
&+(-1)^{\varepsilon}\langle T_n(f_l\otimes\cdots\otimes f_n\otimes f_1\cdots\otimes f_{l-1}),\kappa(b_1\otimes b_2\blacktriangleright g)\rangle\\
&-(-1)^{\varepsilon}\langle T_n(f_l\otimes\cdots\otimes f_n\otimes f_1\cdots\otimes f_{l-1}),\kappa(b_1*b_2\otimes g)\rangle\\
=&(-1)^{\varepsilon}\langle T_n(f_l\otimes\cdots\otimes f_n\otimes f_1\cdots\otimes f_{l-1}\blacktriangleleft b_1)\rhd b_2\\
&+(T_n(f_l\otimes\cdots\otimes f_n\otimes f_1\cdots,f_{l-1})\rhd b_1)*b_2\\
&-T_n(f_l\otimes\cdots\otimes f_n\otimes f_1\cdots\otimes f_{l-1})\rhd(b_1*b_2),g\rangle\\
=&0,
\end{aligned}
$$

where $(-1)^{\varepsilon}$ is the Koszul sign determined by the cyclic permutation. Thus T_n also satisfies Equation (1.3) for all $1<l\leqslant n$.

In conclusion, T_n is strong n-derivative relative to B. □

In the remainder of the paper, we mainly work with interactive pairs whose acting algebras are dg homotopy relative Rota–Baxter algebras. Accordingly, we introduce the following concepts.

Definition 9.6. A *homotopy Rota–Baxter interactive pair*, denoted by $((A,\{T_n\}_{n\geqslant 1}),B)$, is an interactive pair (A,B) where the acting algebra $(A,d_A,\cdot)$ is equipped with a dg homotopy relative Rota–Baxter structure $\{T_n:(A^\vee)^{\otimes n}\to A\}_{n\geqslant 1}$, such that each T_n is an n-derivation relative to B.

If, in addition, each T_n is a strong n-derivation relative to B, then $((A,\{T_n\}_{n\geqslant 1}),B)$ is called a *strong homotopy Rota–Baxter interactive pair*. If each T_n is cyclic (resp. ultracyclic), then the structure is called a *cyclic (resp. ultracyclic) homotopy Rota–Baxter interactive pair.*

9.2 Constructing pre-Calabi–Yau algebras from cyclic homotopy Rota–Baxter algebras

We begin by constructing an A_∞-algebra structure on the space $\partial_{-1}B$, where B is the base algebra of a strong homotopy Rota–Baxter interactive pair.

Lemma 9.7. *Let $\big((A,\{T_n\}_{n\geqslant 1}),B\big)$ be a strong homotopy Rota–Baxter interactive pair. Define a family of operations $\{m_n\}_{n\geqslant 1}$ on the space $\partial_{-1}B:=B\oplus s^{-1}B^\vee$ as follows:*

(i) *For all $b\in B$ and $f\in B^\vee$,*

$$m_1(b,s^{-1}f):=-d_{\partial_{-1}B}(b,s^{-1}f)=-\Big(d_B(b),(-1)^{|f|}s^{-1}f\circ d_B\Big);$$

(ii) *For all $b_1,b_2\in B,f_1,f_2\in B^\vee$,*

$$\begin{aligned}&m_2((b_1,s^{-1}f_1)\otimes(b_2,s^{-1}f_2))\\ :=&\Big(b_1*b_2,\ (-1)^{|b_1|}s^{-1}(b_1\blacktriangleright f_2)+s^{-1}(f_1\blacktriangleleft b_2)\Big);\end{aligned}$$

(iii) *For all $b_1,\dots,b_{n+1}\in B,f_1,\dots,f_n\in B^\vee$,*

$$\begin{aligned}&m_{2n+1}(b_1\otimes s^{-1}f_1\otimes b_2\otimes\cdots\otimes s^{-1}f_n\otimes b_{n+1})\\ :=&(-1)^{\gamma}T_n\left(\kappa(b_1\otimes f_1)\otimes\cdots\otimes\kappa(b_n\otimes f_n)\right)\rhd b_{n+1};\end{aligned}$$

(iv) *For all $b_1,\dots,b_n\in B,f_0,\dots,f_n\in B^\vee$,*

$$\begin{aligned}&m_{2n+1}(s^{-1}f_0\otimes b_1\otimes s^{-1}f_1\otimes\cdots\otimes b_n\otimes s^{-1}f_n)\\ :=&(-1)^{|f_0|+\gamma}s^{-1}\left(f_0\lhd T_n\left(\kappa(b_1\otimes f_1)\otimes\cdots\otimes\kappa(b_n\otimes f_n)\right)\right);\end{aligned}$$

(v) *m_n vanishes in all other cases;*

where

$$\gamma=\sum_{k=1}^{n}(n-k+1)|b_k|+\sum_{k=1}^{n}(n-k)|f_k|.$$

Then $\big(\partial_{-1}B,\{m_n\}_{n\geqslant 1}\big)$ forms an A_∞-algebra.

Proof. The proof involves a detailed and technical computation. For clarity and conciseness, we defer the full argument to Appendix C. □

Corollary 9.8. *Let $(A,A^\vee,\{T_n\}_{n\geqslant 1})$ be a dg homotopy relative Rota–Baxter algebra, and let B be a differential graded left A-module. Then the family of operations $\{m_n\}_{n\geqslant 1}$ defined in Lemma 9.7 equips $\partial_{-1}B$ with an A_∞-algebra structure in which m_2 is trivial.*

We emphasize that the homotopy Rota–Baxter structure plays a central role in constructing the A_∞-algebra structure described above. Even when A is an ordinary (non-homotopy) Rota–Baxter algebra, the induced A_∞-structure on $\partial_{-1}B$ may still be nontrivial. Consider, for instance, a homotopy Rota–Baxter interactive pair (A,B) in which the acting algebra A is a Rota–Baxter algebra with a Rota–Baxter operator T and the base algebra B is a finite-dimensional A-module. According to the formulas in Lemma 9.7, the resulting A_∞-structure $\{m_n\}_{n\geqslant 1}$ on $\partial_{-1}B$ satisfies $m_n = 0$ for all $n \neq 3$, and the only nontrivial operation $m_3 : (\partial_{-1}B)^{\otimes 3} \to \partial_{-1}B$ is given by:

$$m_3(b_1 \otimes s^{-1}f_1 \otimes b_2) = T(\kappa(b_1 \otimes f_1)) \triangleright b_2,$$
$$m_3(s^{-1}f_1 \otimes b_2 \otimes s^{-1}f_2) = s^{-1}f_1 \triangleleft T(\kappa(b_2 \otimes f_2)),$$

for $b_1, b_2 \in B$ and $f_1, f_2 \in B^\vee$, and vanishes in all other cases. Notably, the definition of m_3 explicitly involves the Rota–Baxter operator T.

Furthermore, cyclic homotopy Rota–Baxter operators can produce pre-Calabi–Yau algebra structures.

Theorem 9.9. *Let $\big((A,\{T_n\}_{n\geqslant 1}),B\big)$ be a homotopy Rota–Baxter interactive pair, where the acting algebra A and the base algebra B are locally finite-dimensional.*

(i) If each T_n is cyclic, then B admits a good manageable pre-Calabi–Yau algebra structure.

(ii) If each T_n is ultracyclic, then B admits a good manageable special pre-Calabi–Yau algebra structure.

Proof. Suppose that each T_n is cyclic and an n-derivation relative to B. Then, by Proposition 9.5, each T_n is in fact a strong n-derivation relative to B. By Lemma 9.7, this implies that there is an A_∞-algebra structure on $\partial_{-1}B$.

We now verify that this A_∞-algebra is cyclic under the assumption that the homotopy relative Rota–Baxter structure is cyclic. First, note that m_1 is cyclic. For $n \geqslant 1$, $b_0, \dots, b_n \in B$, and $f_0, \dots, f_n \in B^\vee$, we compute:

$$\begin{aligned}
&\zeta_B\big(m_{2n+1}(b_1 \otimes s^{-1}f_1 \otimes \cdots \otimes b_n \otimes s^{-1}f_n \otimes b_0), s^{-1}f_0\big)\\
&= (-1)^{\gamma}\zeta_B\big(T_n(\kappa(b_1 \otimes f_1) \otimes \cdots \otimes \kappa(b_n \otimes f_n)) \triangleright b_0, s^{-1}f_0\big)\\
&= (-1)^{\gamma+(|f_0|-1)(n-1+|b_0|+\sum\limits_{k=1}^{n}(|b_k|+|f_k|))} f_0\big(T_n(\kappa(b_1 \otimes f_1) \otimes \cdots \otimes \kappa(b_n \otimes f_n)) \triangleright b_0\big)\\
&= (-1)^{2n-1+(|f_0|-1)(n+|b_0|+\sum\limits_{k=1}^{n}(|b_k|+|f_k|))} \zeta_B\big(m_{2n+1}(s^{-1}f_0 \otimes b_1 \otimes \cdots \otimes b_n \otimes s^{-1}f_n), b_0\big).
\end{aligned}$$

By Proposition 8.20, the induced operators $\{\overline{T}_n\}_{n\geqslant 1}$ on $\partial_0 A$ form a cyclic homotopy Rota–Baxter operator. Thus,

$$
\begin{aligned}
&\zeta_B\big(m_{2n+1}(s^{-1}f_0\otimes b_1\otimes\cdots\otimes b_n\otimes s^{-1}f_n),b_0\big)\\
=&(-1)^{|f_0|+\gamma}f_0\big(T_n(\kappa(b_1\otimes f_1)\otimes\cdots\otimes\kappa(b_n\otimes f_n))\triangleright b_0\big)\\
=&(-1)^{|f_0|+\gamma+|f_0|(n-1+|b_0|+\sum_{k=1}^{n}(|b_k|+|f_k|))}\langle T_n(\kappa(b_1\otimes f_1)\otimes\cdots\otimes\kappa(b_n\otimes f_n)),\kappa(b_0\otimes f_0)\rangle\\
=&(-1)^{|f_0|+\gamma+|f_0|(n-1+|b_0|+\sum_{k=1}^{n}(|b_k|+|f_k|))+n+(|b_0|+|f_0|)\sum_{k=1}^{n}(|b_k|+|f_k|)}\\
&\langle T_n(\kappa(b_0\otimes f_0)\otimes\cdots\otimes\kappa(b_{n-1}\otimes f_{n-1})),\kappa(b_n\otimes f_n)\rangle\\
=&(-1)^{2n-1+|b_0|(n+1+|f_0|+\sum_{k=1}^{n}(|b_k|+|f_k|))+\sum_{k=0}^{n-1}(n-k+1)|b_k|+\sum_{k=0}^{n-1}(n-k)|f_k|}\\
&\zeta_A\big(T_n(\kappa(b_0\otimes f_0)\otimes\cdots\otimes\kappa(b_{n-1}\otimes f_{n-1}))\triangleright b_n,s^{-1}f_n\big)\\
=&(-1)^{2n-1+|b_0|(n+1+|f_0|+\sum_{k=1}^{n}(|b_k|+|f_k|))}\zeta_B\big(m_{2n+1}(b_0\otimes s^{-1}f_0\otimes\cdots\otimes b_n),s^{-1}f_n\big).
\end{aligned}
$$

Hence, $\partial_{-1}B$ is a (-1)-cyclic A_∞-algebra containing B as an A_∞-subalgebra; that is, B is a pre-Calabi–Yau algebra. By the construction in Lemma 9.7, this pre-Calabi–Yau algebra is good and manageable.

Now assume further that each T_n is skew-symmetric. For each $n\geqslant 1$, $b_1,\ldots,b_{n+1}\in B$, $f_1,\ldots,f_{n+1}\in B^\vee$, and $\sigma\in\mathfrak{S}_n$, we have:

$$
\begin{aligned}
&\zeta_B\big(m_{2n+1}(b_1\otimes s^{-1}f_1\otimes\cdots\otimes b_n\otimes s^{-1}f_n\otimes b_{n+1}),s^{-1}f_{n+1}\big)\\
=&(-1)^{\gamma+(|f_{n+1}|-1)(n-1+|b_{n+1}|+\sum_{k=1}^{n}(|b_k|+|f_k|))}f_{n+1}\big(T_n(\kappa(b_1\otimes f_1)\otimes\cdots\otimes\kappa(b_n\otimes f_n))\triangleright b_{n+1}\big)\\
=&\chi(\sigma;b_1\otimes f_1\otimes\cdots\otimes b_n\otimes f_n)f_{n+1}\big(T_n(\kappa(b_{\sigma(1)}\otimes f_{\sigma(1)})\otimes\cdots\otimes\kappa(b_{\sigma(n)}\otimes f_{\sigma(n)}))\triangleright b_{n+1}\big)\\
=&\varepsilon(\sigma;b_1\otimes s^{-1}f_1\otimes\cdots\otimes b_n\otimes s^{-1}f_n)\zeta_B\big(m_{2n+1}(b_{\sigma(1)}\otimes s^{-1}f_{\sigma(1)}\otimes\cdots\\
&\cdots\otimes b_{\sigma(n)}\otimes s^{-1}f_{\sigma(n)}\otimes b_{n+1}),s^{-1}f_{n+1}\big).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\zeta_B\big(m_{2n+1}(s^{-1}f_1\otimes b_1\otimes\cdots\otimes s^{-1}f_n\otimes b_n\otimes s^{-1}f_{n+1}),b_{n+1}\big)\\
=&\varepsilon(\sigma;s^{-1}f_1\otimes b_1\otimes\cdots\otimes s^{-1}f_n\otimes b_n)\zeta_B\big(m_{2n+1}(s^{-1}f_{\sigma(1)}\otimes b_{\sigma(1)}\otimes\cdots\\
&\cdots\otimes s^{-1}f_{\sigma(n)}\otimes b_{\sigma(n)}\otimes s^{-1}f_{n+1}),b_{n+1}\big).
\end{aligned}
$$

We already know that m_{2n+1} is cyclic. Moreover, by the skew-symmetry of T_n, we conclude that m_{2n+1} is ultracyclic. Thus, if $\{T_n\}_{n\geqslant 1}$ is ultracyclic, then B is a special pre-Calabi–Yau algebra. □

Remark 9.10. *In fact, the assumption that the acting algebra A is locally finite-dimensional in Theorem 9.9 is not essential. The theorem remains valid even when A is not locally finite-dimensional, and in such cases, the proof can still be carried out through direct computation.*

Corollary 9.11. *Let $(A,d_A,\cdot,\{T_n\}_{n\geqslant 1})$ be a locally finite-dimensional cyclic dg homotopy Rota–Baxter algebra, and let B be a locally finite-dimensional dg module over the dg algebra $(A,d_A,\cdot)$. Then B admits a fine pre-Calabi–Yau algebra structure.*

Proof. Since A is a locally finite-dimensional cyclic dg homotopy Rota–Baxter algebra, it is in particular a dg homotopy relative Rota–Baxter algebra. Let B be a dg A-module. According to Example 9.2(2), the pair (A,B) always forms an interactive pair and is clearly homotopy Rota–Baxter compatible. The result then follows directly from Theorem 9.9. □

Corollary 9.12. *Let B be a finite dimensional graded space, A the graded algebra $\mathrm{End}_{\mathrm{gr}}(B)$ with the composition being multiplication. Then the following four maps given by Lemma 9.7 are bijections:*

$$\left\{\begin{array}{l}\text{pairs } (d_B,\ \{T_n\}_{\geqslant 1}) \text{ where } d_B \text{ is a differential on}\\ B \text{ and } \{T_n : (A^{\vee})^{\otimes n} \to A\}_{n\geqslant 1} \text{ is a cyclic}\\ \text{homotopy relative Rota–Baxter operator}\end{array}\right\} \to \left\{\begin{array}{l}\text{fine pre-Calabi–Yau algebra}\\ \text{structures on } B\end{array}\right\},$$

$$\left\{\begin{array}{l}\text{triples } (d_B, m, \{T_n\}_{n\geqslant 1}) \text{ where } (B, d_B, m) \text{ is a dg}\\ \text{algebra and } \big((A, \{T_n\}_{n\geqslant 1}), B\big) \text{ forms a cyclic}\\ \text{homotopy Rota–Baxter interactive pair}\end{array}\right\} \to \left\{\begin{array}{l}\text{good manageable pre-Calabi–Yau}\\ \text{algebra structures on } B\end{array}\right\},$$

$$\left\{\begin{array}{l}\text{pairs } (d_B, \{T_n\}_{n\geqslant 1}) \text{ where } d_B \text{ is a differential}\\ \text{on } B \text{ and } \{T_n : (A^{\vee})^{\otimes n} \to A\}_{n\geqslant 1} \text{ is an}\\ \text{ultracyclic homotopy Rota–Baxter operator}\end{array}\right\} \to \left\{\begin{array}{l}\text{fine special pre-Calabi–Yau algebra}\\ \text{structures on } B\end{array}\right\},$$

$$\left\{\begin{array}{l}\text{triples } (d_B, m, \{T_n\}_{\geqslant 1}) \text{ where } (B, d_B, m)\\ \text{is a dg algebra and } \{T_n\}_{n\geqslant 1} \text{ makes } (A, B)\\ \text{into an ultracyclic homotopy Rota–Baxter}\\ \text{interactive pair}\end{array}\right\} \to \left\{\begin{array}{l}\text{good manageable special}\\ \text{pre-Calabi–Yau algebra structures on } B\end{array}\right\},$$

where A is always endowed with the induced differential by d_B.

Proof. Each good map m_{2n+1} can be uniquely determined by an operator $\widetilde{T_n} : (B \otimes B^{\vee})^{\otimes n} \to B \otimes B^{\vee}$. Since $\kappa : B \otimes B^{\vee} \to \mathrm{End}_{\mathrm{gr}}((B)^{\vee}$ is an isomorphism, the maps are bijective. □

By Theorem 9.9 and the cyclic completion for homotopy Rota–Baxter algebras Proposition 8.18, we have the following result.

Proposition 9.13. *Let $(A, \{T_n\}_{n\geqslant 1})$ be a locally finite-dimensional dg homotopy Rota–Baxter algebra. Then there is a fine pre-Calabi–Yau structure $\{m_n\}_{n\geqslant 1}$ on any locally finite-dimensional left dg $\partial_0 A$-module M. Moreover, if $\{T_n\}_{n\geqslant 1}$ is skew-symmetric, the induced pre-Calabi–Yau algebra structure on M is fine and special.*

Chapter 10

Homotopy Rota–Baxter algebras and double Poisson structures

In Chapter 9, we constructed a good manageable (resp. good manageable special) pre-Calabi-Yau algebra on the base algebra of a homotopy Rota–Baxter interactive pair endowed with a cyclic (resp. an ultracyclic) homotopy Rota–Baxter operator $\{T_n\}_{n\geqslant 1}$. In [36], Fernández and Herscovich established an equivalence between good manageable special pre-Calabi-Yau algebras and homotopy double Poisson algebras. In the present section, we combine these results to give a direct construction of a homotopy double Poisson algebra from a homotopy Rota–Baxter structure. Specifically, we show that the base algebra of a ultracyclic (resp. cyclic) homotopy Rota–Baxter interactive pair naturally inherits a (resp. cyclic) homotopy double Poisson structure. Moreover, we observe that any module over an ultracyclic homotopy relative Rota–Baxter algebra carries a homotopy double Lie structure, from which it follows that the symmetric algebra on such a module acquires the structure of a homotopy Poisson algebra. As an application, we establish an equivalence between skew-symmetric solutions of the associative Yang-Baxter-infinity equations, ultracyclic homotopy Rota–Baxter algebra structures, fine special pre-Calabi-Yau algebras, and homotopy double Lie algebras.

10.1 Homotopy double Poisson algebras arising from homotopy Rota–Baxter algebras

Definition 10.1. A *cyclic homotopy double Lie algebra* (also called a *cyclic double L_∞-algebra*) is a graded space $V=\oplus_{n\in\mathbb{Z}}V_n$ equipped with a family of homogeneous maps $\{\{-,\cdots,-\}\}_n : V^{\otimes n}\to V^{\otimes n}$ with $|\{\{-,\cdots,-\}\}_n| = n-2$ satisfying the following conditions for all $n\geqslant 1$,

- Cyclic-symmetry: For all elements $\sigma\in\mathfrak{C}_n$(the cyclic group of n elements)

$$\sigma\circ\{\{-,\cdots,-\}\}_n\circ\sigma^{-1} = \operatorname{sgn}(\sigma)\{\{-,\cdots,-\}\}_n;$$

- Double Jacobi$_\infty$ identity:

$$\sum_{i+j=n+1}(-1)^{(j-1)i}\sum_{\sigma\in\mathfrak{C}_n}\operatorname{sgn}(\sigma)\sigma\circ\{\{-,\cdots,-,\{\{-,\cdots,-\}\}_i\}\}_{L,j}\circ\sigma^{-1}=0, \qquad (1.1)$$

  where

$$\{\{-,\cdots,-,\{\{-,\cdots,-\}\}_{i+1}\}\}_{L,j+1} = \left(\{\{-,\cdots,-\}\}_{j+1}\otimes \mathrm{Id}_V^{\otimes i}\right)\circ\left(\mathrm{Id}_V^{\otimes j}\otimes\{\{-,\cdots,-\}\}_{i+1}\right).$$

If, in addition, each map $\{\{-,\cdots,-\}\}_n$ is skew-symmetric, meaning that for all $\sigma \in \mathfrak{S}_n$,

$$\sigma \circ \{\{-,\cdots,-\}\}_n \circ \sigma^{-1} = \operatorname{sgn}(\sigma)\{\{-,\cdots,-\}\}_n \text{ , for all } \sigma \in \mathfrak{S}_n,$$

then $(V, \{\{-,\cdots,-\}\}_n)$is called *double L_∞-algebra* (also known as *homotopy double Lie algebra*).

The following lemma offers an alternative characterization of a homotopy double Lie algebra, which will be used later.

Lemma 10.2. *Let $\{\{\{-,\cdots,-\}\}_n : V^{\otimes n} \to V^{\otimes n}\}_{n\geqslant 1}$ be a family of operations on a graded space $V = \oplus_{n\in\mathbb{Z}} V^n$. For each $k \geqslant 1$, define the opposite bracket $\{\{-,\cdots,-\}\}_k^{\mathrm{op}} := \sigma_k \circ \{\{-,\cdots,-\}\}_k \circ \sigma_k^{-1}$, where $\sigma_k \in \mathfrak{S}_k$ is the order-reversing permutation*

$$\sigma_k = \begin{pmatrix} 1 & 2 & \cdots & k \\ k & k-1 & \cdots & 1 \end{pmatrix} \in \mathfrak{S}_k.$$

Then the family $\{\{\{-,\cdots,-\}\}_n\}_{n\geqslant 1}$ satisfies the double Jacobi$_\infty$ identity if and only if the opposite operations $\{\{\{-,\cdots,-\}\}_{n+1}^{\mathrm{op}}\}_{n\geqslant 0}$ fulfill the following identities:

$$\sum_{i+j=n+1} (-1)^{i(j-1)} \sum_{\sigma\in\mathfrak{C}_n} \operatorname{sgn}(\sigma)\sigma \circ \left(\{\{\{\{-,\cdots,-\}\}_i^{\mathrm{op}}, -,\cdots,-\}\}_{R,j}^{\mathrm{op}} \right) \circ \sigma^{-1} = 0, \tag{1.2}$$

where the right-nested composite is defined by

$$\{\{\{\{-,\cdots,-\}\}_{i+1}^{\mathrm{op}}, -,\cdots,-\}\}_{R,j+1}^{\mathrm{op}} = \left(\mathrm{Id}_V^{\otimes i} \otimes \{\{-,\cdots,-\}\}_{j+1}^{\mathrm{op}} \right) \circ \left(\{\{-,\cdots,-\}\}_{i+1}^{\mathrm{op}} \otimes \mathrm{Id}_V^{\otimes j} \right).$$

Proof. The claim follows by applying the conjugation $\sigma_n \circ (\text{Equation (1.1)}) \circ \sigma_n^{-1}$, which transforms the original double Jacobi$_\infty$ identity into Equation (1.2). □

Definition 10.3.

- A *cyclic homotopy double Poisson algebra* is a graded vector space A equipped with both an associative algebra structure and a cyclic double L_∞-algebra structure, satisfying the *double Leibniz$_\infty$ rule*: for all $n \geq 0$ and homogeneous elements $a_1,\ldots,a_{n-1},a_n',a_n'' \in A$,

$$\begin{aligned} \{\{a_1,\ldots,a_n,a_{n+1}'a_{n+1}''\}\}_n &= \{\{a_1,\ldots,a_n',\}\}_n \cdot a_{n+1}'' \\ &\quad + (-1)^{|a_{n+1}'|(n-2+\sum_{k=1}^{n}|a_k|)} a_{n+1}' \cdot \{\{a_1,\ldots,a_n''\}\}_n, \end{aligned}$$

  where multiplication by a_{n+1}'' and a_{n+1}' is understood to act on the rightmost and leftmost components of the tensor product, respectively.

- A *double Poisson$_\infty$ algebra* (also called a *homotopy double Poisson algebra*) is a graded algebra A equipped with a double L_∞-algebra structure that satisfies the double Leibniz$_\infty$ rule.

Next, we recall the following result of Fernández and Herscovich [36], which establishes a connection between ultracyclic pre-Calabi-Yau algebras and homotopy double Poisson algebras.

Theorem 10.4. *[36, Theorem 6.3] Let $A = \oplus_{n\in\mathbb{Z}} A^n$ be a finite dimensional graded space. For a good manageable special pre-Calabi–Yau structure $\{m_n\}_{n\geqslant 1}$ on A, define a family of maps*

$$\{\{\{-,\cdots,-\}\}_n : A^{\otimes n} \to A^{\otimes n}\}_{n\geqslant 1}$$

by

$$(f_1\otimes\cdots\otimes f_n)\left(\{\{a_1,\cdots,a_n\}\}_n\right)=s^{a_1,\cdots,a_n}_{f_1,\cdots,f_n}\zeta_A\left(m_{2n-1}\left(a_n,s^{-1}f_n,\cdots,a_2,s^{-1}f_2,a_1\right),s^{-1}f_1\right) \tag{1.3}$$

for all homogeneous elements $a_1,\cdots,a_n\in A$ *and* $f_1,\cdots,f_n\in A^\vee$, *where*

$$s^{a_1,\cdots,a_n}_{f_1,\cdots,f_n}=(-1)^{|a_n||f_1|+(n+1)(|a_n|+|f_1|)+\sum\limits_{j=1}^{n}(n-j)|a_j|+\sum\limits_{j=1}^{n}(j-1)|f_j|+\sum\limits_{1\leqslant i<j<n}|a_i||a_j|+\sum\limits_{1<i<j\leqslant n}|f_i||f_j|+\sum\limits_{1<i\leqslant j<n}|f_i||a_j|}.$$

The family of maps $\{\{\{-,\cdots,-\}\}_n\}_{n\geqslant 1}$, *together with the dg algebra structure on A, defines a homotopy double Poisson algebra structure on the graded space A.*

Moreover, the assignment

$$\left\{\begin{array}{l}\text{good manageable special pre-Calabi-Yau}\\ \text{algebra structures } \{m_n\}_{n\geqslant 1} \text{ on } A\end{array}\right\}\to\left\{\begin{array}{l}\text{homotopy double Poisson algebra}\\ \text{structures } \{\{\{-,\cdots,-\}\}_n\}_{n\geqslant 1} \text{ on } A\end{array}\right\}$$

defined by (1.3) *is a bijection.*

In Equation (1.3), the assumption that each operation m_{2n-1} in the pre-Calabi-Yau algebra structure is cyclic ensures that the operation $\{\{-,\cdots,-\}\}_n$ satisfies the cyclic symmetry condition in Definition 10.1 for all $n\geqslant 1$. In fact, when Fernández and Herscovich prove Theorem 10.4 in [36], the assumption that the pre-Calabi-Yau structure is ultracyclic is used solely to guarantee that all the operations $\{\{\{-,\ldots,-\}\}_n\}_{n\geqslant 1}$ are skew-symmetric. In verifying that the family $\{\{\{-,\cdots,-\}\}_n\}_{n\geqslant 1}$ satisfies the double Leibniz$_\infty$ rule and the double Jacobi$_\infty$ identities, only the cyclicity of the pre-Calabi-Yau structure is required. Therefore, without assuming that the pre-Calabi-Yau algebra is special, the bijection in the above theorem extends to a correspondence between the class of good manageable pre-Calabi-Yau structures and the class of cyclic homotopy double Poisson algebra structures. Thus we have

Theorem 10.5. *Let* $A=\oplus_{n\in\mathbb{Z}}A^n$ *be a finite dimensional graded space. Given a good manageable pre-Calabi–Yau structure* $\{m_n\}_{n\geqslant 1}$ *on A, define a family of maps* $\{\{\{-,\cdots,-\}\}_n : A^{\otimes n}\to A^{\otimes n}\}_{n\geqslant 1}$ *as in* (1.3). *Then the family of maps* $\{\{\{-,\cdots,-\}\}_n\}_{n\geqslant 1}$, *together with the dg algebra structure on A, defines a cyclic homotopy double Poisson algebra structure on the graded space A.*

Moreover, the assignment

$$\left\{\begin{array}{l}\text{good manageable pre-Calabi–Yau}\\ \text{algebra structures } \{m_n\}_{n\geqslant 1} \text{ on } A\end{array}\right\}\to\left\{\begin{array}{l}\text{cyclic homotopy double Poisson algebra}\\ \text{structures } \{\{\{-,\cdots,-\}\}_n\}_{n\geqslant 1} \text{ on } A\end{array}\right\}$$

defined by (1.3) *is a bijection.*

As a direct consequence of Theorem 10.4 and Theorem 10.5 , we have the following result:

Corollary 10.6. *The following three maps are bijections via* (1.3)*:*

$$\left\{\begin{array}{l}\text{fine pre-Calabi–Yau algebra}\\ \text{structures } \{m_n\}_{n\geqslant 1} \text{ on } A\end{array}\right\}\to\left\{\begin{array}{l}\text{cyclic homotopy double Lie algebra}\\ \text{structures } \{\{\{-,\cdots,-\}\}_n\}_{n\geqslant 1} \text{ on } A\end{array}\right\},$$

$$\left\{\begin{array}{l}\text{fine special pre-Calabi–Yau}\\ \text{algebra structures } \{m_n\}_{n\geqslant 1} \text{ on } A\end{array}\right\}\to\left\{\begin{array}{l}\text{homotopy double Lie algebra}\\ \{\{\{-,\cdots,-\}\}_n\}_{n\geqslant 1} \text{ on } A\end{array}\right\}.$$

In Theorem 9.9, we constructed pre-Calabi-Yau structures from homotopy Rota–Baxter algebras. By combining this construction with Theorem 10.4 and Theorem 10.5, we obtain the following result, which provides a method for constructing homotopy double Poisson algebras from homotopy Rota–Baxter structures.

Theorem 10.7. *Let $\big((A,\{T_n\}_{n\geqslant 1}),B\big)$ be a homotopy Rota–Baxter interactive pair, where the acting algebra A is finite-dimensional and the base algebra B is locally finite-dimensional.*

Define a sequence of maps $\{\{\{-,\dots,-\}\}_n : B^{\otimes n}\to B^{\otimes n}\}_{n\geqslant 1}$ by setting $\{\{-\}\}_1 = d_B$, and for all $n\geqslant 1$,

$$\{\{-,\dots,-\}\}_{n+1} := \Psi^n(\mathrm{Id}_{A^{\otimes n}}),\tag{1.4}$$

where the map Ψ^n is the composition:

$$\Psi^n : \mathrm{End}_{\mathrm{gr}}(A^{\otimes n}) \cong A^{\otimes n}\otimes (A^\vee)^{\otimes n} \xrightarrow{\mathrm{Id}^{\otimes n}\otimes T_n} A^{\otimes (n+1)} \xrightarrow{\Phi^{\otimes(n+1)}} \mathrm{End}_{\mathrm{gr}}(B)^{\otimes(n+1)} \to \mathrm{End}_{\mathrm{gr}}(B^{\otimes(n+1)}),$$

and $\Phi : A\to \mathrm{End}_{\mathrm{gr}}(B)$ denotes the left A-action on B, i.e., $\Phi(a)(b) := a\rhd b$.

Then,

(i) If each T_n is cyclic, the collection $\{\{\{-,\dots,-\}\}_n\}_{n\geqslant 1}$ defines a cyclic homotopy double Poisson algebra structure on B.

(ii) If each T_n is ultracyclic, the collection $\{\{\{-,\dots,-\}\}_n\}_{n\geqslant 1}$ defines a homotopy double Poisson algebra structure on B.

Proof. Let $\{e_i\}_{i\in I}$ be a homogeneous basis of A and $\{e^i\}_{i\in I}$ be the corresponding dual basis. Then $\mathrm{Id}_{A^{\otimes n}}\in \mathrm{End}_{\mathrm{gr}}(A^{\otimes n})$ corresponds to the element $\sum\limits_{i_1,\cdots,i_n} e_{i_1}\otimes\cdots\otimes e_{i_n}\otimes e^{i_n}\otimes\cdots\otimes e^{i_1}\in A^{\otimes n}\otimes (A^\vee)^{\otimes n}$. Thus, we can write

$$\{\{-,\cdots,-\}\}_{n+1} = \Phi^{\otimes n+1}\Big(\sum_{i_1,\cdots,i_n}(-1)^{(n-1)(\sum\limits_{k=1}^n|e_{i_k}|)} e_{i_1}\otimes\cdots\otimes e_{i_n}\otimes T_n(e^{i_n}\otimes\cdots\otimes e^{i_1})\Big).$$

It remains to verify that the image of the A_∞-structure $\{m_n\}_{n\geqslant 1}$ under the construction given in (1.3), as described in Theorems 10.4 and 10.5, coincides with the family $\{\{\{-,\dots,-\}\}_n\}_{n\geqslant 1}$ defined by (1.4). Let $b_1,\cdots,b_n\in B$ and $f_1,\cdots,f_n\in B^\vee$

$$\begin{aligned}
&(f_1\otimes\cdots\otimes f_{n+1})(\{\{b_1,\cdots,b_{n+1}\}\}_{n+1})\\
=&(f_1\otimes\cdots\otimes f_{n+1})\Phi^{\otimes n+1}\Big(\sum_{i_1,\cdots,i_n}(-1)^{(n-1)(\sum\limits_{k=1}^n|e_{i_k}|)} e_{i_1}\otimes\cdots\otimes e_{i_n}\otimes T_n(e^{i_n}\otimes\cdots\otimes e^{i_1})\Big)(b_1\otimes\cdots\otimes b_{n+1})\\
=&\sum_{i_1,\cdots,i_n}(-1)^{(n-1)(\sum\limits_{k=1}^n(|e_{i_k}|+|b_k|))+\sum\limits_{1\leqslant i<j\leqslant n+1}|b_i||f_j|+\sum\limits_{1\leqslant s<k\leqslant j\leqslant n}(|b_j|+|f_s|)|e_{i_k}|+|f_{n+1}|(\sum\limits_{k=1}^n|e_{i_k}|)}\\
&\quad f_1(e_{i_1}\rhd b_1)\cdots f_n(e_{i_n}\rhd b_n)f_{n+1}\big(T_n(e^{i_n}\otimes\cdots\otimes e^{i_1})\rhd b_{n+1}\big)\\
=&(-1)^{(n-1)(\sum\limits_{k=1}^n|f_k|)+\sum\limits_{1\leqslant i<j\leqslant n+1}|b_i||f_j|+\sum\limits_{1\leqslant s<k\leqslant j\leqslant n}(|b_j|+|f_s|)(|b_k|+|f_k|)+|f_{n+1}|(\sum\limits_{k=1}^n(|b_k|+|f_k|))}\\
&\quad f_{n+1}\Big(T_n\big(\kappa(b_n\otimes f_n),\cdots,\kappa(b_1\otimes f_1)\big)\rhd b_{n+1}\Big)\\
=&(-1)^{(n-1)(\sum\limits_{k=1}^{n+1}|f_k|)+\sum_{1\leqslant i<j\leqslant n+1}|b_i||f_j|+\sum\limits_{1\leqslant s<k\leqslant j\leqslant n}(|b_j|+|f_s|)(|b_k|+|f_k|)+|f_{n+1}||b_{n+1}|}\\
&\quad \langle T_n\big(\kappa(b_n\otimes f_n),\cdots,\kappa(b_1\otimes f_1)\big),\kappa(b_{n+1}\otimes f_{n+1})\rangle\\
=&(-1)^{(n-1)(\sum\limits_{k=1}^{n+1}|f_k|)+\sum\limits_{1\leqslant i<j\leqslant n+1}|b_i||f_j|+\sum\limits_{1\leqslant s<k\leqslant j\leqslant n}(|b_j|+|f_s|)(|b_k|+|f_k|)+|f_{n+1}||b_{n+1}|+n+(|b_{n+1}|+|f_{n+1}|)(\sum\limits_{k=1}^n(|b_k|+|f_k|))}\\
&\quad \langle T_n\big(\kappa(b_{n+1}\otimes f_{n+1}),\cdots,\kappa(b_2\otimes f_2)\big),\kappa(b_1\otimes f_1)\rangle\\
=&(-1)^n s^{b_1,\cdots,b_{n+1}}_{f_1,\cdots,f_{n+1}}\zeta_B(m_{2n+1}(b_{n+1}\otimes s^{-1}f_{n+1}\otimes\cdots\otimes b_2\otimes s^{-1}f_2\otimes b_1),s^{-1}f_1),
\end{aligned}$$

where $\{m_n\}_{n\geqslant 1}$ is defined as Lemma 9.7. Thus, the image of the operation m_{2n-1} under the construction given in (1.3) coincides with $\{\{-,\ldots,-\}\}_n$ up to a sign $(-1)^n$, as defined in (1.4), for all $n \geqslant 1$. By Theorem 9.9, if each T_n is cyclic (resp. ultracyclic), the collection $\{m_n\}_{n\geqslant 1}$ defines a cyclic (resp. ultracyclic) pre-Calabi–Yau algebra structure on B. Consequently, by Theorems 10.4 and 10.5, the family $\{\{\{-,\ldots,-\}\}_n\}_{n\geqslant 1}$ endows B with a homotopy double Poisson algebra structure (resp. cyclic homotopy double Poisson algebra structure). □

As a corollary, we have the following result:

Corollary 10.8. *Let A be a finite-dimensional dg algebra, and let B be a locally finite-dimensional dg left A-module. Suppose $\{T_n : (A^\vee)^{\otimes n} \to A\}_{n\geqslant 1}$ is a homotopy relative Rota–Baxter operator on A.*

If each T_n is ultracyclic (resp. cyclic), then the family of operations $\{\{\{-,\ldots,-\}\}_n\}_{n\geqslant 1}$ defined in Theorem 10.7 endows B with a double L_∞-algebra (resp. cyclic double L_∞-algebra) structure.

Remark 10.9.

(i) In fact, the assumption that B is locally finite-dimensional in Theorem 10.7 is not essential. When B is not locally finite-dimensional, the result still holds, and the proof can be carried out through direct computation.

(ii) The construction in (1.4) serves as a homotopy generalization of the construction in (1.5).

10.2 Homotopy Rota–Baxter algebras and associative Yang–Baxter-infinity equation

In [88], Schedler introduced the notion of the *associative Yang–Baxter-infinity equation* and established a one-to-one correspondence between homotopy double Lie algebra structures and skew-symmetric solutions of this equation.

Definition 10.10. [88] Let A be a graded associative algebra. A solution of *associative Yang–Baxter-infinity equation* is a family of elements $\{r_n \in A^{\otimes n}\}_{n\geqslant 1}$ where each r_n has degree $n-2$, satisfying, for all $n \geqslant 1$,

$$\sum_{i+j=n+1} (-1)^{(j+1)i} \sum_{\sigma\in\mathfrak{C}_n} \operatorname{sgn}(\sigma) r_i^{\sigma(1),\sigma(2),\cdots,\sigma(i)} r_j^{\sigma(i),\sigma(i+1),\sigma(i+2),\cdots,\sigma(n)} = 0,$$

Here we write $r_n = \sum r_n^{[1]} \otimes \cdots \otimes r_n^{[n]}$ for each $n \geqslant 1$, and define

$$r_i^{\sigma(1),\ldots,\sigma(i)} \cdot r_j^{\sigma(i),\ldots,\sigma(n)} := \sigma^{-1}\left(r_i^{1,2,\ldots,i} \cdot r_j^{i,i+1,\ldots,n}\right),$$

where

$$r_i^{1,2,\ldots,i} \cdot r_j^{i,i+1,\ldots,n} := \sum r_i^{[1]} \otimes \cdots \otimes r_i^{[i]} \cdot r_j^{[1]} \otimes r_j^{[2]} \otimes \cdots \otimes r_j^{[j]}.$$

If, for all $n \geqslant 1$, the element r_n satisfies $\operatorname{sgn}(\sigma) r_n = r_n^{\sigma(1),\sigma(2),\cdots,\sigma(n)}$, then the solution is called *skew-symmetric*.

Example 10.11. *Let $\{r_n\}_{n\geqslant 1}$ be a skew-symmetric solution of associative Yang–Baxter-infinity equation. For small n, the associative Yang–Baxter-infinity equation yields the following:*

(i) When $n = 1$, $|r_1| = -1$, $r_1 \cdot r_1 = 0$, which implies that the operator $\partial = [r_1, -] : A \to A$ defines a differential on A;

(ii) *when* $n=2$, $|r_1|=-1$, $|r_2|=0$,

$$r_1^1\cdot r_2^{12}+r_2^{21}\cdot r_1^1=r_1^2\cdot r_2^{21}+r_2^{12}\cdot r_1^2,$$

which shows that $r_2\in A\otimes A$ *is a cycle with respect to the differential* $[r_1,-]$;

(iii) *when* $n=3$, $|r_1|=-1$, $|r_2|=0$, $|r_3|=1$,

$$r_2^{12}\cdot r_2^{23}+r_2^{23}\cdot r_2^{31}+r_2^{31}\cdot r_2^{12}=r_1^1\cdot r_3^{123}+r_1^2\cdot r_3^{231}+r_1^3\cdot r_3^{312}+r_3^{231}\cdot r_1^1+r_3^{312}\cdot r_1^2+r_3^{123}\cdot r_1^3,$$

which shows that r_2 *satisfies the usual associative Yang–Baxter equation up to homotopy provided by* r_3.

Schedler further proved that there is a one-to-one correspondence between homotopy double Lie algebra structures and skew-symmetric solutions to associative Yang–Baxter-infinity equation.

Proposition 10.12. *[88] Let* V *be a graded space over* $\mathbf{k}$. *There is a bijection between the set of homotopy double Lie algebra structures on* V *and skew-symmetric solutions of the associative Yang–Baxter-infinity equation on graded algebra* $\mathrm{End}_{\mathrm{gr}}(V)$.

Combining Corollary 9.12, Corollary 10.6 and Proposition 10.12, we have the following equivalence:

Proposition 10.13. *Let* V *be a finite dimensional graded space over* $\mathbf{k}$. *Then the following data are equivalent:*

(i) *A fine special pre-Calabi-Yau algebra structure on* V;

(ii) *A homotopy double Lie algebra structure* $\{\{\{-,\cdots,-\}\}\}_{n\geqslant 1}$ *on* V;

(iii) *A differential* d *on* V *and an ultracyclic homotopy relative Rota–Baxter operator on dg algebra* $(\mathrm{End}_{\mathrm{gr}}(V),[d,-])$;

(iv) *A skew-symmetric solution to associative Yang–Baxter-infinity equation in* $\mathrm{End}_{\mathrm{gr}}(V)$.

10.3 Homotopy Poisson structures arising from homotopy Rota–Baxter algebras

By Proposition 7.10 we known that the symmetric algebra of an L_∞-algebra carries a homotopy Poisson algebra. Now we will show that this is also true for homotopy double Lie algebras.

Proposition 10.14. *Let* $(V,\{\{\{-,\cdots,-\}\}_n\}_{n\geqslant 1})$ *be a homotopy double Lie algebra. Define a family of operations* $\{l_n\}_{n\geqslant 1}$ *on the graded symmetric algebra* $\mathrm{Sym}(V)$ *as follows: for all homogeneous elements* $u_1^1,\cdots,u_{k_1}^1,\cdots,u_1^n,\cdots,u_{k_n}^n\in V$

$$\begin{aligned}&l_n(u_1^1\cdots u_{k_1}^1\otimes\cdots\otimes u_1^n\cdots u_{k_n}^n)\\ :=&(n-1)!\sum_{1\leqslant q_1\leqslant k_1,\cdots 1\leqslant q_n\leqslant k_n}(-1)^{\sum\limits_{s=1}^{n}\left(\sum\limits_{t=1}^{s-1}(\sum\limits_{j=1}^{q_t-1}|u_j|+\sum\limits_{j=q_t+1}^{k_t}|u_j|)+\sum\limits_{j=1}^{q_s-1}|u_j|\right)|u_{q_s}|+\frac{(n-1)n}{2}}\\ &\{\{u_{q_1}^1,\cdots,u_{q_n}^n\}\}_n^{[1]}\cdots\{\{u_{q_1}^1,\cdots,u_{q_n}^n\}\}_n^{[n]}\cdot u_1^1\cdots\widehat{u_{q_1}^1}\cdots u_{k_1}^1\cdots u_1^n\cdots\widehat{u_{q_n}^n}\cdots u_{k_n}^n.\end{aligned}$$

Then $\left(\mathrm{Sym}(V),\{l_n\}_{n\geqslant 1}\right)$ *defines a homotopy Poisson algebra. Thus* $V^\vee$ *can be regarded as a formal derived Poisson manifold.*

Proof. By the skew-symmetry and the Leibniz$_\infty$ rule satisfied by the homotopy double bracket, it follows that the operators $\{l_n\}_{n\geqslant 1}$ are well-defined on the symmetric algebra $\mathrm{Sym}(V)$. Moreover, it is straightforward to verify that the brackets $\{l_n\}_{n\geqslant 1}$ inherit skew-symmetry and satisfy the Leibniz$_\infty$ rule with respect to the natural multiplication on $\mathrm{Sym}(V)$. Therefore, it remains only to check that they also satisfy the Jacobi$_\infty$ rule.

Since each operation l_n satisfies the Leibniz$_\infty$ rule, it suffices to verify that the family $\{l_n\}_{n\geqslant 1}$ satisfies the Jacobi$_\infty$ identity on the generating space $V \subset \mathrm{Sym}(V)$. Let $x_1,\dots,x_n \in V$, and let μ denote the natural multiplication on the symmetric algebra $\mathrm{Sym}(V)$. Then, using the skew-symmetry of the brackets $\{\{\{-,\dots,-\}\}_n\}_{n\geqslant 1}$ and applying Lemma 10.2, we obtain:

$$\sum_{i=1}^{n}\sum_{\sigma\in \mathrm{Sh}(i,n-i)} \chi(\sigma;x_1,\dots,x_n)(-1)^{i(n-i)} l_{n-i+1}(l_i(x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(i)})\otimes x_{\sigma(i+1)}\otimes\cdots\otimes x_{\sigma(n)}) \tag{3.5}$$

$$\begin{aligned}
=&\sum_{i=1}^{n}(i-1)!(n-i)!\sum_{\sigma\in \mathrm{Sh}(i,n-i)}\sum_{k=1}^{i-1}\mathrm{sgn}(\sigma)(-1)^{i(n-i)+\frac{i(i-1)+(n-i)(n-i+1)}{2}}\mu\circ\Big(\mathrm{Id}^{k-1}\otimes\{\{-,\cdots,-\}\}^{[1]}_{n-i+1}\otimes \mathrm{Id}^{i-k}\\
&\otimes\{\{-,\cdots,-\}\}^{[2]}_{n-i+1}\otimes\cdots\otimes\{\{-,\cdots,-\}\}^{[n-i+1]}_{n-i+1}\Big)\circ\Big(\{\{-,\cdots,-\}\}_i\otimes \mathrm{Id}^{\otimes n-i}\Big)\circ\sigma^{-1}(x_1\otimes\cdots\otimes x_n)\\
=&\sum_{i=1}^{n}(-1)^{i(n-i)+\frac{i(i-1)+(n-i)(n-i+1)}{2}}(i-1)!(n-i)!\sum_{\sigma\in \mathrm{Sh}(i,n-i)}\sum_{\tau\in\mathfrak{C}_i\times \mathrm{Id}^{n-i}}\mathrm{sgn}(\sigma)\mathrm{sgn}(\tau)\\
&\mu\circ(\mathrm{Id}^{\otimes i-1}\otimes\{\{-,\cdots,-\}\}_{n-i+1})\circ(\{\{-,\cdots,-\}\}_i\otimes \mathrm{Id}^{n-1})\circ\tau^{-1}\circ\sigma^{-1}(x_1\otimes\cdots\otimes x_n).
\end{aligned}$$

Note that, for each $1\leqslant i\leqslant n$, the composite map

$$\mu\circ(\mathrm{Id}^{\otimes i-1}\otimes\{\{-,\cdots,-\}\}_{n-i+1})\circ(\{\{-,\cdots,-\}\}_i\otimes \mathrm{Id}^{n-1})$$

is graded symmetric with respect to the first $i-1$ inputs and the last $n-i$ inputs. Thus,

$$\begin{aligned}
&(3.5)\\
=&\sum_{i=1}^{n}(-1)^{i(n-i)+\frac{i(i-1)+(n-i)(n-i+1)}{2}}\sum_{\sigma\in\mathfrak{S}_n}\mathrm{sgn}(\sigma)\mu\circ(\mathrm{Id}^{\otimes i-1}\otimes\{\{-,\cdots,-\}\}_{n-i+1})\circ(\{\{-,\cdots,-\}\}_i\otimes \mathrm{Id}^{n-1})\circ\sigma^{-1}\\
=&\sum_{i=1}^{n}(-1)^{i(n-i)}\sum_{\sigma\in\mathfrak{S}_n}\mathrm{sgn}(\sigma)\mu\circ(\mathrm{Id}^{\otimes i-1}\otimes\{\{-,\cdots,-\}\}^{\mathrm{op}}_{n-i+1})\circ(\{\{-,\cdots,-\}\}^{\mathrm{op}}_i\otimes \mathrm{Id}^{n-1})\circ\sigma^{-1}\\
=&\sum_{i=1}^{n}(-1)^{i(n-i)}\sum_{\sigma\in\mathfrak{S}_n}\mathrm{sgn}(\sigma)\mu\circ\Big(\{\{\{\{-,\cdots,-\}\}^{\mathrm{op}}_i,-,\cdots,-\}\}^{\mathrm{op}}_{R,n-i+1}\Big)\sigma^{-1}(x_1\otimes\cdots\otimes x_n)\\
=&\sum_{\tau\in\mathfrak{S}_{n-1}\times \mathrm{Id}}\mu\circ\left(\sum_{i=1}^{n}(-1)^{i(n-i)}\sum_{\sigma\in\mathfrak{C}_n}\mathrm{sgn}(\sigma)\sigma\cdot\Big(\{\{\{\{-,\cdots,-\}\}^{\mathrm{op}}_i,-,\cdots,-\}\}^{\mathrm{op}}_{R,n-i+1}\Big)\sigma^{-1}\right)\cdot\tau^{-1}(x_1\otimes\cdots\otimes x_n)\\
=&0.
\end{aligned}$$

This completes the proof that the operations $\{l_n\}_{n\geqslant 1}$ satisfy the Jacobi$_\infty$ identity. □

Proposition 10.15. *Let A be a finite-dimensional dg algebra, and let B be a locally finite-dimensional dg left A-module. Suppose there exists an ultracyclic homotopy relative Rota–Baxter operator $\{T_n : (A^\vee)^{\otimes n}\to A\}_{n\geqslant 1}$. Then the symmetric algebra $\mathrm{Sym}(B)$ inherits a homotopy Poisson algebra structure. In particular, the graded dual $B^\vee$ can be regarded as a formal derived Poisson manifold.*

Part III

L-algebras and their ideals: from simplicity to semidirect products

Chapter 11

Preliminaries

In this chapter, we collect the basic concepts and results on L-algebras that will be required in the sequel.

11.1 L-algebras and their ideals

Definition 11.1. An L-*algebra* is a set X equipped with a binary operation

$$(x,y) \mapsto x \cdot y, \qquad x,y \in X$$

and a distinguished element $1 \in X$, satisfying the following conditions:

$$1 \cdot x = x, \quad x \cdot 1 = x \cdot x = 1 \tag{1.1}$$

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z) \tag{1.2}$$

$$x \cdot y = y \cdot x = 1 \Longrightarrow x = y. \tag{1.3}$$

Several notable subclasses of L-algebras arise from additional identities. Let X be an L-algebra.

- X is a KL-*algebra* if $x \cdot (y \cdot x) = 1$, for $x, y \in X$.
- X is a CKL-*algebra* if $x \cdot (y \cdot z) = y \cdot (x \cdot z)$, for every $x, y, z \in X$.
- X is a *Hilbert algebra* if $x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$, for every $x, y, z \in X$.

These classes are related by inclusion: every Hilbert algebra is CKL, and every CKL algebra is KL.

An L-algebra that possesses a smallest element 0 is called a *bounded L-algebra*. In this case, one can define the *negation* by $x^* := x \cdot 0$. Bounded CKL-algebras are known as *Glivenko algebras*. Given an L-algebra X, for each element $x \in X$ we denote by $\sigma_x : X \to X$ the map $\sigma_x(y) = x \cdot y$.

Every L-algebra carries a natural partial order defined by $x \leq y$ if and only if $x \cdot y = 1$. For every element x of an L-algebra X, we denote by $\downarrow x$ the downset $\{y \in X \mid y \leq x\}$ and by $\uparrow x$ the upset $\{y \in X \mid y \geq x\}$. An element x of an L-algebra X is *invariant* if $y \cdot x = x$ for all $y > x$, while it is called *prime* if $x \neq 1$ and $y \cdot x \leq x$ for every $y > x$. If an L-algebra X carries a linear partial order, then X is called *linear*.

Definition 11.2. Let X be an L-algebra and $S \subseteq X$. We say that S is an L-*subalgebra* if it is closed under the L-algebra operation of X.

Definition 11.3. We say that a subset $I \subseteq X$ is an *ideal* of the L-algebra X if it satisfies the following properties:

(I1) If $x \in I$ and $x \cdot y \in I$ then $y \in I$.

(I2) If $x \in I$ then $y \cdot x \in I$ for all $y \in X$.

(I3) If $x \in I$ then $(x \cdot y) \cdot y \in I$ for all $y \in X$.

(I4) If $x \in I$ then $y \cdot (x \cdot y) \in I$ for all $y \in X$.

Condition (I3) already implies that every ideal is closed under the L-algebra operation, so every ideal is an L-subalgebra. Simpler characterizations for ideals exist for special subclasses.

Remark 11.4. *Let X be a* KL*-algebra. Then $I \subseteq X$ is an ideal of X if and only if I satisfies (I1) and (I3). If X is a* CKL*-algebra, then $I \subseteq X$ is an ideal of X if and only if $1 \in I$ and I satisfies (I1).*

If X is an L-algebra, we denote by $\mathscr{I}(X)$ the set of ideals of X. This set is itself an L-algebra with the binary operation defined as

$$I \cdot J = \{x \in X \mid \langle x \rangle \cap I \subset J\}.$$

Note that $(I \cdot J) \cap I \subseteq J$ and for every ideal K such that $K \cap I \subseteq J$ then $K \subseteq I \cdot J$. Moreover, if $I \subseteq J$, then $\langle x \rangle \cap I \subseteq J$, for every $x \in I$. So $I \cdot J = X$.

Using this structure, we can now define the notion of prime ideals. A prime ideal of X is simply a prime element in the L-algebra of ideals $\mathscr{I}(X)$:

Definition 11.5. A proper ideal P of an L-algebra X is *prime* if for every ideal I of X either $I \subseteq P$ or $I \cdot P \subseteq P$.

The prime ideals of $\mathscr{I}(X)$ form a topological space $\mathrm{Spec}(X)$, called the *spectrum* of X, whose open sets are the collections $\{\mathscr{U}_I\}_{I \in \mathscr{I}(X)}$, where

$$\mathscr{U}_I := \{P \in \mathrm{Spec}(X) \mid I \not\subseteq P\}.$$

Furthermore, in [87], the following results are proven.

Theorem 11.6. *Let I and J be ideals of an* L*-algebra X. Then $y \in X$ belongs to $I \vee J$ if and only if there is an element $x \in I$ with $x \equiv y (\bmod J)$.*

Theorem 11.7. *The lattice of ideals of an* L*-algebra X is distributive.*

11.2 Semidirect products of L-algebras

In [82] they introduce the concept of semidirect product of L-algebras that needs also the concept of *operation* of an L-algebra on another one.

Definition 11.8. Let X and Y be L-algebras. Y *operates* on X if there is a map $\rho : Y \to \mathrm{End}(X)$ such that

- $\rho_1 = \mathrm{Id}$
- $\rho_{u \cdot v} \circ \rho_u = \rho_{v \cdot u} \circ \rho_v$ for all $u, v \in Y$.

For example, an L-algebra X operates on itself via $\rho_u(x) = u \cdot x$. With this, we can now form semidirect products of L-algebras.

Definition 11.9. Let X and Y be L-algebras such that Y operates on X via ρ. The *semi-direct product$X \rtimes_\rho Y$* is the L-algebra defined on the set $X \times Y$, with operation

$$(x, u) \cdot (y, v) = \big(\rho_{u \cdot v}(x) \cdot \rho_{v \cdot u}(y), u \cdot v\big).$$

Note that the semidirect product of KL (CKL or Hilbert)-algebras is, in general, no longer a KL (CKL or Hilbert)-algebra. Therefore, we need to restrict the semidirect product in these cases.

Theorem 11.10. *Let X and Y be* L*-algebras such that Y operates on X via ρ. The semi-direct product $X \rtimes_\rho Y$ of* L*-algebras is a* KL*-algebra if and only if X and Y are* KL*-algebras such that $x \cdot \rho_u(x) = 1$ holds for $x \in X$ and $u \in Y$.*

Definition 11.11. Let X and Y be KL-algebras. Y *operates* on X as KL-algebras if Y operates on X via ρ as L-algebras and

$$x \cdot \rho_u(x) = 1,$$

holds for $x \in X$ and $u \in Y$.

Definition 11.12. Let X and Y be CKL-algebras. Y *operates* on X as CKL-algebras if Y operates on X via ρ as KL-algebras and

- $\rho_u \rho_v = \rho_v \rho_u$ for all $u, v \in Y$.
- $\rho_u(x \cdot y) = x \cdot \rho_u(y)$ for all $u \in Y$ and $x \in X$.

Definition 11.13. Let X and Y be CKL-algebras such that Y operates on X via ρ as CKL-algebras. The *symmetric semi-direct product* is the CKL algebra

$$X \rtimes\!\!\!\ltimes_\rho Y = \{(x,u) \in X \rtimes_\rho Y \mid \rho_u(x) = x\}.$$

Definition 11.14. Let X and Y be Hilbert algebras. Y *operates* on X if there is a map $\rho : Y \to \mathrm{End}(X)$ such that Y operates on X as CKL-algebras and $\rho_u^2 = \rho_u$ for all $u \in Y$.

The example given before still works with the hypothesis of being Hilbert. More precisely, any Hilbert algebra X operates on itself as a Hilbert algebra through $\rho_u(x) = u \cdot x$.

Definition 11.15. Let X and Y be Hilbert algebras such that Y operates on X via ρ. The *symmetric semi-direct product* is the Hilbert algebra

$$X \rtimes\!\!\!\ltimes_\rho Y = \{(x,u) \in X \rtimes_\rho Y \mid \rho_u(x) = x\}.$$

11.3 Self-similarity of L-algebras

The concept of self-similarity is introduced in [81] where it also proves the existence of the self-similar closure of any L-algebra.

Definition 11.16. An L-algebra X is *self-similar* if for every $x \in X$ the left multiplication σ_x induces a bijection between $\downarrow x$ and X.

Definition 11.17. Let X be an L-algebra, the *self-similar closure* $S(X)$ of X is a self-similar L-algebra with X as an L-subalgebra which generates $S(X)$ as a monoid.

The construction of the self-similar closure of an L-algebra given in [83] uses the following theorems.

Theorem 11.18. *Let $(X, \cdot)$ be an L-algebra, and let $M(X)$ be the free monoid generated by X with unit 1 and multiplication denoted by juxtaposition. Then the L-algebra operation of X admits a unique extension to $M(X)$ such that*

$$\begin{aligned} ab \cdot c &= a \cdot (b \cdot c) \\ a \cdot bc &= ((c \cdot a) \cdot b)(a \cdot c), \\ 1 \cdot a &= a, \end{aligned}$$

for all $a, b, c \in M(X)$.

Theorem 11.19. *Let X be an* L*-algebra. The self-similar closure of X is defined as the quotient.*

$$S(X) = M(X)/\approx,$$

where $a \approx b$ if and only if

$$(c \cdot a) \cdot d = (c \cdot b) \cdot d,$$

for all $c, d \in M(X)$.

Moreover, L-algebra maps with codomain a self-similar one can be extended to the self-similar closure of the domain. Hence, we have a functor S that is left adjoint to the inclusion of the category of self-similar L-algebras in the category of L-algebras.

Proposition 11.20. *Let $f : X \to H$ be a morphism of L-algebras, where H is self-similar. Then f has a unique extension to a morphism $S(f) : S(X) \to H$ of* L*-algebras. Moreover, every such extension $S(f)$ is multiplicative.*

Proposition 11.21. *Let X be a* KL*-algebra. Then $S(X)$ is a* KL*-algebra.*

However, if X is a CKL-algebra, $S(X)$ is not necessarily a CKL-algebra, as shown by the following counterexample.

Example 11.22. *Let $X = \{1, x, y\}$ with $x \cdot y = y \cdot x = x$. It is easy to prove that X is a* CKL*-algebra. But, on the other hand, $S(X)$ is not CKL as $x \cdot (y \cdot x^2) \neq y \cdot (x \cdot x^2)$:*

$$x \cdot (y \cdot x^2) = x \cdot \big(((x \cdot y) \cdot x)(y \cdot x)\big) = x \cdot \big((x \cdot x)x\big) = x \cdot x = 1;$$
$$y \cdot (x \cdot x^2) = y \cdot \big(((x \cdot x) \cdot x)(x \cdot x)\big) = y \cdot \big((1 \cdot x)1\big) = y \cdot x = x.$$

Chapter 12

Semidirect product and self-similarity

In this chapter, we investigate the interplay between semidirect products and self-similarity.

The following lemma describes the downsets of elements of semidirect products.

Lemma 12.1. *Let X and Y be* L*-algebras such that Y operates on X via ρ. Then for every $(x,u) \in X \rtimes_\rho Y$ we have that*

(i) $\downarrow(x,u) = \{(y,v) \in X \rtimes_\rho Y \mid v \in \downarrow u \text{ and } y \in \downarrow \rho_{u\cdot v}(x)\}$.

(ii) $\downarrow(1,u) = X \times \downarrow u$ *and* $\sigma_{(1,u)}(y,v) = (y, u\cdot v)$, *for every* $(y,v) \in \downarrow(1,u)$.

(iii) $\downarrow(x,1) = \bigcup_{v\in Y} \downarrow \rho_v(x) \times \{v\}$ *and* $\sigma_{(x,1)}(y,v) = (\rho_v(x)\cdot y, v)$, *for every* $(y,v) \in \downarrow(x,1)$.

(iv) $(1,u)\cdot(x,u) = (x,1)$ *and* $X \rtimes_\rho Y = \{(x,1)(1,u) \mid x \in X, u \in Y\}$.

Proof.

(i) Let $(x,u),(y,v) \in X \rtimes_\rho Y$. Then $(y,v) \le (x,u)$ if and only if

$$(1,1) = (y,v)\cdot(x,u) = \big(\rho_{v\cdot u}(y)\cdot \rho_{u\cdot v}(x), v\cdot u\big).$$

The last condition is equivalent to $v\cdot u = 1$ and $\rho_{v\cdot u}(y)\cdot\rho_{u\cdot v}(x) = 1$, i.e. $v\cdot u = 1$ and $y\cdot\rho_{u\cdot v}(x) = 1$.

(ii) Let $(x,v) \in X \rtimes_\rho Y$. Then $(x,v) \le (1,u)$ if and only if

$$(1,1) = (x,v)\cdot(1,u) = (\rho_{v\cdot u}(x)\cdot\rho_{u\cdot v}(1), v\cdot u) = (1, v\cdot u).$$

Thus $\downarrow(1,u) = X \times \downarrow u$. For every $(y,v) \le (1,u)$, we have that $v\cdot u = 1$ and $\rho_{u\cdot v}(1) = 1$. Thus, $\sigma_{(1,u)}(y,v) = (y, u\cdot v)$.

(iii) Let every $(y,v) \in X \rtimes_\rho Y$. Then $(y,v) \le (x,1)$ if and only if $y\cdot\rho_v(x) = 1$.

(iv) The result is directly calculated. □

Thanks to Lemma 12.1, we are now able to settle when a semidirect product of two L-algebras is self-similar and to explicitly compute the monoid operation in this case. Moreover, with an inductive argument, we can extend any action of L-algebras to their self-similar closures.

Proposition 12.2. *Let X and Y be* L*-algebra such that Y operates on X via ρ. Then $X \rtimes_\rho Y$ is self-similar if and only if X and Y are self-similar. Moreover, in this case, the monoid operation on $X \rtimes_\rho Y$ is given by*

$$(z,t)(x,u) = (z\rho_t(x), tu).$$

Proof. Let's first assume that $X \rtimes_\rho Y$ is self-similar. Then, in particular, for every $u \in Y$, the map

$$\sigma_{(1,u)} : \downarrow(1,u) \to X \rtimes_\rho Y$$

is bijective. Hence, by Lemma 12.1 (ii), the map

$$\sigma_u : \downarrow u \to Y$$

is also bijective. Therefore, Y is self-similar. Let us now consider $x \in X$. Since $X \rtimes_\rho Y$ is self-similar, the map

$$\sigma_{(x,1)} : \downarrow(x,1) \to X \rtimes_\rho Y$$

is bijective. Thus, by Lemma 12.1 (iii), there exists a unique $(y,v) \in X \rtimes_\rho Y$ such that $y \in \downarrow \rho_v(x)$ and $(\rho_v(x) \cdot y, v) = (z,1)$. But the v is necessarily equal to 1. Thus we proved that for every $z \in X$ there exists a unique $y \in X$ such that $y \in \downarrow x$ and $x \cdot y = \rho_1(x) \cdot y = z$. Hence, the map

$$\sigma_x : \downarrow x \to X$$

is also bijective.

Suppose now that X and Y are self-similar and let $(x,u),(z,t) \in X \rtimes_\rho Y$. Since Y is self-similar, there exists a unique $v \in \downarrow u$ such that $u \cdot v = \sigma_u(v) = t$. Now, by self-similarity of X, there exists a unique $y \in \downarrow \rho_{u\cdot v}(x) = \downarrow \rho_t(x)$ such that $\sigma_{\rho_{u\cdot v}(x)}(y) = z$. Therefore, by Lemma 12.1 (i), we proved that there exists a unique $(y,v) \in \downarrow(x,u)$ such that

$$\sigma_{(x,u)}(y,v) = \big(\rho_{u\cdot v}(x) \cdot \rho_{v\cdot u}(y), u \cdot v\big) = \big(\sigma_{\rho_t(x)}(y), t\big) = (z,t).$$

Therefore, we proved that $X \rtimes_\rho Y$ is self-similar. □

Proposition 12.3. *Let X and Y be L-algebras such that Y operates on X via ρ. Then $S(Y)$ operates on $S(X)$ via a map $\tilde\rho$ such that $\tilde\rho_u(x) = \rho_u(x)$ for all $x \in X$ and $u \in Y$.*

Proof. By Proposition 11.20, the functor S is the left adjoint of the inclusion of the category of self-similar L-algebras in the category of L-algebras. So for every $u \in Y$ we have a map $S(\rho_u) \in \mathrm{End}(S(X))$ that extends ρ_u. So we have a map $S(\rho) : Y \to \mathrm{End}(S(X))$. Consider now $M(Y)$ the free monoid generated by Y (with identity element 1). By Theorem 11.18, the operation $\cdot$ of Y extends uniquely to $M(Y)$ such that

(i) $ab \cdot c = a \cdot (b \cdot c)$,

(ii) $a \cdot bc = ((c \cdot a) \cdot b)(a \cdot c)$

(iii) $1 \cdot a = a$,

for all $a,b,c \in M(Y)$. Since $M(Y)$ is the free monoid generated by Y, we can extend the map $S(\rho)$ to a map $\rho' \colon M(Y) \to \mathrm{End}(S(X))$. In this way, we have $\rho'_1 = \mathrm{Id}$, and we will prove that the second property,

$$\rho'_{a\cdot b} \circ \rho'_a = \rho'_{b\cdot a} \circ \rho'_b,$$

holds for all $a,b \in M(Y)$.

First, we show that the identity

$$\rho'_{a\cdot y}\circ\rho'_a=\rho'_{y\cdot a}\circ\rho_y \tag{0.1}$$

holds for all $a\in M(Y)$ and $y\in Y$. We proceed by induction on the length of a.

For the base case $a=1$, identity (0.1) becomes

$$\rho'_y\circ\rho'_1=\rho'_1\circ\rho_y,$$

which holds trivially since $\rho'_1=\mathrm{Id}$.

Assume that (0.1) holds for a word $a\in M(Y)$ of length $n\geq 1$. Let $x\in Y$. By the extension property given in Theorem 11.18, we compute:

$$\begin{aligned}\rho'_{xa\cdot y}\circ\rho'_{xa}&=\rho'_{x\cdot(a\cdot y)}\circ\rho_x\circ\rho'_a\\&=\rho'_{(a\cdot y)\cdot x}\circ\rho'_{a\cdot y}\circ\rho'_a\\&=\rho'_{(a\cdot y)\cdot x}\circ\rho'_{y\cdot a}\circ\rho_y\\&=\rho'_{((a\cdot y)\cdot x)\cdot(y\cdot a)}\circ\rho_y\\&=\rho'_{y\cdot(xa)}\circ\rho_y.\end{aligned}$$

Hence, (0.1) is verified for xa as well, completing the induction.

Now we proceed to prove the second property

$$\rho'_{a\cdot b}\circ\rho'_a=\rho'_{b\cdot a}\circ\rho'_b$$

by induction on the length of b. When b has length one, this is just (0.1). Suppose the property holds for a given b of length $n\geq 1$. Let $x\in Y$. Then:

$$\begin{aligned}\rho'_{a\cdot(xb)}\circ\rho'_a&=\rho'_{((b\cdot a)\cdot x)\cdot(a\cdot b)}\circ\rho'_a\\&=\rho'_{(b\cdot a)\cdot x}\circ\rho'_{a\cdot b}\circ\rho'_a\\&=\rho'_{x\cdot(b\cdot a)}\circ\rho_x\circ\rho'_b\\&=\rho'_{xb\cdot a}\circ\rho'_{xb},\end{aligned}$$

completing the induction step. Hence, the second property holds for all $a,b\in M(Y)$.

We now verify that it can also be defined on $S(X)=M(Y)/\approx$. Let $a,b\in M(Y)$ such that $a\approx b$. Then, in particular $a\cdot b=b\cdot a=1$, hence

$$\rho'_a=\rho'_1\rho'_a=\rho'_{a\cdot b}\rho'_a=\rho'_{b\cdot a}\rho'_b=\rho'_1\rho'_b=\rho'_b.$$

Therefore we have a well-defined map $\tilde\rho: S(X)\to\mathrm{End}(S(X))$ that extends ρ and such that $\tilde\rho_{a\cdot b}\tilde\rho_a=\tilde\rho_{b\cdot a}\tilde\rho_b$ for all $a,b\in S(X)$. □

We are now ready to prove the main result of this section.

Theorem 12.4. *Let X and Y be L-algebras such that Y operates on X via ρ. Then*

$$S(X\rtimes_\rho Y)=S(X)\rtimes_{\tilde\rho}S(Y).$$

Proof. By Proposition 12.2, we know that $S(X)\rtimes_{\tilde\rho}S(Y)$ is a self-similar L-algebra. Since the natural maps $X\to S(X)$ and $Y\to S(Y)$ are injective, the induced map

$$X\rtimes_\rho Y\longrightarrow S(X)\rtimes_{\tilde\rho}S(Y)$$

is also injective. Moreover, since X generates $S(X)$ as a monoid and Y generates $S(Y)$ as a monoid, for all $(a,b) \in S(X) \rtimes_{\tilde{\rho}} S(Y)$ there exist elements $x_1,\ldots,x_n \in X$ and $u_1,\ldots,u_m \in Y$ such that

$$a = x_1x_2\cdots x_n \quad \text{and} \quad b = u_1u_2\cdots u_m.$$

Thus, by Proposition 12.2,

$$\begin{aligned}(a,b) &= (a,1)(1,b) = (x_1x_2\cdots x_n,1)(1,u_1u_2\cdots u_m)\\ &= (x_1,1)(x_2,1)\cdots(x_n,1)(1,u_1)(1,u_2)\cdots(1,u_m).\end{aligned}$$

Since $(x_i,1),(1,u_j) \in X \rtimes_\rho Y$ for every $1 \leq i \leq n$ and $1 \leq j \leq m$, we proved that $S(X) \rtimes_{\tilde{\rho}} S(Y)$ is generated by $X \rtimes_\rho Y$ as a monoid. □

Corollary 12.5. *Let X and Y be KL-algebras such that Y operates on X via ρ as KL-algebras. Then $S(X) \rtimes_{\tilde{\rho}} S(Y)$ is also a KL-algebra.*

Proof. By Proposition 11.21, $S(X \rtimes_\rho Y) = S(X) \rtimes_{\tilde{\rho}} S(Y)$ is a KL-algebra. □

Chapter 13

Ideals of Semidirect Products of L-Algebras

This chapter is dedicated to the study of ideals of a semidirect product. Hence, we will consider two L-algebras X and Y such that Y acts on X via $\rho : Y \to \mathrm{End}(X)$.

13.1 Structures of ideals of semidirect products of L-algebras

Definition 13.1. Let K be an ideal of $X \rtimes_\rho Y$. We define

$$K_Y = \{y \in Y \mid (1,y) \in K\};$$
$$K_X = \{x \in X \mid (x,1) \in K\}.$$

Similarly, if X and Y are CKL-algebras such that Y operates on X via ρ and L is an ideal of $X \infty_\rho Y$, we define

$$L_Y = \{y \in Y \mid (1,y) \in L\};$$
$$L_X = \{x \in X \mid (x,1) \in L\}.$$

Lemma 13.2. *Let $K \subset X \rtimes_\rho Y$ be an ideal. Then K_Y is an ideal of Y and K_X is an ideal of X.*

Moreover, if X and Y are CKL*-algebras such that Y operates on X via ρ and $L \subset X \infty_\rho Y$ is an ideal, then L_Y is an ideal of Y and L_X is an ideal of X.*

Proof. Consider the following maps:

$$f : X \to (X \rtimes_\rho Y)/K;\; x \mapsto [(x,1)]_K \qquad g : Y \to (X \rtimes_\rho Y)/K;\; y \mapsto [(1,y)]_K.$$

It is easy to show that they are both L-algebra morphisms. Moreover, $\ker f = K_X$ and $\ker g = K_Y$. Therefore, K_X is an ideal of X and K_Y is an ideal of Y.

Moreover, since $(x,1)$ and $(1,u)$ are elements of $X \infty_\rho Y$ for every $x \in X$ and $u \in Y$, we can apply the same strategy to prove that L_X is an ideal of X and L_Y is an ideal of Y. □

Lemma 13.3. *Let $K \subset X \rtimes_\rho Y$ be an ideal. Then $(x,u) \in K$ if and only if $x \in K_X$ and $u \in K_Y$.*

Moreover, if X and Y are CKL*-algebras such that Y operates on X via ρ and $L \subset X \infty_\rho Y$ is an ideal, then $(x,u) \in L$ if and only if $x \in L_X$ and $u \in L_Y$.*

Proof. Suppose that $(x,u) \in K$. Then, by Lemma 12.1 (iv), $(1,u)\cdot(x,u) = (x,1)$, so $(x,1) \in K$. Moreover, $(x,u)\cdot(1,u) = (1,1) \in K$, hence $(1,u)$ is also in K. Suppose that $(x,1)$ and $(1,u)$ are in K. Then, by Lemma 12.1 (iv), $(1,u)\cdot(x,u) = (x,1) \in K$, so $(x,u) \in K$.

The same proof also works for the CKL-algebras case. □

Corollary 13.4. *Let $K \subseteq X \rtimes_\rho Y$ be an ideal. We have that*

$$K = K_X \rtimes_{\rho|_{K_Y}} K_Y.$$

If X and Y are CKL-algebra such that Y operates on X via ρ and $L \subset X \infty_\rho Y$ is an ideal, then

$$L = L_X \infty_{\rho|_{L_Y}} L_Y.$$

Proof. We need to show that $\rho_u \in \mathrm{End}(K_X)$ for every $u \in K_Y$. So let $u \in K_Y$ and $x \in K_X$, then $\big(\rho_u(x),1\big) = (1,u)\cdot(x,1)$ and, since $(1,u),(x,1) \in K$ we obtain that $\big(\rho_u(x),1\big) \in K$, i.e. $\rho_u(x) \in K_X$. Therefore $\rho_u \in \mathrm{End}(K_X)$, so K_Y operates on K_X via the restriction of ρ.

Moreover, by Lemma 13.3, K and $K_X \rtimes_{\rho|_{K_Y}} K_Y$ coincide as subsets of $X \rtimes_\rho Y$.

The same proof also works for the CKL-algebras case. □

Note that the converse of Corollary 13.4 is not true, as shown by the following.

Example 13.5. *Let $X = \{1,x,y\}$ as in Example 11.22, i.e. $x\cdot y = y\cdot x = x$ and let $Y = \{1,u\}$ be the L-algebra (unique of size 2 up to isomorphism) with logical unit 1. Then $\mathrm{End}(X) = \{id,\sigma\}$, where σ is the map that sends every element to 1. Moreover X operates on Y via $\rho : X \to \mathrm{End}(Y)$, where $\rho_1 = id$ and $\rho_2 = \sigma$.*

X has two ideals:

$$I_1 = \{1\} \text{ and } I_2 = X.$$

Y has only two ideals:

$$J_1 = \{1\} \text{ and } J_2 = Y.$$

But he semidirect product $X \rtimes_\rho Y$ has 3 ideals:

$$\begin{aligned} K_1 &= \{(1,1)\} = I_1 \rtimes_\rho J_1; \\ K_2 &= \{(x,1),(y,1),(1,1)\} = I_2 \rtimes_\rho J_1; \\ K_3 &= X \rtimes_\rho Y = I_2 \rtimes_\rho J_2. \end{aligned}$$

So, $I_1 \rtimes_\rho J_2$ is not an ideal. For example

$$(1,u)\cdot(x,1) = (\rho_{u\cdot 1}(1)\cdot\rho_{1\cdot u}(x), u\cdot 1) = (1\cdot\rho_u(x),1) = (1,1) \in I_1 \rtimes_\rho J_2$$

but $(x,1) \notin \{1\} \rtimes_\rho J_2 = I_1 \rtimes_\rho J_2$.

Now we know that ideals of the semidirect products are semidirect products of ideals in the respective components. However, as shown in Example 13.5, there exist ideals in the individual components that cannot be used to form an ideal of the semidirect product.

The following proposition provides an equivalent characterization for when such pairs of ideals give rise to an ideal of the semidirect product.

Proposition 13.6. *Let X and Y be* L*-algebras such that Y operates on X via ρ. Let I be an ideal of X, U be an ideal of Y. Then $I \rtimes_{\rho|_U} U$ is an ideal of $X \rtimes_\rho Y$ if and only if, for each $u \in U$, $x \in I$, $v \in Y$, and $y \in X$*

(I'1) $\rho_v(I) \subseteq I$;

(I'2) $(x \cdot \rho_u(y)) \cdot y,\ y \cdot (x \cdot \rho_u(y)) \in I$.

Proof. We fix that $u \in U$, $x \in I$, $v \in Y$, and $y \in X$. And, we denote a condition (I'3) for ρ as: $\rho_u^{-1}(I) \subseteq I$ for $u \in U$.

If $I \rtimes_{\rho|_U} U$ satisfies (I1), $(1,v) \cdot (x,1) \in I \rtimes_{\rho|_U} U$. Then we have $\rho_v(x) \in I$, that is (I'1). If ρ satisfies (I'1), we have $(y,v) \cdot (x,u) = (\rho_{v\cdot u}(y) \cdot \rho_{u\cdot v}(x), v \cdot u) \in I \rtimes_{\rho|_U} U$. Thus, (I1) for $I \rtimes_{\rho|_U} U$ is equivalent to (I'1) for ρ.

Assume that condition (I'1) holds for ρ. Let

$$(x,u) \cdot (y,v) = (\rho_{u\cdot v}(x) \cdot \rho_{v\cdot u}(y), u \cdot v) \in I \rtimes_{\rho|_U} U.$$

Since I is an ideal, we have $\rho_{v\cdot u}(y) \in I$. Take v to be 1, $\rho_u(y) \in I$. Thus, (I2) for $I \rtimes_{\rho|_U} U$ is equivalent to the condition (I'3).

For $u \in U$, $x \in I$, $v \in Y$, and $y \in X$, we have

$$\begin{aligned} ((x,u) \cdot (y,v)) \cdot (y,v) &= (\rho_{u\cdot v}(x) \cdot \rho_{v\cdot u}(y), u \cdot v) \cdot (y,v) \\ &= \Big(\underbrace{\big(\rho_{(u\cdot v)\cdot v}\rho_{u\cdot v}(x) \cdot \rho_{(u\cdot v)\cdot v}\rho_{v\cdot u}(y)\big) \cdot \rho_{v\cdot(u\cdot v)}(y)}_{A}, (u \cdot v) \cdot v \Big). \end{aligned}$$

We denote the first component of $((x,u) \cdot (y,v)) \cdot (y,v)$ as A. Notice that, since $(u \cdot v) \cdot v \in U$, $((x,u) \cdot (y,v)) \cdot (y,v) \in I \rtimes_{\rho|_U} U$ if and only if $A \in I$. Now, we suppose $A \in I$ and choose $x = \rho_u(x')$, $v = 1$, $y = \rho_u(y')$ with $x' \in I$, $y' \in Y$. Then $A = \rho_u((x' \cdot \rho_u(y')) \cdot y') \in I$. By (I'3), we have $(x' \cdot \rho_u(y')) \cdot y' \in I$. Meanwhile, we have

$$\begin{aligned} (y,v) \cdot ((x,u) \cdot (y,v)) &= (y,v) \cdot (\rho_{u\cdot v}(x) \cdot \rho_{v\cdot u}(y), u \cdot v) \\ &= \Big(\underbrace{\rho_{v\cdot(u\cdot v)}(y) \cdot \big(\rho_{(u\cdot v)\cdot v}\rho_{u\cdot v}(x) \cdot \rho_{(u\cdot v)\cdot v}\rho_{v\cdot u}(y)\big)}_{B}, v \cdot (u \cdot v) \Big). \end{aligned}$$

Similarly, $y' \cdot (x' \cdot \rho_u(y')) \in I$. Thus, (I'2) holds for ρ, if $I \rtimes_{\rho|_U} U$ is an ideal of $X \rtimes_\rho Y$. Now, if $I \rtimes_{\rho|_U} U$ is an ideal of $X \rtimes_\rho Y$, the three conditions hold.

Let now assume that (I'1), and (I'2) hold for ρ and for every $x \in I$ and $u \in U$. Firstly, we show that $\rho_{v\cdot u}(y) \cdot \rho_{v\cdot(u\cdot v)}(y)$ and $\rho_{v\cdot(u\cdot v)}(y) \cdot \rho_{v\cdot u}(y) \in I$. Denote the following expressions, respectively, by C, D, E, and F:

$$\begin{aligned} C &= \rho_{v\cdot(u\cdot v)}(y) \cdot \big(x \cdot \rho_{((u\cdot v)\cdot v)\cdot(v\cdot u)}\rho_{v\cdot(u\cdot v)}(y)\big), \\ D &= \big(x \cdot \rho_{((u\cdot v)\cdot v)\cdot(v\cdot u)}\rho_{v\cdot(u\cdot v)}(y)\big) \cdot \rho_{v\cdot(u\cdot v)}(y), \\ E &= \rho_{v\cdot u}(y) \cdot \big(x \cdot \rho_{((u\cdot v)\cdot v)\cdot(v\cdot u)}\rho_{v\cdot u}(y)\big), \\ F &= \big(x \cdot \rho_{((u\cdot v)\cdot v)\cdot(v\cdot u)}\rho_{v\cdot u}(y)\big) \cdot \rho_{v\cdot u}(y). \end{aligned}$$

By (I'1) and (I'2), we have that $C, D, E, F \in I$. Thus, by

$$C \cdot (\rho_{v\cdot u}(y) \cdot \rho_{v\cdot(u\cdot v)}(y)) = D \cdot F$$

and

$$E(x) \cdot (\rho_{v\cdot u}(y) \cdot \rho_{v\cdot(u\cdot v)}(y)) = F \cdot D,$$

we have $\rho_{v\cdot u} \cdot \rho_{v\cdot(u\cdot v)}(y)$ and $\rho_{v\cdot(u\cdot v)}(y) \cdot \rho_{v\cdot u}(y) \in I$.

Denote $\big(\rho_{(u\cdot v)\cdot v}\rho_{u\cdot v}(x)\cdot\rho_{(u\cdot v)\cdot v}\rho_{v\cdot u}(y)\big)\cdot\rho_{v\cdot u}(y)$ by G. Notice that

$$G\cdot A=\big(\rho_{v\cdot u}\cdot(\rho_{(u\cdot v)\cdot v}\rho_{u\cdot v}(x)\cdot\rho_{(u\cdot v)\cdot v}\rho_{v\cdot u}(y))\big)\cdot\big(\rho_{v\cdot u}(y)\cdot\rho_{v\cdot(u\cdot v)}(y)\big),$$

and $\rho_{v\cdot u}(y)\cdot\rho_{v\cdot(u\cdot v)}(y), G\in I$. Thus, $A\in I$.

Denote $\rho_{v\cdot u}(y)\cdot\big(\rho_{(u\cdot v)\cdot v}\rho_{u\cdot v}(x)\cdot\rho_{(u\cdot v)\cdot v}\rho_{v\cdot u}(y)\big)$ by H. Similarly,

$$\begin{aligned}H\cdot B=&\big(\big(\rho_{(u\cdot v)\cdot v}\rho_{u\cdot v}(x)\cdot\rho_{(u\cdot v)\cdot v}\rho_{v\cdot u}(y)\big)\cdot\rho_{v\cdot u}(y)\big)\\&\cdot\big(\big(\rho_{(u\cdot v)\cdot v}\rho_{v\cdot u}(y)\cdot\rho_{(u\cdot v)\cdot v}\rho_{u\cdot v}(x)\big)\cdot\big(\rho_{(u\cdot v)\cdot v}(\rho_{v\cdot(u\cdot v)}(y)\cdot\rho_{v\cdot u}(y))\big)\big).\end{aligned}$$

Since $\rho_{v\cdot(u\cdot v)}(y)\cdot\rho_{v\cdot u}(y)$ and $G\in I$, we can show that $B\in I$.

Take x to be 1. From (I'2), we have $\rho_u(y)\cdot y\in I$ for all $u\in U$. Thus, if $\rho_u(y)\in I$, then we obtain $y\in I$. Therefore, (I'2) implies (I'3) for ρ. and also implies (I2) for $I\rtimes_{\rho|_U}U$. □

As a consequence, we obtain an explicit upper bound for the number of ideals of the semidirect product of two L-algebras.

Corollary 13.7. *Let X and Y be* L*-algebras such that Y operates on X via ρ. Let U be an ideal of Y. Then $\{1\}\rtimes U$ is an ideal of $X\rtimes_\rho Y$ if and only if $\rho_u=\mathrm{Id}_X$ for all $u\in U$. In particular,*

$$|\mathscr{I}(X\rtimes_\rho Y)|\le|\mathscr{I}(X)||\mathscr{I}(Y)|,$$

and the equality holds if and only if $X\rtimes_\rho Y=X\times Y$.

Proof. If $\rho_u=\mathrm{Id}_X$ for all $u\in U$, then it is easy to check that the conditions (I'1) and (I'2) of Proposition 13.6 are satisfied.

Vice versa, if $\{1\}\rtimes U$ is an ideal of $I\rtimes_{\rho|_U}U$, then, by condition (I'2) of Proposition 13.6, we get that $(1\cdot\rho_u(y))\cdot y, y\cdot(1\cdot\rho_u(y))\in\{1\}$ for all $u\in U$ and $y\in X$. Therefore, for all $u\in U$ and $y\in X$

$$\rho_u(y)\cdot y=1=y\cdot\rho_u(y).$$

which implies that $\rho_u(y)=y$ for all $u\in U$ and $y\in X$.

The second property derives from the fact that if $\rho_y\neq\mathrm{Id}$, then $\{1\}\rtimes_\rho Y$ is not an ideal of $X\rtimes_\rho Y$. □

By Lemma 13.2, Corollary 13.4, Proposition 13.6 and Definition 11.13, we can immediately obtain the following result.

Theorem 13.8. *Let X and Y be* L*-algebras such that Y operates on X via ρ. Then K is an ideal of $X\rtimes_\rho Y$ if and only if ρ induces an operation*

$$\tilde\rho: Y/K_Y\longrightarrow\mathrm{End}(X/K_X)$$

such that

$$(X\rtimes_\rho Y)/(K_X\rtimes_{\rho|_{K_Y}}K_Y)\;\cong\;X/K_X\rtimes_{\tilde\rho}Y/K_Y.$$

Proof. By Lemma 13.2 and Corollary 13.4, we have

$$K=K_X\rtimes_{\rho|_{K_Y}}K_Y,$$

where K_X is an ideal of X and K_Y is an ideal of Y.

Next, set $I=K_X$ and $U=K_Y$.

By (I'1) of Proposition 13.6, the map

$$\rho^I: Y\to\mathrm{End}(X/I)$$

is well-defined. Since $I \rtimes \{1\}$ is an ideal of $X \rtimes_\rho Y$, the quotient

$$(X \rtimes_\rho Y)/(I \rtimes \{1\}) \cong X/I \rtimes_{\rho^I} Y$$

is an L-algebra. By (I'2) of Proposition 13.6, for all $u \in U$ and $y \in X/I$, we have

$$\rho_u^I(y) \cdot y = [1]_I = y \cdot \rho_u^I(y),$$

which implies $\rho_u^I = \mathrm{Id}_{X/I}$ for all $u \in U$. Moreover, for all $v, w \in U$,

$$\rho_v^I = \rho_{v\cdot w}^I \circ \rho_v^I = \rho_{w\cdot v}^I \circ \rho_w^I = \rho_w^I,$$

so the induced map

$$\tilde{\rho} : Y/U \to \mathrm{End}(X/I)$$

is well-defined.

Conversely, if ρ^I is well-defined, then (I'1) holds for ρ. Since $\rho_u^I = \mathrm{Id}_{X/I}$ for all $u \in U$, it follows that

$$(x \cdot \rho_u(y)) \cdot y, \quad y \cdot (x \cdot \rho_u(y)) \in I$$

for all $x, y \in X$ and $u \in U$. □

We can also directly imply the following corollary.

Corollary 13.9. *Let X and Y be* L*-algebras such that Y operates on X via ρ. Let I be an ideal of X, U be an ideal of Y. The following statements are equivalent:*

(i) $I \rtimes_{\rho|_U} U$ *is an ideal of* $X \rtimes_\rho Y$*;*

(ii) ρ *can induce an operation* $\tilde{\rho} : Y/U \to \mathrm{End}(X/I)$ *such that*

$$(X \rtimes_\rho Y)/(I \rtimes_{\rho|_U} U) \cong X/I \rtimes_{\tilde{\rho}} Y/U;$$

(iii) ρ *induces an operation* $\rho^I : Y \to \mathrm{End}(X/I)$ *such that*

$$X/I \rtimes_{\rho^I|_U} U = X/I \times U;$$

(iv) ρ *can induce an operation* $\rho^I : Y \to \mathrm{End}(X/I)$ *such that* $\{[1]_I\} \times U$ *is an ideal of* $X/I \rtimes_{\rho^I} Y$*.*

We now focus on the L-algebra structure of the set of ideals. The next example shows how, in general, it does not commute with the semidirect product.

Example 13.10. *Using the same* L*-algebras as in Example 13.5, we obtain*

$$\langle 1 \rangle = \{1\} = I_1, \qquad \langle u \rangle = Y = J_2, \qquad \langle (1,u) \rangle = K_3 = I_2 \rtimes J_2 = I_2 \rtimes \langle u \rangle.$$

Moreover, since $K_2 \cdot K_1 = K_1$, we obtain that

$$(I_2 \rtimes J_1) \cdot (I_1 \rtimes J_1) = K_2 \cdot K_1 = K_1 = I_1 \rtimes J_1 \neq I_1 \rtimes J_2 = (I_2 \cdot I_1) \rtimes (J_1 \cdot J_1).$$

However, we can still deduce some structural properties about the product of two ideals of the semidirect product and completely determine it in some specific cases.

Lemma 13.11. *Let X and Y be* L*-algebras such that Y operates on X via ρ. Then the following statements hold.*

(i) *Let* $(x,u) \in X \rtimes_\rho Y$ *and let* $K = \langle (x,u) \rangle$ *be the ideal generated by* (x,u). *Then*

$$K = K_X \rtimes \langle u \rangle,$$

and $\langle x \rangle \subseteq K_X$.

(ii) *Let* $I \rtimes U$ *and* $J \rtimes V$ *be ideals of* $X \rtimes_\rho Y$, *and let*

$$L = (I \rtimes U) \cdot (J \rtimes V).$$

Then

$$L_X \subseteq I \cdot J \qquad \text{and} \qquad L_Y \subseteq U \cdot V.$$

(iii) *Let* $I \rtimes U$ *be an ideal of* $X \rtimes_\rho Y$. *Then*

$$(X \rtimes \{1\}) \cdot (I \rtimes U) = I \rtimes V,$$

where V *is the maximal ideal of* Y *such that* $I \rtimes V$ *is an ideal of* $X \rtimes_\rho Y$, *and where*

$$V = \ker\big(\rho^I : Y \to \mathrm{End}(X/I)\big).$$

Proof.

(i) By Theorem 13.8, $X \rtimes \langle u \rangle$ is an ideal of $X \rtimes_\rho Y$. So, we have $K \subseteq X \rtimes \langle u \rangle$. Thus, $K_Y \subseteq \langle u \rangle$. In addition, K_Y is an ideal of Y that contains u. We have $K_Y = \langle u \rangle$.

(ii) By definition, $(x,u) \in L$ if and only if $\langle (x,u) \rangle \cap (I \rtimes U) \subseteq J \rtimes V$. By Part *1.*, this means that $\langle (x,u) \rangle_X \cap I \subseteq J$ and $\langle u \rangle \cap U \subseteq V$, i.e.

$$L = \{(x,u) \in (I \cdot J) \times (U \cdot V) \mid \langle (x,u) \rangle_X \cap I \subseteq J\}.$$

(iii) Let L be $(X \rtimes \{1\}) \cdot (I \rtimes U)$. By Part *(2)*, we have $L_X \subseteq X \cdot I = I$. Moreover, $I \rtimes U \subseteq L$. Hence $L_X = I$ and L is the greatest ideal such that $L \cap (X \rtimes \{1\}) \subseteq (I \rtimes U)$, i.e. such that $L_X \subseteq I$. Therefore, we prove the thesis. □

13.2 ρ-ideals and ρ-spectrum

Definition 13.12. Let X and Y be L-algebras such that Y operates on X via ρ.

- An ideal I of X is called ρ*-ideal* if I satisfies (I'1) i.e. $\rho_v(I) \subseteq I$ for every $v \in Y$.
- A proper ρ-ideal I of X is called ρ*-prime* if for every ρ-ideals I_1 and I_2 of X and such that $I_1 \cap I_2 \subseteq I$, then either $I_1 \subseteq I$ or $I_2 \subseteq I$.

We denote by $\rho\mathscr{I}(X)$ the poset of ρ-ideals of X, and by ρ-spectrum $\rho\,\mathrm{Spec}(X)$ the space of ρ-prime ideals.

Note that the name is not an accident. Indeed, a proper ideal I is prime if and only if for every ideals I_1 and I_2 of X such that $I_1 \cap I_2 \subseteq I$, either $I_1 \subseteq I$ or $I_2 \subseteq I$. So if I is prime and satisfies (I'1), it is also ρ-prime.

Lemma 13.13. *Let J and I be two ρ-ideals of X. Let $\rho^I : Y \to \mathrm{End}(X/I)$ and $\rho^J : Y \to \mathrm{End}(X/J)$ be the maps induced by ρ. Then the following statements hold:*

(i) $I \rtimes \ker(\rho^I)$ *(and $J \rtimes \ker(\rho^I)$) is an ideal of the semi-direct product $X \rtimes_\rho Y$.*

(ii) *If $J \subseteq I$, then* $\ker(\rho^J) \subseteq \ker(\rho^I)$.

(iii) *If $U \subset \ker(\rho^I)$ is an ideal of Y, then $I \rtimes U$ is an ideal of $X \rtimes_\rho Y$.*

Proof.

(i) By Proposition 13.6, $I \rtimes \ker(\rho^I)$ is an ideal of $X \rtimes_\rho Y$ if and only if I satisfies (I'2). Let $u \in \ker(\rho^I)$, $x \in I$ and $y \in X$, then in X/I we have that

$$\big[(x \cdot \rho_u(y)) \cdot y\big]_I = ([x]_I \cdot [\rho_u(y)]_I) \cdot [y]_I = (1 \cdot \rho_u^I([y]_I)) \cdot [y]_I = (1 \cdot [y]_I) \cdot [y]_I = 1.$$

Hence $(x \cdot \rho_u(y)) \cdot y \in I$. Similarly,

$$\big[y \cdot (x \cdot \rho_u(y))\big]_I = [y]_I \cdot ([x]_I \cdot [\rho_u(y)]_I) = [y]_I \cdot (1 \cdot \rho_u^I([y]_I)) = [y]_I \cdot (1 \cdot [y]_I) = 1,$$

i.e. $y \cdot (x \cdot \rho_u(y)) \in I$. Thus, we proved that (I'2) is satisfied too, i.e. $I \rtimes \ker(\rho^I)$ is an ideal of $X \rtimes_\rho Y$.

(ii) Take $u \in \ker(\rho^I)$, then for every $x \in X$

$$\rho_u^J\big([x]_J\big) = [\rho_u(x)]_J = [x]_J.$$

Hence, $\rho_u(x) \in [x]_J$ for every $x \in X$. Since $J \subseteq I$, then $[x]_J \subseteq [x]_I$ for every $x \in X$. Therefore $\rho_u(x) \in [x]_I$ for every $x \in X$. So $[\rho_u(x)]_I = [x]_I$ for every $x \in X$, which means that $\rho_u^I = id_{X/J}$, i.e. $u \in \ker(\rho^I)$.

(iii) This is clear that I and U satisfy conditions (I'1) and (I'2). □

Corollary 13.14. *Let X and Y be* L*-algebras such that Y operates on X via ρ. Then*

$$|\mathscr{I}(X \rtimes_\rho Y)| = \sum_{I \in \rho\mathscr{I}(X)} |\{U \leq \ker(\rho^I) \mid U \in \mathscr{I}(Y)\}|.$$

Proposition 13.15. *Let X and Y be* L*-algebras such that Y operates on X via ρ. Then*

$$\rho\mathscr{I}(X) = \{I \in \mathscr{I}(X) \mid I \rtimes_\rho \{1\} \in \mathscr{I}(X \rtimes_\rho Y)\}.$$

Moreover, $\rho\mathscr{I}(X)$ is a complete sublattice of $\mathscr{I}(X)$, and is distributive.

Proof. Let I be a ρ-ideal and $U = \ker(\rho^I)$. By Lemma 13.13, therefore, $I \rtimes_\rho \{1\}$ is an ideal of $X \rtimes_\rho Y$. Vice versa, if $I \rtimes_\rho \{1\}$ is an ideal of $X \rtimes_\rho Y$, I satisfies (I'1). Therefore, $\rho\mathscr{I}(X) = \{I \mid I \rtimes_\rho \{1\} \in \mathscr{I}(X \rtimes_\rho Y)\}$.

Next, we will show that $\rho\mathscr{I}(X)$ is a complete sublattice.

Let $\{I_\alpha \mid \alpha \in \mathscr{Z}\}$ be a set of ρ-ideals and $v \in Y$. Then $\rho_v(\cap_{\alpha\in\mathscr{Z}} I_\alpha) \subseteq I_\alpha$, for each $\alpha \in \mathscr{Z}$. Thus, the ρ-ideals are closed with respect to intersections.

Let I_1 and I_2 be ρ-ideals, $y \in I_1 \vee I_2$ and $v \in Y$. By Theorem 11.6, there exists an element $x \in I_1$ with $x \equiv y \pmod{I_2}$. Since $\rho_v(I_1) \subseteq I_1$ and $\rho_v(I_2) \subseteq I_2$, then $\rho_v(x) \in I_1$ and $\rho_v(x) \equiv \rho_v(y) \pmod{I_2}$. Then $\rho_v(y) \in I_1 \cap I_2$. Thus, the ρ-ideals are closed with respect to the joints. Therefore, $\rho\mathscr{I}(X)$ is a complete sublattice of $\mathscr{I}(X)$. □

Proposition 13.16.

(i) *Let U be a prime ideal of Y, then $X \rtimes U$ is a prime ideal of $X \rtimes_\rho Y$.*

(ii) *Let I be a ρ-prime ideal of X and $U = \ker(\rho^I)$. Then $I \rtimes U$ is a prime ideal of $X \rtimes_\rho Y$.*

Proof.

(i) By Theorem 13.8, $(X \rtimes_\rho Y)/(X \rtimes U) \cong Y/U$. Since U is a prime ideal, Y/U is subdirectly irreducible. Thus $X \rtimes U$ is a prime ideal of $X \rtimes_\rho Y$.

(ii) Let $I_1 \rtimes U_1$ and $I_1 \rtimes U_1$ be ideals of $X \rtimes_\rho Y$, such that

$$(I_1 \rtimes U_1) \cap (I_2 \rtimes U_2) \subseteq I \rtimes U.$$

Thus, we have $I_1 \cap I_2 \subseteq I$. Since I_1 and I_2 satisfy (I'1), then either $I_1 \subseteq I$ or $I_2 \subseteq I$. By Lemma 13.11 (iii), either $I_1 \rtimes U_1 \subseteq I \rtimes U$ or $I_2 \rtimes U_2 \subseteq I \rtimes U$. Therefore, $I \rtimes U$ is a prime ideal. □

Theorem 13.17. *Let P be an ideal of X and Q be an ideal of Y. $P \rtimes Q$ is a prime ideal of $X \rtimes_\rho Y$ if and only if one of the following holds:*

(i) *$P = X$ and Q is a prime ideal of Y;*

(ii) *P is a ρ-prime ideal of X and $Q = \ker(\rho^P)$.*

Moreover, in this case,

$$\operatorname{Spec}(X \rtimes_\rho Y) \cong \rho \operatorname{Spec}(X) \sqcup \operatorname{Spec}(Y).$$

Proof. By Proposition 13.16, we know that in both cases we obtain a prime ideal of $X \rtimes_\rho Y$.

Now suppose that $P \rtimes Q$ is a prime ideal. Since $X \rtimes \{1\}$ is an ideal of $X \rtimes_\rho Y$, we must have either

$$X \rtimes \{1\} \subseteq P \rtimes Q \qquad \text{or} \qquad (X \rtimes \{1\}) \cdot (P \rtimes Q) \subseteq P \rtimes U.$$

In the first case, $X \rtimes \{1\} \subseteq P \rtimes Q$, which implies $P = X$. Moreover, Q is a prime ideal of Y because Y/P is subdirectly irreducible: indeed, $(X \rtimes_\rho Y)/(P \rtimes Q)$ is subdirectly irreducible and, by Theorem 13.8,

$$(X \rtimes_\rho Y)/(X \rtimes P) \cong Y/P.$$

In the second case, $P \rtimes Q \supseteq (X \rtimes \{1\}) \cdot (P \rtimes Q)$ and by Lemma 13.11 (iii),

$$(X \rtimes \{1\}) \cdot (P \rtimes Q) = P \rtimes \ker(\rho^P),$$

where $V = \ker(\rho^P)$ is an ideal of Y that is maximal such that $P \rtimes V$ is an ideal of $X \rtimes_\rho Y$. But inclusion $P \rtimes Q \supseteq P \rtimes V$ forces $Q = \ker(\rho^P)$.

Consider ideals I_1 and I_2 of X that satisfy (I'1) and such that $P_1 \cap P_2 \subseteq P$. Then $P_1 \cap P_2$ also satisfies (I'1). By Lemma 13.13, we have

$$\ker(\rho^P) \subseteq \ker(\rho^{P_1 \cap P_2}) = \ker(\rho^{P_1}) \cap \ker(\rho^{P_2}).$$

Hence,

$$\big(P_1 \rtimes \ker(\rho^{P_1})\big) \cap \big(P_2 \rtimes \ker(\rho^{P_2})\big) \subseteq \big(P \rtimes \ker(\rho^P)\big).$$

Since $P \rtimes \ker(\rho^P)$ is prime, we must have either

$$P_1 \rtimes \ker(\rho^{P_1}) \subseteq P \rtimes \ker(\rho^P) \quad \text{or} \quad P_2 \rtimes \ker(\rho^{P_2}) \subseteq P \rtimes \ker(\rho^P).$$

Thus either $P_1 \subseteq P$ or $P_2 \subseteq P$. This proves that P is a ρ-prime ideal. □

Remark 13.18. *The bijection*

$$f : \mathrm{Spec}(X \rtimes_\rho Y) \longrightarrow \rho\,\mathrm{Spec}(X) \sqcup \mathrm{Spec}(Y),$$

defined by

$$f(X \rtimes Q) = Q \quad \textit{and} \quad f(P \rtimes Q) = P \textit{ for } P \neq X,$$

is an open map, where the subspace $\rho\,\mathrm{Spec}(X)$ is endowed with the subspace topology inherited from $\mathrm{Spec}(X)$.

Proof. Let $I \rtimes U$ be an ideal of $X \rtimes_\rho Y$ with $I \neq X$, and let $P \rtimes Q$ be a prime ideal of $X \rtimes_\rho Y$. By Lemma 13.13,

$$I \rtimes U \subseteq P \rtimes Q \iff I \subseteq P.$$

Therefore,

$$f(\mathscr{U}_{I \rtimes U}) = (\mathscr{U}_I \cap \rho\,\mathrm{Spec}(X)) \sqcup \mathscr{U}_U, \qquad f(\mathscr{U}_{X \rtimes_\rho V}) = \rho\,\mathrm{Spec}(X) \sqcup \mathscr{U}_V, \quad (V \in \mathscr{I}(Y)),$$

which shows that f is an open map. □

By Theorem 13.17, we can explicitly describe the prime spectrum of the semidirect product of KL-algebras as follows.

Proposition 13.19. *Let X and Y be* KL*-algebras such that Y operates on X via ρ as* KL*-algebras. Then*

$$\rho\mathscr{I}(X) = \mathscr{I}(X).$$

In this case,

$$\rho\,\mathrm{Spec}(X) = \mathrm{Spec}(X) \quad \textit{and} \quad \mathrm{Spec}(X \rtimes_\rho Y) \cong \mathrm{Spec}(X) \sqcup \mathrm{Spec}(Y).$$

Proof. By Definition 11.11, we have

$$x \cdot \rho_u(x) = 1,$$

for all $x \in X$ and $u \in Y$. It follows that for any $u \in Y$,

$$x \cdot \rho_u(x) = 1 \in \langle x \rangle.$$

By condition (I1), we conclude that

$$\rho_u(x) \in \langle x \rangle \quad \text{for all } u \in Y.$$

Therefore, every ideal of X is automatically a ρ-ideal. By Theorem 13.17, we obtain the following characterizations:

$$\rho\mathscr{I}(X) = \mathscr{I}(X), \quad \text{and} \quad \rho\,\mathrm{Spec}(X) = \mathrm{Spec}(X).$$

□

Proposition 13.20. *Let X and Y be CKL-algebras such that Y acts on X via ρ as CKL-algebras. Let L be an ideal of the symmetric semidirect product $X \rtimes\!\!\!\ltimes_\rho Y$. Define*

$$\widetilde{L} = L_X \rtimes_{\rho|_{L_Y}} L_Y \subseteq X \rtimes_\rho Y.$$

Then the assignments

$$L \longmapsto \widetilde{L} \quad \textit{and} \quad K \cap (X \rtimes\!\!\!\ltimes_\rho Y) \longleftarrow\!\!\shortmid K$$

establish a bijective correspondence between the ideals of the symmetric semidirect product $X \rtimes\!\!\!\ltimes_\rho Y$ and those of the semidirect product $X \rtimes_\rho Y$.

Proof. Let L be an ideal of $X \propto_\rho Y$. By Theorem 13.19, every ideal of X is a ρ-ideal, hence L_X satisfies condition (I'1).

Let $x \in L_X$, $u \in L_Y$, and $y \in X$. By Definition 11.12, we have

$$y \cdot (x \cdot \rho_u(y)) = \rho_u\big(y \cdot (x \cdot y)\big) \in L_X.$$

Moreover, since $(x,u) \in L$, $(y,1) \in X \propto_\rho Y$, and L is an ideal of $X \propto_\rho Y$, it follows that

$$\big((x,u) \cdot (y,1)\big) \cdot (y,1) = \big((x \cdot \rho_u(y)) \cdot y, 1\big) \in L.$$

Hence $(x \cdot \rho_u(y)) \cdot y \in L_X$, showing that condition (I'2) holds for $\widetilde{L}$. Therefore, by Proposition 13.6, $\widetilde{L}$ is an ideal of $X \rtimes_\rho Y$.

Since $X \propto_\rho Y$ is an L-subalgebra of $X \rtimes_\rho Y$, for any ideal K of $X \rtimes_\rho Y$, the intersection

$$K \cap (X \propto_\rho Y)$$

is an ideal of $X \propto_\rho Y$. These two constructions are inverses of each other, establishing the claimed bijective correspondence. ☐

Chapter 14

Simple linear L-algebras and CKL-algebras

14.1 Simple linear L-algebras

In [29, Lemma 4.3], Dietzel, Menchón, and Vendramin have shown the following lemma for linear L-algebras.

Lemma 14.1. *Let X be a linear algebra. For any $x,y,z \in X$ with $x \geq y > z$, one has*

$$x \cdot y \;>\; x \cdot z.$$

Moreover, every linear L-algebra is also a KL-algebra, i.e. $x \cdot y \geq y$ for all $x,y \in X$.

Proposition 14.2. *Let X be a linear L-algebra with*

$$X = \{x_0 > x_1 > \cdots > x_{n-1}\},$$

and suppose that x_{i+1} is an invariant element of X. Let

$$I = \uparrow x_i := \{x_j \mid j \leq i\}.$$

Then

$$x \cdot y = y \qquad \text{for all } x \in I,\ y \in X \setminus I.$$

Proof. We show that $x_{i+1}, \ldots, x_{n-1}$ are invariant under the action of I. Proceed by induction. Since x_{i+1} is invariant, the base case holds. Assume that x_k is invariant under I for some $k > i+1$. Then by Lemma 14.1,

$$x_k = x \cdot x_k > x \cdot x_{k+1} \geq x_{k+1}$$

for all $x \in I$. Thus $x \cdot x_{k+1} = x_{k+1}$ for all $x \in I$, proving the induction step. □

Using this result, we can give a characterization of ideals and prime ideals of a linear L-algebra.

Theorem 14.3. *Let X be a linear L-algebra with*

$$X = \{x_0 > x_1 > \cdots > x_{n-1}\},$$

and let $I \subseteq X$. Then I is an ideal of X if and only if

$$I = \uparrow x_i := \{x_j \mid j \le i\}$$

for some $i \in \{0,\dots,n-1\}$, and moreover either $i = n-1$ or x_{i+1} is an invariant element.

Proof. Assume first that I is an ideal of X, and let x_i be the minimal element of I. By (I1), I is upward closed, hence $I = \uparrow x_i$.

Suppose now that $i < n-1$ and choose any $y > x_{i+1}$. Then $y \in I$ while $x_{i+1} \notin I$, and by (I1) we must have $y \cdot x_{i+1} \notin I$. Thus $y \cdot x_{i+1} \le x_{i+1}$. On the other hand, by Lemma 14.1,

$$x_{i+1} \le y \cdot x_{i+1}.$$

Hence $y \cdot x_{i+1} = x_{i+1}$ for all $y > x_{i+1}$, showing that x_{i+1} is invariant.

Conversely, suppose that $I = \uparrow x_i$ and that x_{i+1} is invariant. By Remark 11.4, it suffices to verify (I1) and (I3).

(I1) Let $x \in I$ and suppose $x \cdot y \in I$. By Proposition 14.2, if $y \notin I$ then $x \cdot y = y \notin I$, a contradiction. Therefore $y \in I$.

(I3) If $x \in I$ and $y \notin I$, then by Proposition 14.2,

$$(x \cdot y) \cdot y = y \cdot y = 1 \in I.$$

If $x \in I$ and $y \in I$, then since X is a KL-algebra,

$$(x \cdot y) \cdot y \ge y \ge x_i,$$

hence $(x \cdot y) \cdot y \in I$.

Therefore, I is an ideal of X. □

Corollary 14.4. *Let $X = \{x_0 > x_1 > \cdots > x_{n-1}\}$ be a linear L-algebra, then*

$$\mathscr{I}(X) = \{\uparrow x_i \mid i = n-1 \text{ or } x_{i+1} \text{ is invariant}\}$$

and $\mathrm{Spec}(X) = \mathscr{I}(X) \setminus \{X\}$.

Proof. The first claim is exactly what is proven in Theorem 14.3.

Let P be a proper ideal of X. We want to prove that it is prime. By the previous property, $P = \uparrow x_k$ for some $k \in \{0,\dots,n-2\}$ with x_{k+1} invariant. Consider now any other ideal $I = \uparrow x_i$, then

$$\begin{aligned} I \cdot P &= \{x_j \mid \langle x_j \rangle \cap I \subseteq P\} = \{x_j \mid \langle \uparrow x_j \rangle \cap \uparrow x_i \subseteq \uparrow x_k\} = \{x_j \mid \uparrow x_{\min(j,i)} \subseteq \uparrow x_k\} \\ &= \{x_j \mid \min(j,i) \le k\} = \begin{cases} X & \text{if } i \le k \\ \uparrow x_k & \text{if } i > k \end{cases} = \begin{cases} X & \text{if } i \le k \\ P & \text{if } i > k \end{cases} \end{aligned}$$

Therefore, $i \le k$ and so $I \subseteq P$ or $i > k$ and $I \cdot P = P$. Hence, P is a prime ideal. □

We now introduce a family of L-algebras $\{\mathbf{A}_n\}_{n \ge 1}$ and use the previous theorem to establish that each of these L-algebras is simple.

Proposition 14.5. *Let $n > 1$ and $\mathbf{A}_n$ be the set $\{x_0, x_1, \dots, x_{n-1}\}$ with multiplication defined as $x_i \cdot x_j = x_{\max(j-i,0)}$ for all $i, j \in \{0,\dots,n-1\}$. Then $\mathbf{A}_n$ is a simple linear CKL-algebra with $x_0 > x_1 > \cdots > x_{n-1}$.*

Proof. It is easy to check that

$$\max(\max(k-i,0)-\max(j-i,0)) = \max(\max(k-j,0)-\max(i-j,0))$$

for every $i,j,k \geq 0$, hence $(x_i \cdot x_j)\cdot(x_i \cdot x_k) = (x_j \cdot x_i)\cdot(x_j \cdot x_k)$ for every $x_i,x_j,x_k \in \mathbf{A}_n$. Moreover $x_0 \cdot x_i = x_i$ and $x_i \cdot x_0 = x_i \cdot x_i = x_0$ for all $x_i \in \mathbf{A}_n$ and if $x_i \cdot x_j = x_j \cdot x_i = x_0$, then $i \leq j \leq i$, i.e. $x_i = x_j$. Therefore, $\mathbf{A}_n$ is an L-algebra with $1 = x_0$ and $x_0 < x_1 < \cdots < x_{n-1}$.

Moreover,

$$x_i \cdot (x_j \cdot x_k) = \begin{cases} x_{k-j-i} & \text{if } k > i+j \\ x_0 & \text{otherwise} \end{cases} = x_j \cdot (x_i \cdot x_k), \text{ for every } x_i,x_j,x_k \in \mathbf{A}_n,$$

so $\mathbf{A}_n$ is a CKL-algebra.

Finally, to prove that it is simple, using Theorem 14.3, it is enough to show that there are no invariant elements apart from x_0 and x_1. Note that $x_{i-1} > x_i$ for every $i > 1$ and $x_{i-1} \cdot x_i = x_1 \neq x_i$, i.e. x_i is not invariant for every $i > 1$. □

Remark 14.6. *This construction of $\mathbf{A}_n$ can be extended in a natural way to countable sets. Consequently, there also exists a simple linear L-algebra structure $\mathbf{A}_\infty$ on a countable set.*

Lemma 14.7. *Let $n > 1$ and $X = \{x_0 > x_1 > \cdots x_{n-1} > x_n\}$ be a linear L-algebra. Then $Y = X \setminus \{x_n\}$ is an L-subalgebra of X. Moreover, I is an ideal of Y for every proper ideal $I \subset X$.*

Proof. Let I be a proper ideal of X, then $x_n \notin I$ and, more precisely, by Theorem 14.3, $I = \uparrow x_i$ for some $i < n$ and x_{i+1} is invariant in X.

If $i = n-1$, then $I = Y$, which is an ideal of Y.

Otherwise, $i < n-1$ and $x_{i+1} \in Y$ and x_{i+1} is invariant also in Y. Therefore, by Theorem 14.3, I is an ideal of Y. □

The previous lemma allows us to use the inductive construction of linear algebras proved in [29]. More precisely, [29, Proposition 4.4] is the following.

Proposition 14.8. *Let $X = \{x_0 > x_1 > \cdots > x_{n-1}\}$ be a linear L-algebra and let $p \in X$ be the smallest invariant element of X. Consider now the poset*

$$L_{n+1} = \{x_0 > x_1 > \cdots x_{n-1} > x_n\}$$

and take $c \in L_{n+1}$ such that $p \cdot x_{n-1} > c$. Then there exists a unique L-algebra structure X' on L_{n+1} such that X is an L-subalgebra of X' and such that $p \cdot x_n = c$.

Theorem 14.9. *Let $n > 1$ and $X = \{x_0 > x_1 > \cdots > x_{n-1}\}$ be a linear L-algebra. If X is simple, then X is isomorphic to $\mathbf{A}_n$.*

Proof. We prove the thesis by induction. For $n = 2$, the claim is trivial.

Let $n > 1$ and $X = \{x_0 > x_1 > \cdots x_{n-1} > x_n\}$ be a linear simple L-algebra. Then, by Lemma 14.7, $Y = \{x_0 > x_1 > \cdots > x_{n-1}\}$ is a linear simple L-algebra too. Hence, by inductive hypothesis, $x_i \cdot x_j = x_{\max(j-i,0)}$ for all $i,j \in \{0,\ldots,n-1\}$. It remains to check that $x_i \cdot x_n = x_{n-i}$ for all $i \in \{0,\ldots,n-1\}$.

Notice that in Y the smallest invariant element is x_1 and, since $x_n < x_1$, by Lemma 14.1, $x_1 \cdot x_n < x_1 \cdot x_{n-1} = x_{n-2}$. Moreover $x_1 \cdot x_n$ cannot be x_n otherwise, by Lemma 14.1, $x \cdot x_n = x_n$ for every $x \neq x_n$ i.e. x_n is invariant, which is against the fact that X is a linear simple L-algebra. Therefore $x_1 \cdot x_n = x_{n-1}$ and, by Proposition 14.8, there is a unique L-algebra structure on X such that Y is a L-subalgebra and $x_1 \cdot x_n = x_{n-1}$, which is precisely S_{n+1}. □

14.2 Tail$^+$ CKL-algebras

In the remaining, we will extend Theorem 14.9 to a subclass of CKL-algebra, namely tail$^+$ CKL-algebras.

Definition 14.10. Let X be an L-algebra, and let z be a minimal element of X. The upset $\uparrow z$ of z is called a *tail* if it is a linear subset of X.

A finite L-algebra X is called a *tail$^+$ L-algebra* if it has a tail or if it contains L-subalgebras

$$Y \subseteq Y_0 \subseteq X$$

such that:

(i) Y has a tail;

(ii) the set $Y_0 \setminus Y = \{z_0\}$ consists of a single element, which is the smallest element of Y_0;

(iii) the complement $X \setminus Y$ is a linear poset.

In particular, any L-algebra X whose Hasse diagram forms a directed tree is an L-algebra with n tails, where n denotes the number of leaves of the tree.

Proposition 14.11. *Let X be a CKL-algebra with a minimal element $z \in X$. If the corresponding upset*

$$I := \uparrow z = \{x \in X \mid z \leq x\}$$

is a tail. Then I is an ideal of X.

Proof. Assume that I is a proper subset of X.

We first show that $z \cdot y \notin I$ for all $y \notin I$. Let $y \notin I$. There exists a minimal element x such that $y < x$ and $z < x$. Then, we have

$$\begin{aligned} y \cdot z &= (y \cdot x) \cdot (y \cdot z) \\ &= (x \cdot y) \cdot (x \cdot z). \end{aligned}$$

Since $y \nleq z$, it follows that $x \cdot y \nleq x \cdot z$. Similarly, since

$$z \cdot y = (x \cdot z) \cdot (x \cdot y),$$

we also have $x \cdot z \nleq x \cdot y$. Hence, since $x \cdot z \in I$ and I is linear, it follows that $x \cdot y \notin I$. Moreover, since X is a CKL-algebra, we obtain

$$z \cdot (x \cdot y) = x \cdot (z \cdot y).$$

Since $z \nleq x \cdot y$, it follows that $x \nleq z \cdot y$. Note that $z \cdot y \in \uparrow y$. Thus,

$$z \cdot y \in \uparrow y \setminus \uparrow x,$$

which implies $z \cdot y \notin I$.

Next, we show that each I satisfies property (I1). Suppose $x, x \cdot y \in I$. Since X is a CKL-algebra, we have

$$x \cdot (z \cdot y) = z \cdot (x \cdot y) = 1.$$

Hence $z \leq x \leq z \cdot y$, which means $z \cdot y \in I$. From the first part of this proof, it follows that $y \in I$. Therefore, I is an ideal of X. □

By Proposition 14.11, we can directly obtain the following result.

Example 14.12. *Let X be a set $\{1,x,y,z\}$ with the following multiplication table:*

	x	y	z	1
x	1	y	x	1
y	x	1	z	1
z	1	y	1	1
1	x	y	z	1

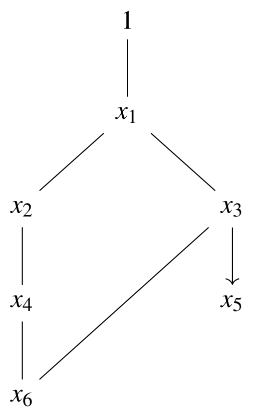
It can be verified that X is a CKL-algebra with the partial order $1 > y$ and $1 > x > z$. By Proposition 14.11, we obtain two ideals of X:

$$I_1 = \{1,x,z\} \quad \text{and} \quad I_2 = \{1,y\}.$$

Example 14.13. *Let $X = \{1,x_1,x_2,x_3,x_4,x_5,x_6\}$ be a set equipped with the following multiplication table:*

	x_1	x_2	x_3	x_4	x_5	x_6	1
x_1	1	x_2	x_3	x_4	x_5	x_6	1
x_2	1	1	x_3	x_4	x_5	x_6	1
x_3	1	x_2	1	x_4	x_3	x_4	1
x_4	1	1	x_3	1	x_5	x_3	1
x_5	1	x_2	1	x_4	1	x_4	1
x_6	1	1	1	1	x_3	1	1
1	x_1	x_2	x_3	x_4	x_5	x_6	1

It can be verified that X is a CKL-algebra. The corresponding strict partial order on X is represented by the following Hasse diagram:



By Proposition 14.11, the tail $\uparrow x_5 = \{1,x_1,x_3,x_5\}$ is an ideal of X. In contrast, the upset $\uparrow x_6$ is not an ideal of X.

Lemma 14.14. *Let X be a Glivenko algebra with the smallest element $0 \in X$. Then $Y = X \setminus \{0\}$ is a CKL-subalgebra. Moreover, I is an ideal of Y if and only if I is an ideal of X or $I = Y$.*

Proof. To prove that Y is a CKL-subalgebra, it is enough to notice that if $x,y \in Y$, then $0 < y \leq x \cdot y$, hence $x \cdot y \in Y$.

Let now I be an ideal of Y that is not an ideal of X. Then there exists $x \in I$ such that the negation $x^* \in I$. We claim that $I = Y$. Let $y \in Y$, then $x^* \in I$ and $x \cdot y \in Y$, but

$$x^* \cdot (x \cdot y) = (0 \cdot x) \cdot (0 \cdot y) = 1.$$

So $x \cdot y \in I$, since I is an ideal of Y. But now we have $x \in I$, $y \in Y$ such that $x \cdot y \in I$, thus $y \in I$. □

Theorem 14.15. *Let $n > 1$ and let X be a tail$^+$ CKL-algebra of size n. If X is simple, then X is linear, hence it is isomorphic to $\mathbf{A}_n$.*

Proof. First, we start with the case when X is a simple CKL-algebra with a tail. By Proposition 14.11, X has a unique minimal element. Hence, the partial order of X is linear, and X is isomorphic to $\mathbf{A}_n$.

Let X be a tail$^+$ CKL-algebra. By Lemma 14.14 and induction, X is also linear and isomorphic to $\mathbf{A}_n$. □

Moreover, a CKL simple L-algebra cannot have more than one connected component in the Hasse diagram of $X \setminus \{1\}$ as the following proposition states.

Proposition 14.16. *Let X be a CKL-algebra and let C be a connected component of the Hasse diagram of $X \setminus \{1\}$. Then $C \sqcup \{1\}$ is an ideal of X.*

Proof. Thanks to Remark 11.4, we only need to prove property (I1) of the definition of ideal.

Let $x \in C \sqcup \{1\}$ and $y \in X$ such that $x \cdot y \in C \sqcup \{1\}$. Then either $x = 1$ or $x \in C$.

If $x = 1$, we have directly $y = x \cdot y \in C \sqcup \{1\}$.

Otherwise, $x \in C$. Since $x \cdot y \in C \sqcup \{1\}$, we have two cases again: either $x \cdot y = 1$ or $x \cdot y \in C$. If $x \cdot y = 1$, then $x \leq y$. Hence, y is connected to x in the Hasse diagram. So $y = 1$ or $y \in C$. Assume now that $x \cdot y \in C$. Using that X is CKL, hence KL, we get that $y \leq x \cdot y$. Thus, y is connected to x in the Hasse diagram. So $y = 1$ or $y \in C$. In any case, we proved that $y \in C \sqcup \{1\}$. □

Chapter 15

Symmetric semidirect products and Hilbert algebras

In this chapter, we mainly study the ideals, semidirect products of Hilbert algebras, and the structure of linear Hilbert algebras.

Lemma 15.1. *Let X be a Hilbert algebra, and let $z \in X$. Then the upset $\uparrow z$ is the ideal $\langle z \rangle$ generated by z.*

Proof. Note that X is also a CKL-algebra, so $I \subseteq X$ is an ideal of X if and only if $1 \in I$ and I satisfies (I1).

(i) Clearly $1 \in \uparrow z$.

(ii) If $x, x \cdot y \in \uparrow z$, then
$$z \cdot y = 1 \cdot (z \cdot y) = (z \cdot x) \cdot (z \cdot y) = z \cdot (x \cdot y) = 1,$$
i.e. $y \in \uparrow z$.

Then $\uparrow z$ is an ideal of X. Thus, $\langle z \rangle \subseteq \uparrow z$.

For each $x \in \uparrow z$, we have $z \cdot x = 1 \in \langle z \rangle$. By condition (I1), we conclude that $x \in \langle z \rangle$. Therefore, $\langle z \rangle = \uparrow z$. □

Proposition 15.2. *Let X be a finite Hilbert algebra and let I be an ideal of X. Denote by $\min(I)$ the set of all minimal elements of I. Then*
$$I = \bigcup_{z \in \min(I)} \uparrow z.$$

Moreover, let $\min(X) = \{m_1, \ldots, m_n\}$ be the set of all minimal elements of X, and for each $1 \le i \le n$ define
$$P_i = \bigcup_{m \in \min(X) \setminus \{m_i\}} \uparrow m.$$
If P is a proper ideal of X such that the $P_i \subseteq P$ for some $1 \le i \le n$ and $X \setminus P$ is linear, then P is a prime ideal of X.

Proof. By Lemma 15.1, we have $\langle z_i \rangle = \uparrow z_i \subseteq I$ for each $z_i \in \min(I)$. Hence
$$\bigcup_{z \in \min(I)} \uparrow z \subseteq I.$$

Conversely, let $x \in I$. Since I is finite, it has minimal elements, and every element of I lies above some minimal element of I. Thus, there exists $z_j \in \min(I)$ such that $z_j \leq x$, i.e. $x \in \uparrow z_j$. Therefore

$$I \subseteq \bigcup_{z \in \min(I)} \uparrow z.$$

Combining the two inclusions yields

$$I = \bigcup_{z \in \min(I)} \uparrow z,$$

as claimed.

Let P be a proper ideal such that $P_i \subseteq P$ for some $1 \leq i \leq n$ and $X \setminus P$ is linear. Then

$$P = P_i \cup \uparrow z_i, \qquad \text{where } m_i < z_i.$$

Assume that $I \nsubseteq P$. Then there exists a minimal element $z \in I$ such that

$$m_i \leq z < z_i.$$

Since $I \setminus P$ is linear, we have

$$\begin{aligned} I \cdot P &= \{x \in X \mid \uparrow x \cap I \subseteq P\} \\ &\subseteq \{x \in X \mid \uparrow x \cap \uparrow z \subseteq P\} \\ &= \{x \in X \mid \uparrow x \subseteq P\} \\ &\subseteq P. \end{aligned}$$

Thus $I \cdot P \subseteq P$ for every ideal I with $I \nsubseteq P$. Therefore, P is a prime ideal. □

Using Lemma 15.1, it is now easy to show that there is only one simple Hilbert algebra.

Proposition 15.3. *Let X be a Hilbert algebra. X is simple if and only if $|X| \leq 2$.*

Proof. Let $z \in X$, then, by Lemma 15.1, $\uparrow z = \{x \in X \mid z \leq x\}$ is an ideal of X.

Assume now, by contradiction, that X is simple and $|X| > 2$, then there exist $z_1, z_2 \in X \setminus \{1\}$ distinct elements. But $\uparrow z_1$ and $\uparrow z_2$ are non-trivial ideals, so $\uparrow z_1 = X = \uparrow z_2$, which is a contradiction because we would have $z_1 < z_2 < z_1$. □

Proposition 15.4. *Let $\mathrm{LH}_n = \{x_0, x_1, \ldots, x_{n-1}\}$ with multiplication defined by*

$$x_i \cdot x_j = \begin{cases} x_0 = 1, & \text{if } i \geq j, \\ x_j, & \text{if } i < j. \end{cases}$$

Then LH_n is a linear Hilbert algebra.

Proof. For all $x_i, x_j, x_k \in \mathrm{LH}_n$, we have

$$x_i \cdot (x_j \cdot x_k) = \begin{cases} x_k, & \text{if } j < k \text{ and } i < k, \\ 1, & \text{otherwise.} \end{cases}$$

On the other hand,

$$(x_i \cdot x_j) \cdot (x_i \cdot x_k) = (x_j \cdot x_i) \cdot (x_j \cdot x_k) = \begin{cases} x_k, & \text{if } j < k \text{ and } i < k, \\ 1, & \text{otherwise.} \end{cases}$$

Hence, the defining identity of a Hilbert algebra,

$$x_i \cdot (x_j \cdot x_k) = (x_i \cdot x_j) \cdot (x_i \cdot x_k) = (x_j \cdot x_i) \cdot (x_j \cdot x_k),$$

holds for all $x_i, x_j, x_k \in \mathrm{LH}_n$. Therefore, LH_n is a Hilbert algebra. □

Proposition 15.5. *Let $n > 1$ and $X = \{x_0 > x_1 > \cdots > x_{n-1}\}$ be a linear Hilbert algebra. Then X is isomorphic to* LH_n.

Proof. By the definition of L-algebra, $x_0 = 1$ is an invariant element.

Let now $j \in \mathbb{N}_{\geq 1}$. Since X is Hilbert, by Lemma 15.1 $\uparrow x_{j-1}$ is an ideal of X. Moreover, since X is also linear, by Theorem 14.3, x_j is an invariant element. Therefore, we proved the thesis. □

Corollary 15.6. *Let X be a linear Hilbert algebra of size n, and let I be an ideal of X. Then:*

(i) There exists an ρ such that I operates on X/I via ρ as Hilbert algebras, and

$$X \cong I \rtimes_\rho (X/I).$$

(ii) Conversely, if there exists a ρ such that I operates on Y via ρ as Hilbert algebras and

$$X \cong I \rtimes_\rho Y,$$

then $Y \cong X/I$.

Proof. Let I be a proper ideal of X. By Theorem 14.3, we have

$$I = \uparrow x_i := \{x_j \mid j \leq i\},$$

for some $i \in \{0, \ldots, n-1\}$.

By Proposition 15.5 and Theorem 14.3, it follows that $X/I \cong \mathrm{LH}_{n-i+1}$. Define $\rho : X/I \to \mathrm{End}(I)$ by

$$\rho_{[u]_I}(x) = 1, \quad \text{for all } x \in I,\ [u]_I \in X/I.$$

Then

$$I \rtimes_\rho (X/I) = (\{1\} \times X/I) \cup (I \times \{1\})$$

is a linear Hilbert algebra of size n. By Proposition 15.5, we obtain the isomorphism

$$X \cong I \rtimes_\rho (X/I).$$

Conversely, by Proposition 13.20 and Proposition 13.19, we have

$$|\,\mathrm{Spec}(I \rtimes_\rho Y)| = |\,\mathrm{Spec}(I \ltimes_\rho Y)| = |\,\mathrm{Spec}(I)| + |\,\mathrm{Spec}(Y)|.$$

By Corollary 14.4 and Proposition 15.5, we know that

$$|\mathrm{LH}_n| = |\,\mathrm{Spec}(\mathrm{LH}_n)| + 1.$$

Hence, $|Y| = n - i + 1$. Since Y is isomorphic to a Hilbert subalgebra of X, it follows that

$$Y \cong \mathrm{LH}_{n-i+1} \cong X/I.$$ □

We now focus on Hilbert algebras that arise as extensions, via symmetric semidirect products, of the simple Hilbert algebra $\mathbf{A}_2 = \{1 > 0\}$.

Proposition 15.7. *Let X be a Hilbert algebra and $\mathbf{A}_2$ be the simple Hilbert algebra such that $\mathbf{A}_2$ acts on X via ρ as Hilbert algebras. Let $I_0 = \ker \rho_0$. Then*

$$|\mathscr{I}(X \rtimes\!\!\!\ltimes_\rho \mathbf{A}_2)| = |\mathscr{I}(X)| + |\mathscr{I}(X/I_0)|.$$

Proof. Let $I_0 = \ker \rho_0$. By Proposition 13.19, I_0 is an ρ-ideal of X. Thus, we can induce X/I_0 to operate on $\mathbf{A}_2$ via ρ^{I_0}. Let $\bar{I}$ be an ideal of X/I_0. For all $y \in X/I_0$, then

$$\rho_0^{I_0}(\rho_0^{I_0}(y)\cdot y) = \rho_0^{I_0}(y)\cdot\rho_0^{I_0}(y) = 1$$

and

$$\rho_0^{I_0}(y\cdot\rho_0^{I_0}(y)) = \rho_0^{I_0}(y)\cdot\rho_0^{I_0}(y) = 1$$

Since $\ker \rho_0^{I_0} = 1$, then $\rho_0^{I_0}(y)\cdot y = y\cdot\rho_0^{I_0}(y) = 1$, which means $\rho_0^{I_0} = \mathrm{Id}_{X/I_0}$. Then, we have

$$(X/I_0) \rtimes_{\rho^{I_0}} \mathbf{A}_2 = (X/I_0) \times \mathbf{A}_2.$$

By Proposition 13.20,

$$|\mathscr{I}(X \rtimes\!\!\!\ltimes_\rho \mathbf{A}_2)| = |\mathscr{I}(X)| + |\mathscr{I}(X/I_0)|.$$ □

Corollary 15.8. *Let X be a Hilbert algebra such that $\mathbf{A}_2$ operates on X via ρ as Hilbert algebras. Then*

$$|\mathscr{I}(X \rtimes\!\!\!\ltimes_\rho \mathbf{A}_2)| = |\mathscr{I}(X)| + 1$$

if and only if $X \rtimes\!\!\!\ltimes_\rho \mathbf{A}_2 = X \sqcup \{(1,0)\}$ is bounded with smallest element $(1,0)$.

Proof. $|\mathscr{I}(X/I_0)| = 1$ if and only $\rho_0(x)$ for all $x \in X$. □

Example 15.9. *Let $X = \{x_0 > x_1 > \cdots > x_{n-1}\}$ be a linear Hilbert algebra. For each $0 \le k < n$, define a map*

$$\rho^{(k)} : \mathbf{A}_2 \longrightarrow \mathrm{End}(X)$$

by setting $\rho_1^{(k)} = \mathrm{Id}_X$ and defining $\rho_0^{(k)} : X \to X$ as

$$\rho_0^{(k)}(x_i) = \begin{cases} x_i, & \text{if } i > k, \\ 1, & \text{if } i \le k. \end{cases}$$

It is straightforward to verify that for each $1 \le k < n$, $\mathbf{A}_2$ acts on X via $\rho^{(k)}$ as Hilbert algebras. By Proposition 15.5, the quotient $X/\ker\rho_0^{(k)}$ is isomorphic to the Hilbert chain LH_{n-k}. Therefore, by Proposition 15.7, we obtain

$$|\mathscr{I}(X \rtimes\!\!\!\ltimes_{\rho^{(k)}} \mathbf{A}_2)| = 2n - k.$$

Appendix A

Proof of Proposition 4.11

The purpose of this appendix is to prove Proposition 4.11.

Firstly, let's introduce some notations we will use in the following proof. For an element

$$(\mathfrak{X}_1,\ldots,\mathfrak{X}_n,x_{n+1}) \in \underbrace{\wedge^2\mathfrak{g}\otimes\cdots\otimes\wedge^2\mathfrak{g}}_{n-1}\wedge\mathfrak{g}$$

with $\mathfrak{X}_i = x_i\wedge y_i$, we write it as an ordered sequence $(z_1,z_2,\ldots,z_{2n},z_{2n+1})$, where

$$z_1 = x_1,\ z_2 = y_1,\ldots,z_{2k-1} = x_k,\ z_{2k} = y_k,\ldots,z_{2n-1} = x_n,\ z_{2n} = y_n,\ z_{2n+1} = x_{n+1}.$$

For arbitrary permutation $\tau \in S_{2n+1}$, its action on the element $(z_1,\ldots,z_{2n+1})$ is given as

$$\tau(z_1,z_2,\ldots,z_{2n},z_{2n+1}) = (z_{\tau^{-1}(1)},z_{\tau^{-1}(2)},\ldots,z_{\tau^{-1}(2n)},z_{\tau^{-1}(2n+1)}).$$

Define an operation $[i-1,i,i+1]_{\mathfrak{g}}$ from $\underbrace{\wedge^2\mathfrak{g}\otimes\cdots\otimes\wedge^2\mathfrak{g}}_{n-1}\wedge\mathfrak{g}$ to $\underbrace{\wedge^2\mathfrak{g}\otimes\cdots\otimes\wedge^2\mathfrak{g}}_{n-2}\wedge\mathfrak{g}$ as

$$[i-1,i,i+1]_{\mathfrak{g}}\big(z_1,z_2,\ldots,z_{2n},z_{2n+1}\big) = \big(z_1,z_2,\ldots,z_{i-2},[z_{i-1},z_i,z_{i+1}]_{\mathfrak{g}},z_{i+2},\ldots,z_{2n+1}\big).$$

Proposition 4.11 The map $\Phi^\bullet : \mathrm{C}^\bullet_{3\text{-Lie}}(\mathfrak{g},M) \to \mathrm{C}^\bullet_{\mathsf{RBO}^\lambda}(\mathfrak{g},M)$ is a chain map.

Proof. Let $n \geqslant 1$. For arbitrary $f \in \mathrm{C}^n_{3\text{-Lie}}(\mathfrak{g},M)$ and $\mathfrak{X}_1\otimes\mathfrak{X}_2\otimes\cdots\otimes\mathfrak{X}_n\wedge x_{n+1} \in (\mathfrak{g}^{\wedge 2})^{\otimes n}\otimes\mathfrak{g}^{\wedge 3}$ with $\mathfrak{X}_i = x_i\wedge y_i, \forall 1\leqslant i\leqslant n$,

$$\begin{aligned}
&\Big(\Phi^{n+1}\delta^n(f)\Big)\Big(\mathfrak{X}_1,\ldots,\mathfrak{X}_n,x_{n+1}\Big)\\
=\ &\Big(\delta^n(f)\circ(T,\cdots,T,T)\Big)\Big(\mathfrak{X}_1,\ldots,\mathfrak{X}_n,x_{n+1}\Big) - \sum_{k=0}^{2n}\lambda^{2n-k}\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant 2n+1}\\
&\Big(T_M\circ\delta^n(f)\circ(\mathrm{Id}^{(i_1-1)},T,\mathrm{Id}^{(i_2-i_1-1)},T,\cdots,T,\mathrm{Id}^{(2n+1-i_k)})\Big)\Big(\mathfrak{X}_1,\ldots,\mathfrak{X}_n,x_{n+1}\Big)
\end{aligned}$$

$$= \sum_{1\le j<k\le n}(-1)^j f\Big(T(x_1)\wedge T(y_1),\cdots,\hat{\mathfrak{X}}_j,\cdots,T(x_{k-1})\wedge T(y_{k-1}),\Big([T(x_j),T(y_j),T(x_k)]_{\mathfrak{g}}\wedge T(y_k)$$
$$+T(x_k)\wedge[T(x_j),T(y_j),T(y_k)]_{\mathfrak{g}}\Big),T(x_{k+1})\wedge T(y_{k+1}),\cdots,T(x_n)\wedge T(y_n),T(x_{n+1})\Big)$$
$$+\sum_{j=1}^{n}(-1)^j f\Big(T(x_1)\wedge T(y_1),\cdots,\hat{\mathfrak{X}}_j,\cdots,T(x_n)\wedge T(y_n),[T(x_j),T(y_j),T(x_{n+1})]_{\mathfrak{g}}\Big)$$
$$+\sum_{j=1}^{n}(-1)^{j+1}\rho\Big(T(x_j),T(y_j)\Big)f\Big(T(x_1)\wedge T(y_1),\cdots,\hat{\mathfrak{X}}_j,\cdots,T(x_n)\wedge T(y_n),T(x_{n+1})\Big)$$
$$+(-1)^{n+1}\rho\Big(T(y_n),T(x_{n+1})\Big)f\Big(T(x_1)\wedge T(y_1),\cdots,T(x_{n-1})\wedge T(y_{n-1}),T(x_n)\Big)$$
$$+(-1)^{n+1}\rho\Big(T(x_{n+1}),T(x_n)\Big)f\Big(T(x_1)\wedge T(y_1),\cdots,T(x_{n-1})\wedge T(y_{n-1}),T(y_n)\Big)$$
$$-\sum_{k=0}^{2n}\lambda^{2n-k}\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant 2n+1}\sum_{1\le j<k\le n}(-1)^j\Big(T_M\circ f\circ[2k-3,2k-2,2k-1]_{\mathfrak{g}}\circ$$
$$(2j-1,\cdots,2k-2)^{-2}\circ(\mathrm{Id}^{(i_1-1)},T,\mathrm{Id}^{(i_2-i_1-1)},T,\cdots,T,\mathrm{Id}^{(2n+1-i_k)})\Big)\Big(\mathfrak{X}_1,\cdots,\mathfrak{X}_n,x_{n+1}\Big)$$
$$-\sum_{k=0}^{2n}\lambda^{2n-k}\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant 2n+1}\sum_{1\le j<k\le n}(-1)^j\Big(T_M\circ f\circ[2k-2,2k-1,2k]_{\mathfrak{g}}\circ$$
$$(2j-1,\cdots,2k-1)^{-2}\circ(\mathrm{Id}^{(i_1-1)},T,\mathrm{Id}^{(i_2-i_1-1)},T,\cdots,T,\mathrm{Id}^{(2n+1-i_k)})\Big)\Big(\mathfrak{X}_1,\cdots,\mathfrak{X}_n,x_{n+1}\Big)$$
$$-\sum_{k=0}^{2n}\lambda^{2n-k}\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant 2n+1}\sum_{j=1}^{n}(-1)^j\Big(T_M\circ f\circ[2n-1,2n,2n+1]_{\mathfrak{g}}\circ$$
$$(2j-1,\cdots,2n)^{-2}\circ(\mathrm{Id}^{(i_1-1)},T,\mathrm{Id}^{(i_2-i_1-1)},T,\cdots,T,\mathrm{Id}^{(2n+1-i_k)})\Big)\Big(\mathfrak{X}_1,\cdots,\mathfrak{X}_n,x_{n+1}\Big)$$
$$-\sum_{k=0}^{2n}\lambda^{2n-k}\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant 2n+1}\sum_{j=1}^{n}(-1)^{j+1}\Big(T_M(\rho\cdot f)\circ(1,\cdots,2j)^2\circ$$
$$(\mathrm{Id}^{(i_1-1)},T,\mathrm{Id}^{(i_2-i_1-1)},T,\cdots,T,\mathrm{Id}^{(2n+1-i_k)}))\Big(\mathfrak{X}_1,\cdots,\mathfrak{X}_n,x_{n+1}\Big)$$
$$-\sum_{k=0}^{2n}\lambda^{2n-k}\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant 2n+1}(-1)^{n+1}\Big\{T_M\circ(\rho\cdot f)\circ\Big((1,\cdots,2n+1)+(1,\cdots,2n+1)\circ(1,\cdots,2n-1)\Big)\circ$$
$$\Big(\mathrm{Id}^{(i_1-1)},T,\mathrm{Id}^{(i_2-i_1-1)},T,\cdots,T,\mathrm{Id}^{(2n+1-i_k)}\Big)\Big\}\Big(\mathfrak{X}_1,\cdots,\mathfrak{X}_n,x_{n+1}\Big)$$

On the other hand, we have

$$\partial^n\Phi^n(f)(\mathfrak{X}_1,\ldots\mathfrak{X}_n,x_{n+1})$$
$$=\sum_{1\le j<k\le n}(-1)^j\Phi^n(f)\Big(\mathfrak{X}_1,\cdots,\hat{\mathfrak{X}}_j,\cdots,\mathfrak{X}_{k-1},[x_j,y_j,x_k]_T\wedge y_k+x_k\wedge[x_j,y_j,y_k]_T,\mathfrak{X}_{k+1},\cdots,\mathfrak{X}_n,x_{n+1}\Big)$$
$$+\sum_{j=1}^{n}(-1)^j\Phi^n(f)\Big(\mathfrak{X}_1,\cdots,\hat{\mathfrak{X}}_j,\cdots,\mathfrak{X}_n,[x_j,y_j,x_{n+1}]_T\Big)$$
$$+\sum_{j=1}^{n}(-1)^{j+1}\rho\Big(T(x_j),T(y_j)\Big)\Phi^n(f)\Big(\mathfrak{X}_1,\cdots,\hat{\mathfrak{X}}_j,\cdots,\mathfrak{X}_n,x_{n+1}\Big)$$
$$-\sum_{j=1}^{n}(-1)^{j+1}T_M\Big(\rho(T(x_j),y_j)\Phi^n(f)(\mathfrak{X}_1,\cdots,\hat{\mathfrak{X}}_j,\cdots,\mathfrak{X}_n,x_{n+1})\Big)$$
$$-\sum_{j=1}^{n}(-1)^{j+1}T_M\Big(\rho(x_j,T(y_j))\Phi^n(f)(\mathfrak{X}_1,\cdots,\hat{\mathfrak{X}}_j,\cdots,\mathfrak{X}_n,x_{n+1})\Big)$$

$$
\begin{aligned}
&-\sum_{j=1}^{n}(-1)^{j+1}\lambda\; T_M\Big(\rho(x_j,y_j)\Phi^n(f)(\mathfrak{X}_1,\cdots,\hat{\mathfrak{X}}_j,\cdots,\mathfrak{X}_n,x_{n+1})\Big)\\
&+(-1)^{n+1}\Big(\rho(T(y_n),T(x_{n+1}))\Phi^n(f)(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n-1},x_n)+\rho(T(x_{n+1}),T(x_n))\Phi^n(f)(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n-1},y_n)\Big)\\
&-(-1)^{n+1}T_M\Big(\rho(T(y_n),x_{n+1})\Phi^n(f)(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n-1},x_n)+\rho(T(x_{n+1}),x_n)\Phi^n(f)(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n-1},y_n)\Big)\\
&-(-1)^{n+1}T_M\Big(\rho(y_n,T(x_{n+1}))\Phi^n(f)(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n-1},x_n)+\rho(x_{n+1},T(x_n))\Phi^n(f)(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n-1},y_n)\Big)\\
&-(-1)^{n+1}\lambda\; T_M\Big(\rho(y_n,x_{n+1})\Phi^n(f)(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n-1},x_n)+\rho(x_{n+1},x_n)\Phi^n(f)(\mathfrak{X}_1,\cdots,\mathfrak{X}_{n-1},y_n)\Big)\\
=&\sum_{1\le j<k\le n}(-1)^j f\Big(T(x_1)\wedge T(y_1),\cdots,\hat{\mathfrak{X}}_j,\cdots,T(x_{k-1})\wedge T(y_{k-1}),[T(x_j),T(y_j),T(x_k)]_{\mathfrak{g}}\wedge T(y_k)\\
&\qquad+T(x_k)\wedge[T(x_j),T(y_j),T(y_k)]_{\mathfrak{g}},T(x_{k+1})\wedge T(y_{k+1}),\cdots,T(x_n)\wedge T(y_n)\wedge T(x_{n+1})\Big)\\
&-\sum_{k=0}^{2n}\lambda^{2n-k}\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant 2n+1}\;\sum_{1\le j<k\le n}(-1)^jT_M\circ f\circ[2k-1,2k,2k+1]_{\mathfrak{g}}\circ\\
&\quad(2j-1,\cdots,2k)^{-2}\circ(\mathrm{Id}^{(i_1-1)},T,\mathrm{Id}^{(i_2-i_1-1)},T,\cdots,T,\mathrm{Id}^{(2n+1-i_k)})\Big(\mathfrak{X}_1,\cdots,\mathfrak{X}_n,x_{n+1}\Big)\\
&-\sum_{k=0}^{2n}\lambda^{2n-k}\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant 2n+1}\;\sum_{1\le j<k\le n}(-1)^jT_M\circ f\circ[2k,2k+1,2k+2]_{\mathfrak{g}}\circ(2j-1,\cdots,2k+1)^{-2}\circ\\
&\quad(\mathrm{Id}^{(i_1-1)},T,\mathrm{Id}^{(i_2-i_1-1)},T,\cdots,T,\mathrm{Id}^{(2n+1-i_k)})\Big(\mathfrak{X}_1,\cdots,\mathfrak{X}_n,x_{n+1}\Big)\\
&+\sum_{j=1}^{n}(-1)^j f\Big(T(x_1)\wedge T(y_1),\cdots,\hat{\mathfrak{X}}_j,\cdots,T(x_n)\wedge T(y_n)\wedge[T(x_j),T(y_j),T(x_{n+1})]_{\mathfrak{g}}\Big)\\
&-\sum_{k=0}^{2n}\lambda^{2n-k}\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant 2n+1}\;\sum_{1\le j<k\le n}(-1)^jT_M\circ f\circ[2k,2k+1,2k+2]_{\mathfrak{g}}\circ\big(2j-1,\cdots,2k+1\big)^{-2}\circ\\
&\qquad\circ(\mathrm{Id}^{(i_1-1)},T,\mathrm{Id}^{(i_2-i_1-1)},T,\cdots,T,\mathrm{Id}^{(2n+1-i_k)})\Big(\mathfrak{X}_1,\cdots,\mathfrak{X}_n,x_{n+1}\Big)\\
&+\sum_{j=1}^{n}(-1)^{j+1}\rho\big(T(x_j),T(y_j)\big)\cdot f\Big(T(x_1)\wedge T(y_1),\cdots,\hat{\mathfrak{X}}_j,\cdots,T(x_n)\wedge T(y_n)\wedge T(x_{n+1})\Big)\\
&-\sum_{k=0}^{2n}\lambda^{2n-k}\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant 2n+1}\;\sum_{j=1}^{n}(-1)^{j+1}T_M\circ(\rho\cdot f)\circ(1,\cdots,2j)^2\circ\\
&\qquad(\mathrm{Id}^{(i_1-1)},T,\mathrm{Id}^{(i_2-i_1-1)},T,\cdots,T,\mathrm{Id}^{(2n+1-i_k)})\Big(\mathfrak{X}_1,\cdots,\mathfrak{X}_n,x_{n+1}\Big)\\
&-\sum_{k=0}^{2n}\lambda^{2n-k}\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant 2n+1}(-1)^{n+1}T_M\circ(\rho\cdot f)\circ\Big((1,\cdots,2n+1)+(1,\cdots,2n+1)\circ(1,\cdots,2n-1)\Big)\circ\\
&\qquad\Big(\mathrm{Id}^{(i_1-1)},T,\mathrm{Id}^{(i_2-i_1-1)},T,\cdots,T,\mathrm{Id}^{(2n+1-i_k)}\Big)\Big(\mathfrak{X}_1,\cdots,\mathfrak{X}_n,x_{n+1}\Big)\\
&+(-1)^{n+1}\rho\big(T(y_n),T(x_{n+1})\big)\cdot f\Big(T(x_1)\wedge T(y_1),\cdots,T(x_{n-1})\wedge T(y_{n-1}),T(x_n)\Big)\\
&+(-1)^{n+1}\rho\big(T(x_{n+1}),T(x_n)\big)\cdot f\Big(T(x_1)\wedge T(y_1),\cdots,T(x_{n-1})\wedge T(y_{n-1}),T(y_n)\Big).
\end{aligned}
$$

So we have $\partial^n\Phi^n=\Phi^{n+1}\delta^n$. □

Appendix $\mathcal{B}$

Proof of Proposition 8.16

Proof. We just to check that $\{T^{M^\vee}_{i,j}\}_{i,j\geqslant 0}$ satisfies Equation (2.8), that is, we need to check the following identity:

$$\textbf{(I)}+\textbf{(II)}+\textbf{(III)}+\textbf{(IV)}=0,$$

where

$$\textbf{(I)} \quad = -\sum_{\substack{i_1+\cdots+i_p+l=m,\\ j_1+\cdots+j_q+k=n\\ p,q,l,k\geqslant 0}} (-1)^{\alpha} m^{M^\vee}_{p,q}\circ\left(T_{i_1}\otimes\cdots\otimes T_{i_p}\otimes T^{M^\vee}_{l,k}\otimes T_{j_1}\otimes\cdots\otimes T_{j_q}\right);$$

$$\textbf{(II)} \quad = \sum_{\substack{i_1+\cdots+i_p+l=m,\\ j_1+\cdots+j_q+k=n,\\ i_1,\cdots,i_p,j_1,\cdots,j_q\geqslant 1\\ p,\ q,\ l,\ k\geqslant 0;}} (-1)^{\beta_1} T^{M^\vee}_{l,k}\circ\left(\mathrm{Id}_A^{\otimes l}\otimes m^{M^\vee}_{p,q}\circ(T_{i_1}\otimes\cdots\otimes T_{i_p}\otimes\mathrm{Id}_M\otimes T_{j_1}\otimes\cdots\otimes T_{j_q})\otimes\mathrm{Id}_A^{\otimes k}\right);$$

$$\textbf{(III)} \quad = \sum_{\substack{i_1+\cdots+i_p+l+1=m\\ j_1+\cdots+j_q+k=n\\ i_1,\cdots,i_p,j_1,\cdots,j_q\geqslant 0\\ v,l,k,p,q\geqslant 0}} (-1)^{\beta_2} T^{M^\vee}_{l,k}\circ\left(\mathrm{Id}_A^{\otimes l}\otimes m^{M^\vee}_{p+1,q}\circ(T_{i_1}\otimes\cdots\otimes T_{i_v}\otimes\mathrm{Id}_A\otimes T_{i_{v+1}}\cdots\right.$$
$$\left.\cdots\otimes T_{i_p}\otimes T^{M^\vee}_{r,t}\otimes T_{j_1}\otimes\cdots\otimes T_{j_q})\otimes\mathrm{Id}_A^{\otimes k}\right);$$

$$\textbf{(IV)} \quad = \sum_{\substack{i_1+\cdots+i_p+l+1=m\\ j_1+\cdots+j_q+k=n\\ i_1,\cdots,i_p,j_1,\cdots,j_q\geqslant 0\\ v,l,k,p,q\geqslant 0}} (-1)^{\beta_3} T^{M^\vee}_{l,k}\circ\left(\mathrm{Id}_A^{\otimes i}\otimes m^{M^\vee}_{p,q+1}\circ(T_{i_1}\otimes\cdots\otimes T_{i_p}\otimes T^{M^\vee}_{r,t}\otimes T_{j_1}\otimes\cdots\right.$$
$$\left.\cdots\otimes T_{j_v}\otimes\mathrm{Id}_A\otimes T_{j_{v+1}}\otimes\cdots T_{j_q})\otimes\mathrm{Id}_A^{\otimes k}\right).$$

We have that:

Term (I):

$$-\sum_{\substack{i_1+\cdots+i_p+l=m,\\ j_1+\cdots+j_q+k=n\\ p,q,l,k\geqslant 0}} (-1)^{\alpha} m^{M^\vee}_{p,q}\circ\Big(T_{i_1}\otimes\cdots\otimes T_{i_p}\otimes T^{M^\vee}_{l,k}\otimes T_{j_1}\otimes\cdots\otimes T_{j_q}\Big)(a_1\otimes\cdots\otimes a_m\otimes f\otimes b_1\otimes\cdots\otimes b_n)$$

$$=\sum_{\substack{i_1+\cdots+i_p+l=m,\\ j_1+\cdots+j_q+k=n,\\ i_1,\cdots,i_p,j_1,\cdots,j_q\geqslant 1\\ p,\ q,\ l,\ k\geqslant 0;}} (-1)^{\beta_1+\theta} f\circ T^M_{l,k}\circ\Big(\mathrm{Id}_A^{\otimes l}\otimes m_{p,q}\circ(T_{i_1}\otimes\cdots\otimes T_{i_p}\otimes \mathrm{Id}_M\otimes T_{j_1}\otimes\cdots\otimes T_{j_q})\otimes\mathrm{Id}_A^{\otimes k}\Big)$$

$$(b_1\otimes\cdots\otimes b_n\otimes\mathrm{Id}_M\otimes a_1\otimes\cdots\otimes a_m),$$

Term (II):

$$\sum_{\substack{i_1+\cdots+i_p+l=m,\\ j_1+\cdots+j_q+k=n,\\ i_1,\cdots,i_p,j_1,\cdots,j_q\geqslant 1\\ p,\ q,\ l,\ k\geqslant 0;}} (-1)^{\beta_1} T^{M^\vee}_{l,k}\circ\Big(\mathrm{Id}_A^{\otimes l}\otimes m^{M^\vee}_{p,q}\circ(T_{i_1}\otimes\cdots\otimes T_{i_p}\otimes\mathrm{Id}_M\otimes T_{j_1}\otimes\cdots\otimes T_{j_q})\otimes\mathrm{Id}_A^{\otimes k}\Big)$$

$$(a_1\otimes\cdots\otimes a_m\otimes f\otimes b_1\otimes\cdots\otimes b_n)$$

$$=-\sum_{\substack{i_1+\cdots+i_p+l=m,\\ j_1+\cdots+j_q+k=n\\ p,q,l,k\geqslant 0}} (-1)^{\alpha+\theta} f\circ m_{p,q}\circ\Big(T_{i_1}\otimes\cdots\otimes T_{i_p}\otimes T^M_{l,k}\otimes T_{j_1}\otimes\cdots\otimes T_{j_q}\Big)(b_1\otimes\cdots$$

$$\cdots\otimes b_n\otimes\mathrm{Id}_M\otimes a_1\otimes\cdots\otimes a_m)(b_1\otimes\cdots\otimes b_n\otimes\mathrm{Id}_M\otimes a_1\otimes\cdots\otimes a_m),$$

Term (III):

$$\sum_{\substack{i_1+\cdots+i_p+l+1=m\\ j_1+\cdots+j_q+k=n\\ i_1,\cdots,i_p,j_1,\cdots,j_q\geqslant 0\\ v,l,k,p,q\geqslant 0}} (-1)^{\beta_2} T^{M^\vee}_{l,k}\circ\Big(\mathrm{Id}_A^{\otimes l}\otimes m^{M^\vee}_{p+1,q}\circ\Big(T_{i_1}\otimes\cdots\otimes T_{i_v}\otimes\mathrm{Id}_A\otimes T_{i_{v+1}}\otimes\cdots\otimes T_{i_p}\otimes$$

$$T^{M^\vee}_{r,t}\otimes T_{j_1}\otimes\cdots\otimes T_{j_q}\Big)\otimes\mathrm{Id}_A^{\otimes k}\Big)(a_1\otimes\cdots\otimes a_m\otimes f\otimes b_1\otimes\cdots\otimes b_n)$$

$$=\sum_{\substack{i_1+\cdots+i_p+l+1=m\\ j_1+\cdots+j_q+k=n\\ i_1,\cdots,i_p,j_1,\cdots,j_q\geqslant 0\\ v,l,k,p,q\geqslant 0}} (-1)^{\beta_3+\theta} f\circ T^M_{l,k}\circ\Big(\mathrm{Id}_A^{\otimes i}\otimes m_{p,q+1}\circ\Big(T_{i_1}\otimes\cdots\otimes T_{i_p}\otimes T^M_{r,t}\otimes T_{j_1}\otimes\cdots\otimes T_{j_v}\otimes$$

$$\mathrm{Id}_A\otimes T_{j_{v+1}}\otimes\cdots T_{j_q}\Big)\otimes\mathrm{Id}_A^{\otimes k}\Big)(b_1\otimes\cdots\otimes b_n\otimes\mathrm{Id}_M\otimes a_1\otimes\cdots\otimes a_m),$$

Term (IV):

$$\sum_{\substack{i_1+\cdots+i_p+l+1=m\\ j_1+\cdots+j_q+k=n\\ i_1,\cdots,i_p,j_1,\cdots,j_q\geqslant 0\\ v,l,k,p,q\geqslant 0}} (-1)^{\beta_3} T^{M^\vee}_{l,k}\circ\Big(\mathrm{Id}_A^{\otimes i}\otimes m^{M^\vee}_{p,q+1}\circ(T_{i_1}\otimes\cdots\otimes T_{i_p}\otimes T^{M^\vee}_{r,t}\otimes T_{j_1}\otimes\cdots\otimes T_{j_v}\otimes\mathrm{Id}_A\otimes T_{j_{v+1}}\otimes\cdots$$

$$\cdots T_{j_q})\otimes\mathrm{Id}_A^{\otimes k}\Big)(a_1\otimes\cdots\otimes a_m\otimes f\otimes b_1\otimes\cdots\otimes b_n)$$

$$=\sum_{\substack{i_1+\cdots+i_p+l+1=m\\ j_1+\cdots+j_q+k=n\\ i_1,\cdots,i_p,j_1,\cdots,j_q\geqslant 0\\ v,l,k,p,q\geqslant 0}} (-1)^{\beta_2+\theta} f\circ T^M_{l,k}\circ\Big(\mathrm{Id}_A^{\otimes l}\otimes m_{p+1,q}\circ\Big(T_{i_1}\otimes\cdots\otimes T_{i_v}\otimes\mathrm{Id}_A\otimes T_{i_{v+1}}\otimes\cdots\otimes T_{i_p}\otimes T^M_{r,t}\otimes T_{j_1}\otimes\cdots$$

$$\cdots\otimes T_{j_q}\Big)\otimes\mathrm{Id}_A^{\otimes k}\Big)(b_1\otimes\cdots\otimes b_n\otimes\mathrm{Id}_M\otimes a_1\otimes\cdots\otimes a_m),$$

where

$$\theta = (\sum_{s=1}^{m} |a_s|)(\sum_{s=1}^{n} |b_s|) + |f|(\sum_{s=1}^{m} |a_s| + m + n + 1) + (m+n+1)(n+1).$$

Taking the sum, one can easily see that

$$(\mathbf{I}) + (\mathbf{II}) + (\mathbf{III}) + (\mathbf{IV}) = 0,$$

since $\{T^{M}_{i,j}\}_{i,j\geqslant 0}$ subjects to Equation (2.8). Thus $\{T^{M^\vee}_{i,j}\}_{i,j\geqslant 0}$ satisfies Equation (2.8). $\square$

Appendix C

Proof of Lemma 9.7

Before proving Lemma 9.7, we first introduce the following lemma, which will be used extensively in the proof.

Lemma C.1. *Let $(A, d_A, \cdot)$ be a dg algebra and $(B, \rhd)$ a left dg A-module. Then*

$$\kappa : B \otimes B^\vee \to A^\vee$$

is a dg A-bimodule morphism.

Moreover, if (A, B) is an interactive pair, then κ is also a right dg B-module morphism.

Proof. For any $b \in B$, $f \in B^\vee$ and $a_1, a_2 \in A$,

$$\begin{aligned}
\kappa\left((a_2 \rhd b) \otimes f\right)(a_1) &= (-1)^{(|a_2|+|b|(|f|+|a_1|)} f\left((a_1 \cdot a_2) \rhd b\right) \\
&= (-1)^{|a_2|(|b|+|f|+|a_1|)} \kappa(b \otimes f)(a_1 \cdot a_2) \\
&= \left(a_2 \rhd \kappa(b \otimes f)\right)(a_1).
\end{aligned}$$

Similarly, we also have

$$\kappa(b \otimes (f \lhd a_2)) = \kappa(b \otimes f) \lhd a_2, d_{A^\vee}\left(\kappa(b \otimes f)\right) = \kappa\left(d_B(b) \otimes f\right) + (-1)^{|b|} \kappa\left(b \otimes d_{B^\vee}(f)\right).$$

Thus, κ is a dg A-bimodule morphism.

Now, we assume that (A, B) is an interactive pair. For any $b_1, b_2 \in B$, $f \in B^\vee$ and $a \in A$,

$$\begin{aligned}
\kappa\left(b_1 \otimes f \blacktriangleleft b_2\right)(a) &= (-1)^{|b_1|(|f|+|b_2|+|a|}(f \blacktriangleleft b_2)(a \rhd b_1) \\
&= (-1)^{|b_1|(|f|+|b_2|+|a|} f\left(b_2 * (a \rhd b_1)\right) \\
&= (-1)^{|b_1|(|f|+|b_2|+|a|} f\left((b_2 \blacktriangleright a \rhd) b_1\right) \\
&= \left(\kappa(b_1 \otimes f) \blacktriangleleft b_2\right)(a),
\end{aligned}$$

where "$*$" stands for the multiplication on B and "$\blacktriangleleft$" stands for the induced right action of B on $A^\vee$. Thus, κ is also a right dg B-module morphism. □

Proof of Lemma 9.7. We proceed to verify that the Stasheff identities for the operations $\{m_n\}_{n \geqslant 1}$, introduced in Lemma 9.7, hold trivially in every case. We divide it into the following five cases.

Case I: For $b_1,\dots,b_{n+1}\in B$ and $f_1,\dots,f_n\in B^\vee$, by Lemma C.1 , we have

$$
\begin{aligned}
&\sum_{\substack{i+j+k=2n+1,\\ i,k\geqslant 0, j\geqslant 1}} (-1)^{i+jk} m_{i+k+1}(\mathrm{Id}^{\otimes i}\otimes m_j\otimes \mathrm{Id}^{\otimes k})(b_1\otimes s^{-1}f_1\otimes\cdots\otimes s^{-1}f_n\otimes b_{n+1})\\
=&\sum_{\substack{i+j=n,\\ s+k=2i,\\ i,j,k\geqslant 0}} m_{2i+1}\circ\Big(\mathrm{Id}^{\otimes s}\otimes m_{2j+1}\otimes \mathrm{Id}^{\otimes k}\Big)(b_1\otimes s^{-1}f_1\otimes\cdots\otimes s^{-1}f_n\otimes b_{n+1})\\
=&\sum_{\substack{i+j=n,\\ i,j\geqslant 1,\\ i-1\geqslant p\geqslant 0}} (-1)^{p+\sum_{k=1}^{p}(|b_k|+|f_k|)} m_{2i+1}\Big(b_1\otimes s^{-1}f_1\otimes\cdots\otimes b_p\otimes s^{-1}f_p\otimes m_{2j+1}\big(b_{p+1}\otimes s^{-1}f_{s+j}\otimes b_{s+j+1}\big)\\
&\qquad\otimes s^{-1}f_{p+j+1}\otimes\cdots\otimes b_{n+1}\Big)\\
&+\sum_{\substack{i+j=n,\\ i,j\geqslant 1,\\ i-1\geqslant p\geqslant 0}} (-1)^{p+\sum\limits_{k=1}^{p}(|b_k|+|f_k|)+|b_{p+1}|} m_{2i+1}\Big(b_1\otimes s^{-1}f_1\otimes\cdots\otimes s^{-1}f_p\otimes b_{p+1}\otimes m_{2j+1}\big(s^{-1}f_{p+1}\otimes\cdots\\
&\qquad\cdots\otimes s^{-1}f_{p+j+1}\big)\otimes b_{p+j+2}\otimes\cdots\otimes b_{n+1}\Big)\\
&+\sum_{\substack{i+j=n,\\ i,j\geqslant 1}} (-1)^{i+\sum\limits_{k=1}^{i}(|b_k|+|f_k|)} m_{2i+1}\Big(b_1\otimes s^{-1}f_1\otimes\cdots\otimes b_i\otimes s^{-1}f_i\otimes m_{2j+1}(b_{i+1}\otimes s^{-1}f_{i+1}\otimes\cdots\otimes s^{-1}f_{n-1}\otimes b_{n+1})\Big)\\
&+\sum_{0\leqslant p\leqslant n-1} (-1)^{p+\sum\limits_{k=1}^{p}(|b_k|+|f_k|)+1} m_{2n+1}\Big(b_1\otimes s^{-1}f_1\otimes\cdots\otimes b_p\otimes s^{-1}f_p\otimes d_B(b_{p+1})\otimes s^{-1}f_{p+1}\otimes\cdots\otimes b_{n+1}\Big)\\
&+\sum_{0\leqslant p\leqslant n-1} (-1)^{p+\sum\limits_{k=1}^{p}(|b_k|+|f_k|)+|b_{p+1}|} m_{2n+1}\Big(b_1\otimes s^{-1}f_1\otimes\cdots\otimes s^{-1}f_p\otimes b_{p+1}\otimes s^{-1}d_{B^\vee}(f_{p+1})\otimes\cdots\otimes b_{n+1}\Big)\\
&-d_B m_{2n+1}\Big(b_1\otimes s^{-1}f_1\otimes\cdots\otimes s^{-1}f_n\otimes b_{n+1}\Big)\\
&+(-1)^{n-1+\sum\limits_{k=1}^{n}(|b_k|+|f_k|)} m_{2n+1}\Big(b_1\otimes s^{-1}f_1\otimes\cdots\otimes s^{-1}f_n\otimes d_B(b_{n+1})\Big)\\
=&\sum_{\substack{i+j=n;\\ i,j\geqslant 1;\\ i-1\geqslant p\geqslant 0}} (-1)^{\gamma_1} m_{2i+1}\Big(b_1\otimes s^{-1}f_1\otimes\cdots\otimes b_p\otimes s^{-1}f_p\otimes T_j\Big(\kappa(b_{p+1}\otimes f_{p+1})\otimes\cdots\otimes\kappa(b_{p+j}\otimes f_{p+j})\Big)\triangleright b_{p+j+1}\\
&\qquad\otimes s^{-1}f_{p+j+1}\otimes\cdots\otimes b_{n+1}\Big)\\
&+\sum_{\substack{i+j=n;\\ i,j\geqslant 1;\\ i-1\geqslant p\geqslant 0}} (-1)^{\gamma_2} m_{2i+1}\Big(b_1\otimes s^{-1}f_1\otimes\cdots\otimes s^{-1}f_p\otimes b_{p+1}\otimes s^{-1}f_{p+1}\triangleleft T_j\Big(\kappa(b_{p+2}\otimes f_{p+2})\otimes\cdots\\
&\qquad\cdots\otimes\kappa(b_{p+j+1}\otimes f_{p+j+1})\Big)\otimes s^{-1}f_{p+j+2}\otimes\cdots\otimes b_{n+1}\Big)\\
&+\sum_{\substack{i+j=n,\\ i,j\geqslant 1}} (-1)^{\gamma_3} m_{2i+1}\Big(b_1\otimes s^{-1}f_1\otimes\cdots\otimes b_i\otimes s^{-1}f_i\otimes T_j(\kappa(b_{i+1}\otimes f_{i+1})\otimes\cdots\otimes\kappa(b_n\otimes f_n))\triangleright b_{n+1}\Big)\\
&+\sum_{0\leqslant p\leqslant n-1} (-1)^{n-1+\sum\limits_{k=1}^{p}(|b_k|+|f_k|)+\gamma} T_n\Big(\kappa(b_1\otimes f_1)\otimes\cdots\otimes\kappa(d_B(b_{p+1})\otimes f_{p+1})\otimes\cdots\otimes\kappa(b_n,f_n)\Big)\triangleright b_{n+1}\\
&+\sum_{0\leqslant p\leqslant n-1} (-1)^{n-1+\sum\limits_{k=1}^{p}(|b_k|+|f_k|)+\gamma+|b_{p+1}|} T_n\Big(\kappa(b_1\otimes f_1)\otimes\cdots\otimes\kappa(b_{p+1}\otimes d_{B^\vee}(f_{p+1}))\otimes\cdots\otimes\kappa(b_n\otimes f_n)\Big)\triangleright b_{n+1}\\
&-(-1)^{\gamma} d_B\Big(T_n(\kappa(b_1\otimes f_1)\otimes\cdots\otimes\kappa(b_n\otimes f_n))\triangleright b_{n+1}\Big)\\
&+(-1)^{n-1+\sum\limits_{k=1}^{n}(|b_k|+|f_k|)+\gamma} T_n\Big(\kappa(b_1\otimes f_1)\otimes\cdots\otimes\kappa(b_n\otimes f_n)\Big)\triangleright d_B(b_{n+1})
\end{aligned}
$$

$$
\begin{aligned}
=&\sum_{\substack{i+j=n;\\ i,j\geqslant 1;\\ i-1\geqslant p\geqslant 0}} (-1)^{\gamma_4} T_i\Big(\kappa(b_1\otimes f_1)\otimes\cdots\otimes\kappa(b_s\otimes f_s)\otimes m^l\Big(T_j\Big(\kappa(b_{s+1}\otimes f_{s+1})\otimes\cdots\otimes\kappa(b_{s+j}\otimes f_{s+j})\Big)\\
&\qquad\qquad \otimes\kappa(b_{s+j+1}\otimes f_{s+j+1})\Big)\otimes\cdots\otimes\kappa(b_n\otimes f_n)\Big)\triangleright b_{n+1}\\
&+\sum_{\substack{i+j=n,\\ i,j\geqslant 1\\ i-1\geqslant s\geqslant 0}} (-1)^{\gamma_5} T_i\Big(\kappa(b_1\otimes f_1)\otimes\cdots\otimes\kappa(b_s\otimes f_s)\otimes m^r\Big(\kappa(b_{s+1}\otimes f_{s+1})\otimes T_j\Big(\kappa(b_{s+2}\otimes f_{s+2})\otimes\cdots\\
&\qquad\qquad \cdots\otimes\kappa(b_{s+j+1}, f_{s+j+1})\Big)\Big)\otimes\cdots\otimes\kappa(b_n\otimes f_n)\Big)\triangleright b_{n+1}\\
&-\sum_{\substack{i+j=n,\\ i,j\geqslant 1}} (-1)^{\gamma_6} m\Big(T_i\Big(\kappa(b_1\otimes f_1)\otimes\cdots\otimes\kappa(b_i\otimes f_i)\Big)\otimes T_j\Big(\kappa(b_{i+1}\otimes f_{i+1})\otimes\cdots\otimes\kappa(b_n\otimes f_n)\Big)\Big)\triangleright b_{n+1}\\
&-(-1)^{\gamma} d_A\Big(T_n\Big(\kappa(b_1\otimes f_1)\otimes\cdots\otimes\kappa(b_n\otimes f_n)\Big)\Big)\triangleright b_{n+1}\\
&+(-1)^{n-1+\sum_{k=1}^{s}(|b_k|+|f_k|)+\gamma} T_n\Big(\kappa(b_1\otimes f_1)\otimes\cdots\otimes d_{A^\vee}(\kappa(b_s\otimes f_s))\otimes\cdots\otimes\kappa(b_n\otimes f_n)\Big)\triangleright b_{n+1}\\
=&(-1)^{\gamma}\Bigg(\Big(\sum_{s+k+j+1=n}(-1)^{s+(j-1)(i-s)} T_i\circ\Big(\mathrm{Id}^{\otimes s}\otimes m^l\circ(T_j\otimes\mathrm{Id})\otimes\mathrm{Id}^{\otimes k}\Big)\\
&\quad+\sum_{s+k+j+1=n}(-1)^{s+(j-1)(i-s)}(-1)^{1-j} T_i\circ\big(\mathrm{Id}^{\otimes s}\otimes m^r\circ(\mathrm{Id}\otimes T_j)\otimes\mathrm{Id}^{\otimes k}\big)-\sum_{i+j=n}(-1)^{1+i} m\circ\big(T_i\otimes T_j\big)\\
&\quad+\sum_{s+k+1=n}(-1)^{n-1} T_n\circ\big(\mathrm{Id}^{\otimes s}\otimes d_{A^\vee}\otimes\mathrm{Id}^{\otimes k}\big)-d_A\circ T_n\Big)\Big(\kappa(b_1\otimes f_1)\otimes\cdots\otimes\kappa(b_n\otimes f_n)\Big)\Bigg)\triangleright b_{n+1}\\
=&0,
\end{aligned}
$$

where

$$
\begin{aligned}
\gamma &= \sum_{k=1}^{n}(n-k+1)|b_k|+\sum_{k=1}^{n}(n-k)|f_k|;\\
\gamma_1 &= p+\sum_{k=1}^{p}(|b_k|+|f_k|)+\sum_{k=p+1}^{p+j}(p+j-k+1)|b_k|+\sum_{k=p+1}^{p+j}(p+j-k)|f_k|;\\
\gamma_2 &= p+\sum_{k=1}^{p+1}(|b_k|+|f_k|)+\sum_{k=p+2}^{p+j+1}(p+j-k+1)|b_k|+\sum_{k=p+1}^{p+j}(p+j-k)|f_k|;\\
\gamma_3 &= i+\sum_{k=1}^{i}(|b_k|+|f_k|)+\sum_{k=i+1}^{n}(n-k+1)|b_k|+\sum_{k=i+1}^{n}(n-k)|f_k|;\\
\gamma_4 &= p+(j-1)(i-p)+(j-1)(\sum_{k=1}^{p}(|b_k|+|f_k|))+\sum_{k=1}^{p+j}(n-k+1)|b_k|+\sum_{k=1}^{p+j}(n-k)|f_k|;\\
\gamma_5 &= s+(j-1)(i-s-1)+(j-1)(\sum_{k=1}^{s}(|b_k|+|f_k|))+\sum_{k=1}^{s+j}(n-k+1)|b_k|+\sum_{k=1}^{s+j}(n-k)|f_k|;\\
\gamma_6 &= i+1+(j-1)(\sum_{k=1}^{i}(|b_k|+|f_k|))+\sum_{k=i+1}^{n}(n-k+1)|b_k|+\sum_{k=i+1}^{n}(n-k)|f_k|.
\end{aligned}
$$

Case II: For brevity, we omit the detailed calculations, which are analogous to those in Case I. For

$b_1,\dots,b_{n+1}\in B$ and $f_0,\dots,f_n\in B^\vee$,

$$
\begin{aligned}
&\sum_{\substack{i+j=n,\\ s+k=2i,i,j,k\geqslant 0}} m_{2i+1}\circ\Big(\mathrm{Id}^{\otimes s}\otimes m_{2j+1}\otimes \mathrm{Id}^{\otimes k}\Big)(s^{-1}f_0\otimes b_1\otimes s^{-1}f_1\otimes\cdots\otimes b_n\otimes s^{-1}f_n)\\
&=(-1)^{\gamma}s^{-1}f_0\lhd\Bigg(\Big(-\sum_{s+k+j+1=n}(-1)^{s+(j-1)(i-s)}T_i\circ\Big(\mathrm{Id}^{\otimes s}\otimes m_l\circ(T_j\otimes\mathrm{Id})\otimes\mathrm{Id}^{\otimes k}\Big)\\
&-\sum_{s+k+j+1=n}(-1)^{s+(j-1)(i-s-1)}T_i\circ\Big(\mathrm{Id}^{\otimes s}\otimes m_r\circ(\mathrm{Id}\otimes T_j)\otimes\mathrm{Id}^{\otimes k}\Big)+\sum_{i+j=n}(-1)^{1+i}m\circ\Big(T_i\otimes T_j\Big)\\
&-\sum_{s+k+1=n}(-1)^{n-1}T_n\circ\Big(\mathrm{Id}^{\otimes s}\otimes d_{A^\vee}\otimes\mathrm{Id}^{\otimes k}\Big)+d_A\circ T_n\Big)\Big(\kappa(b_1\otimes f_1)\otimes\cdots\otimes\kappa(b_n\otimes f_n)\Big)\Bigg)\\
&=0.
\end{aligned}
$$

Since (A,B) is an interactive pair, by Lemma C.1, then

$$\kappa(b_1\otimes f)\blacktriangleleft b_2=\kappa\big(b_1\otimes sm_2(s^{-1}f\otimes b_2)\big),\ \forall b_1,b_2\in B,\ f\in B^\vee. \tag{0.1}$$

Next, we will use Equation (0.1) to verify three cases where the Stasheff identity holds with a nontrivial m_2 involved.

Case III: For $n\geqslant 1$, $b_1,\dots,b_{n+2}\in B$ and $f_1,\dots,f_n\in B^\vee$,

$$
\begin{aligned}
&\sum_{\substack{i+j+k=2n+2,\\ i,k\geqslant 0,j\geqslant 1}}(-1)^{i+jk}m_{i+1+k}\circ\Big(\mathrm{Id}^{\otimes i}\otimes m_j\otimes\mathrm{Id}^{\otimes k}\Big)\Big(b_1\otimes s^{-1}f_1\otimes\cdots\otimes s^{-1}f_n\otimes b_{n+1}\otimes b_{n+2}\Big)\\
=&\Big(m_{2n+1}\circ(\mathrm{Id}^{\otimes 2n}\otimes m_2)-m_{2n+1}\circ(\mathrm{Id}^{\otimes 2n-1}\otimes m_2\otimes\mathrm{Id})-m_2(m_{2n+1}\otimes\mathrm{Id})\Big)\\
&\Big(b_1\otimes s^{-1}f_1\otimes\cdots\otimes s^{-1}f_n\otimes b_{n+1}\otimes b_{n+2}\Big)\\
=&(-1)^{\gamma}T_n\Big(\kappa(b_1\otimes f_1)\otimes\cdots\otimes\kappa(b_n\otimes f_n)\Big)\rhd m_2(b_{n+1}\otimes b_{n+2})\\
&-(-1)^{\gamma}T_n\Big(\kappa(b_1\otimes f_1)\otimes\cdots\otimes\kappa(b_n\otimes sm_2(s^{-1}f_n\otimes b_{n+1}))\Big)\rhd b_{n+2}\\
&-(-1)^{\gamma}m_2\Big(T_n\big(\kappa(b_1\otimes f_1)\otimes\cdots\otimes\kappa(b_n\otimes f_n)\big)\rhd b_{n+1}\otimes b_{n+2}\Big)\\
=&(-1)^{\gamma}T_n\Big(\kappa(b_1\otimes f_1)\otimes\cdots\otimes\kappa(b_n\otimes f_n)\Big)\rhd(b_{n+1}*b_{n+2})\\
&-(-1)^{\gamma}T_n\Big(\kappa(b_1\otimes f_1)\otimes\cdots\otimes\kappa(b_n\otimes f_n\blacktriangleleft b_{n+1})\Big)\rhd b_{n+2}\\
&-(-1)^{\gamma}\Big(T_n\Big(\kappa(b_1\otimes f_1)\otimes\cdots\otimes\kappa(b_n\otimes f_n)\Big)\rhd b_{n+1}\Big)*b_{n+2}\\
=&(-1)^{\gamma}T_n\Big(\kappa(b_1\otimes f_1)\otimes\cdots\otimes\kappa(b_n\otimes f_n)\Big)\rhd(b_{n+1}*b_{n+2})\\
&-(-1)^{\gamma}T_n\Big(\kappa(b_1\otimes f_1)\otimes\cdots\otimes\kappa(b_n\otimes f_n)\blacktriangleleft b_{n+1}\Big)\rhd b_{n+2}\\
&-(-1)^{\gamma}\Big(T_n\Big(\kappa(b_1\otimes f_1)\otimes\cdots\otimes\kappa(b_n\otimes f_n)\Big)\rhd b_{n+1}\Big)*b_{n+2}.
\end{aligned}
$$

Thus, the Stasheff identity holding for the element $a_1\otimes s^{-1}f_1\otimes\cdots\otimes s^{-1}f_n\otimes b_{n+1}\otimes b_{n+2}$ is equivalent to that T_n is an n-derivation relative to B.

Case IV: For $b_0,\dots,b_{n+1}\in B$ and $f_1,\dots,f_n\in B^\vee$,

$$\begin{aligned}
&\sum_{\substack{i+j+k=2n+2,\\ i,k\geqslant 0, j\geqslant 1}}(-1)^{i+jk}m_{i+1+k}\circ\Big(\mathrm{Id}^{\otimes i}\otimes m_j\otimes \mathrm{Id}^{\otimes k}\Big)(b_0\otimes b_1\otimes s^{-1}f_1\otimes b_2\otimes s^{-1}f_2\dots b_n\otimes s^{-1}f_n\otimes b_{n+1})\\
=\quad&\Big(-m_{2n+1}\big(\mathrm{Id}\otimes m_2\otimes \mathrm{Id}^{\otimes 2n-1}\big)+m_{2n+1}\big(m_2\otimes \mathrm{Id}^{\otimes 2n}\big)-m_2\big(\mathrm{Id}\otimes m_{2n+1}\big)\Big)\\
&(b_0\otimes b_1\otimes s^{-1}f_1\otimes b_2\otimes s^{-1}f_2\otimes\cdots b_n\otimes s^{-1}f_n\otimes b_{n+1})\\
=\quad&-(-1)^{\gamma+n|b_0|}T_n\Big(\kappa(b_0\otimes b_1\lhd s^{-1}f_1)\otimes\kappa(b_2\otimes s^{-1}f_2)\otimes\cdots\otimes\kappa(b_n\otimes s^{-1}f_n)\Big)\rhd b_{n+1}\\
&+(-1)^{\gamma+n|b_0|}T_n\Big(\kappa(b_0* b_1\otimes s^{-1}f_1)\otimes\kappa(b_2\otimes s^{-1}f_2)\otimes\cdots\otimes\kappa(b_n\otimes s^{-1}f_n)\Big)\rhd b_{n+1}\\
&-(-1)^{\gamma+|b_0|}\Big(b_0\blacktriangleright T_n\Big(\kappa(b_1\otimes s^{-1}f_1)\otimes\kappa(b_2\otimes s^{-1}f_2)\otimes\cdots\otimes\kappa(b_n\otimes s^{-1}f_n)\Big)\Big)\rhd b_{n+1}
\end{aligned}$$

Thus, the Stasheff identity holding for the element

$$b_0\otimes b_1\otimes s^{-1}f_1\otimes b_2\otimes s^{-1}f_2\otimes\cdots b_n\otimes s^{-1}f_n\otimes b_{n+1}$$

is equivalent to that T_n satisfies Equation (1.2) .

Case V: For $1<l\leqslant n$, $b_1,\dots,b_{n+2}\in B$ and $f_1,\dots,f_n\in B^\vee$,

$$\begin{aligned}
&\sum_{\substack{i+j+k=2n+2,\\ i,k\geqslant 0, j\geqslant 1}}(-1)^{i+jk}m_{i+1+k}\circ\Big(\mathrm{Id}^{\otimes i}\otimes m_j\otimes \mathrm{Id}^{\otimes k}\Big)\Big(b_1\otimes s^{-1}f_1\otimes\cdots\otimes s^{-1}f_{l-1}\otimes b_l\otimes b_{l+1}\otimes s^{-1}f_l\otimes\cdots\\
&\qquad\qquad\cdots\otimes s^{-1}f_n\otimes b_{n+2}\Big)\\
=&\Big(-m_{2n+1}\big(\mathrm{Id}^{\otimes 2l-3}\otimes m_2\otimes \mathrm{Id}^{\otimes 2(n-l)+2}\big)+m_{2n+1}\big(\mathrm{Id}^{\otimes 2l-2}\otimes m_2\otimes \mathrm{Id}^{\otimes 2(n-l)+1}\big)\\
&-m_{2n+1}\big(\mathrm{Id}^{\otimes 2l-1}\otimes m_2\otimes \mathrm{Id}^{\otimes 2(n-l)}\big)\Big)\Big(b_1\otimes s^{-1}f_1\otimes\cdots\otimes b_l\otimes b_{l+1}\otimes s^{-1}f_l\otimes\cdots\otimes s^{-1}f_n\otimes b_{n+2}\Big)\\
=&-m_{2n+1}\Big(b_1\otimes\cdots\otimes m_2(s^{-1}f_{l-1}\otimes b_l)\otimes b_{l+1}\otimes\cdots\otimes s^{-1}f_n\otimes b_{n+2}\Big)\\
&+m_{2n+1}\Big(b_1\otimes\cdots\otimes s^{-1}f_{l-1}\otimes m_2(b_l\otimes b_{l+1})\otimes s^{-1}f_l\otimes\cdots\otimes s^{-1}f_n\otimes b_{n+2}\Big)\\
&-m_{2n+1}\Big(b_1\otimes\cdots\otimes b_l\otimes m_2(b_{l+1}\otimes s^{-1}f_l)\otimes b_{l+2}\cdots\otimes s^{-1}f_n\otimes b_{n+2}\Big)\\
=&-(-1)^{\gamma+\sum_{k=l+1}^{n+1}|b_k|}T_n\Big(\kappa(b_1\otimes f_1)\otimes\cdots\otimes\kappa(b_{l-1}\otimes f_{l-1})\blacktriangleleft b_l\otimes\kappa(b_{l+1}\otimes f_l)\otimes\cdots\otimes\kappa(b_{n+1}\otimes f_n)\Big)\rhd b_{n+2}\\
&+(-1)^{\gamma+\sum_{k=l+1}^{n+1}|b_k|}T_n\Big(\kappa(b_1\otimes f_1)\otimes\cdots\otimes\kappa(b_{l-1}\otimes f_{l-1})\otimes\kappa(b_l* b_{l+1}\otimes f_l)\otimes\cdots\otimes\kappa(b_{n+1}\otimes f_n)\Big)\rhd b_{n+2}\\
&-(-1)^{\gamma+\sum_{k=l+1}^{n+1}|b_k|}T_n\Big(\kappa(b_1\otimes f_1)\otimes\cdots\otimes\kappa(b_{l-1}\otimes f_{l-1})\otimes\kappa(b_l\otimes b_{l+1}\blacktriangleright f_l)\otimes\cdots\otimes\kappa(b_{n+1}\otimes f_n)\Big)\rhd b_{n+2}.
\end{aligned}$$

So, we can see that for each $1<l\leqslant n$ the Stasheff identity holding for the element

$$b_1\otimes s^{-1}f_1\otimes\cdots\otimes s^{-1}f_{l-1}\otimes b_l\otimes b_{l+1}\otimes s^{-1}f_l\otimes\cdots\otimes s^{-1}f_n\otimes b_{n+2}$$

is equivalent to that T_n satisfies Equation (1.3) for l.

In conclusion, $(\partial_{-1}B,\{m_n\}_{n\geqslant 1})$ is an A_∞ algebra. $\square$

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Index