# Pointed Hopf algebras over simple groups
# VIII. Pointed Hopf algebras in positive characteristic

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**Key points assuming** $\operatorname{char}\Bbbk = 0$**.**

**1.** If $H$ is a fin.-dim. cocommutative Hopf algebra, then $H \simeq \Bbbk G$.

**2.** If $G$ is a finite group, then ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ is semisimple.

**3.** If $(V, c^{\mathfrak{q}})$ is a braided vector space of diagonal type, then (the connected components of) $\mathfrak{q} \in$ list of [H].

**4.** If $G$ is a finite simple group, then $\Bbbk G$ collapses.

**5.** If $G$ is a finite group, $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ and $R$ is a finite-dimensional pre-Nichols algebra of $V$, then $R \simeq \mathscr{B}(V)$.

## 1. Cocommutative fin.-dim. Hopf algebras, $\operatorname{char} \Bbbk = p > 0$.

**Example.** $H = \Bbbk[T]$ is a Hopf algebra with $\Delta(T) = T \otimes 1 + 1 \otimes T$.

**Lemma.** $\mathcal{P}(H) = \oplus_{n \geq 0} \Bbbk T^{p^n}$.
*Proof.* If $A$ is a commutative $\Bbbk$-algebra and $a, b \in A$, then $(a+b)^{p^n} = a^{p^n} + b^{p^n}$ for all $n \geq 0$. Hence $\supseteq$; $\subseteq$ follows from the Newton's binomial formula and the fact $p | \binom{n}{k} \implies n = p^j$.

**Corollary.** $\mathcal{B}(n) = \Bbbk[T]/\langle T^{p^n} \rangle$ is a cocommutative Hopf algebra of dimension $p^n$ with coradical $\Bbbk$ and $\mathcal{P}(\mathcal{B}(n)) = \oplus_{0 \leq j \leq n-1} \Bbbk T^{p^j}$. Thus, $\mathcal{B}(n)$ is Nichols only if $n = 1$; $\mathcal{B}(1) \simeq \mathscr{B}(V, \tau)$ where $\dim V = 1$ and $\tau$ is the flip.

Thus **1, 3, 5.** do not hold.

**Example.** Let $\mathcal{R}$ be the graded dual of $\Bbbk[T]$, with basis $(e_j)_{j\geq 0}$, $\langle e_j, T^k \rangle = \delta_{j,k}$. Then $\langle e_j \cdot e_h, T^k \rangle = \langle e_j \otimes e_h, \Delta(T^k) \rangle$

$$= \langle e_j \otimes e_h, \sum_{0\leq i\leq k} \binom{k}{i} T^i \otimes T^{k-i} \rangle = \sum_{0\leq i\leq k} \binom{k}{i} \delta_{j,i}\delta_{h,k-i} = \delta_{j+h,k}\binom{k}{j}.$$

Hence $e_j \cdot e_h = \binom{j+h}{j} e_{j+h}$.

**Exercise.** $\Delta(e_j) = \sum_{0\leq i\leq j} e_i \otimes e_{j-i}$.
Show that the subalgebra of $\mathcal{R}$ generated by $e_1$ is a Nichols algebra.
Compute all Hopf subalgebras of $\mathcal{R}$.

**Exercise.** Let $N \in \mathbb{Z}_{>0}$; $H = \Bbbk[T_1, \dots, T_N]$ is a Hopf algebra with $\Delta(T_j) = T_j \otimes 1 + 1 \otimes T_j$. Extend the discussion above to this example.

**Example.** A restricted Lie algebra is a pair $(\mathfrak{g}, (\ )^{[p]})$, where $\mathfrak{g}$ is a Lie algebra and $(\ )^{[p]} : \mathfrak{g} \to \mathfrak{g}$ is a map, called the *$p$-operation*, that satisfies

$$(\lambda x)^{[p]} = \lambda^p x^{[p]}, \ \mathrm{ad}_{x^{[p]}} = (\mathrm{ad}_x)^p, \ (x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_i s_i(x,y),$$

for all $x, y \in \mathfrak{g}$, $\lambda \in \Bbbk$. The *restricted enveloping algebra* of $(\mathfrak{g}, (\ )^{[p]})$ is

$$\mathfrak{u}(\mathfrak{g}) := U(\mathfrak{g})/\langle x^p - x^{[p]} : x \in \mathfrak{g}\rangle,$$

where $U(\mathfrak{g})$ is the enveloping algebra of $\mathfrak{g}$. The Hopf algebra $\mathfrak{u}(\mathfrak{g})$ has coradical $\Bbbk$; if $\dim \mathfrak{g} < \infty$, then

$$\dim \mathfrak{u}(\mathfrak{g}) = (\dim \mathfrak{g})^p < \infty.$$

There are Lie algebras without $p$-structure but they can be embedded in restricted Lie algebras.

But there are more cocommutative finite-dimensional, even connected, Hopf algebras.

**2. Semisimplicity of ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$, $G$ a finite group.**
It is known that ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ is semisimple if and only if $p \not| |G|$. Indeed, ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ is semisimple if and only if $\Bbbk G^g$ is semisimple for all $g \in G$.

If ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ is not semisimple, then any $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}_{fd}$ is a sum of indecomposables, but these usually can not be classified (wild representation type).

Still, one may try to classify the simple objects in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$; since the socle of an indecomposable is simple, one may hope for some control. However the determination of the simple objects in $\operatorname{Rep} G$ is usually much harder than in characteristic 0.

**The restricted Jordan plane ($p$ odd).** Let $\mathcal{V}(1,2)$ be a braided vector space with basis $\{x, y\}$ and braiding given by

$$
\begin{aligned}
c(x \otimes x) &= x \otimes x, & c(y \otimes x) &= x \otimes y, \\
c(x \otimes y) &= (y + x) \otimes x, & c(y \otimes y) &= (y + x) \otimes y.
\end{aligned}
$$

Actually $V = \mathcal{V}(1,2) \in {}^{\Bbbk\mathbb{Z}/p}_{\Bbbk\mathbb{Z}/p}\mathcal{YD}$: if $g$ is generator of $\mathbb{Z}/p$, take $V = V_g$ and let $g$ act on $V$ by a Jordan block.

**Theorem.** (Cibils, Lauve & Witherspoon) The Nichols algebra $\mathscr{B}(V)$ is the quotient of $T(V)$ by the ideal generated by

$$
x^p, \qquad y^p, \qquad yx - xy + \frac{1}{2}x^2.
$$

That is, $\mathscr{B}(V) \simeq J/\langle x^p,\ y^p\rangle$, where $J = T(V)/\langle yx - xy + \frac{1}{2}x^2\rangle$; this is the *restricted* Jordan plane.

**The super Jordan plane (& its restricted version in odd char).** Assume that $\operatorname{char} \Bbbk \neq 2$. Let $\mathcal{V}(-1,2)$ be a braided vector space with basis $\{x, y\}$ and braiding

$$\begin{aligned} c(x \otimes x) &= -x \otimes x, & c(y \otimes x) &= -x \otimes y, \\ c(x \otimes y) &= (-y + x) \otimes x, & c(y \otimes y) &= (-y + x) \otimes y. \end{aligned}$$

Let $x_{21} = x_2x_1 + x_1x_2$. The super Jordan plane is the algebra

$$sJ = \Bbbk\langle x_1, x_2 | x_1^2, x_2x_{21} - x_{21}x_2 - x_1x_{21}\rangle$$

**Theorem.** (A.–Angiono–Heckenberger)

- If $\operatorname{char} \Bbbk = 0$, then $\mathscr{B}(V) \simeq sJ$. (and $V \in {}^{\Bbbk\mathbb{Z}}_{\Bbbk\mathbb{Z}}\mathcal{YD}$).

- If $\operatorname{char} \Bbbk = p > 2$, then $\mathscr{B}(V) \simeq sJ/\langle x_2^{2p}, x_{21}^p\rangle$, $V \in {}^{\Bbbk\mathbb{Z}/2p}_{\Bbbk\mathbb{Z}/2p}\mathcal{YD}$ and $\dim \mathscr{B}(V) = 4p^2$. This is the restricted super Jordan plane in characteristic $p$.

More generally, let $\mathcal{V}(\epsilon,\ell)$, where $\epsilon \in \Bbbk^\times$ and $\ell \in \mathbb{Z}_{\geq 2}$, is a braided vector space with basis $(x_i)_{i\in\mathbb{I}_\ell}$ such that for $i,j \in \mathbb{I}_\ell$, $1 < j$:

$$c(x_i \otimes x_1) = \epsilon x_1 \otimes x_i, \qquad c(x_i \otimes x_j) = (\epsilon x_j + x_{j-1}) \otimes x_i;$$

this is called a *block*. It can be realized in ${}^{\Bbbk\mathbb{Z}}_{\Bbbk\mathbb{Z}}\mathcal{YD}$ where for a generator $g$ of $\mathbb{Z}$, $\mathcal{V}(\epsilon,\ell) = \mathcal{V}(\epsilon,\ell)_g$, and $g$ acts by a Jordan block.

**Proposition, $p$ odd.** (A.–Angiono–Heckenberger)
$\dim \mathscr{B}(\mathcal{V}(\epsilon,2)) < \infty$ if and only if $\epsilon = \pm 1$.

**Corollary.** If $\ell > 2$ and $\epsilon \neq \pm 1$, then $\dim \mathscr{B}(\mathcal{V}(\epsilon,\ell)) = \infty$.

**Problem.** If $\ell > 2$ and $\epsilon = \pm 1$, then $\dim \mathscr{B}(\mathcal{V}(\epsilon,\ell)) =?$

**Problem.** When GK-dim $\mathscr{B}(\mathcal{V}(\epsilon,\ell)) < \infty$?

**The Nichols algebra $\mathscr{B}(\mathfrak{L}_q(1,\mathscr{G}))$..** Let $q \in \Bbbk^{\times}$, $a \in \mathbb{F}_p{}^{\times}$ and

$$\mathtt{r} \in \{1-p, 2-p, \dots, -2, -1\} \quad \text{such that} \quad \mathtt{r} \equiv 2a \mod p.$$

The *ghost* is $\mathscr{G} := -\mathtt{r} \in \{1, \dots, p-1\}$; since $p$ is odd, $\mathscr{G}$ gives $a$.

The braided vector space $\mathfrak{L}_q(1,\mathscr{G})$ has basis $\mathscr{b} = \{x_1, y_1, x_2\}$ and

$$(c(b \otimes b'))_{b,b' \in \mathscr{b}} = \begin{pmatrix} x_1 \otimes x_1 & (y_1 + x_1) \otimes x_1 & q\, x_2 \otimes x_1 \\ x_1 \otimes y_1 & (y_1 + x_1) \otimes y_1 & q\, x_2 \otimes y_1 \\ q^{-1} x_1 \otimes x_2 & q^{-1}(y_1 + a x_1) \otimes x_2 & x_2 \otimes x_2 \end{pmatrix}.$$

Thus $V_1 := \Bbbk x_1 + \Bbbk y_1 \simeq \mathcal{V}(1,2)$ and $V_2 := \Bbbk x_2$ satisfy

$$c : V_i \otimes V_j = V_j \otimes V_i, \qquad i, j \in \{1, 2\}.$$

Hence $V_1$ and $V_2$ are braided subspaces of $V$.

Set $z_0 := x_2$, $z_{n+1} := y_1 z_n - q z_n y_1$, $n > 0$.

**Lemma.** [A-Angiono-Heckenberger] The Nichols algebra $\mathscr{B}(\mathfrak{L}_q(1,\mathscr{G}))$ is generated by $x_1, y_1, x_2$ with relations

$$\begin{aligned} &y_1 x_1 - x_1 y_1 + \tfrac{1}{2} x_1^2, && x_1^p,\ y_1^p. \\ x_1 x_2 &= q\, x_2 x_1, && \\ z_{1+\mathscr{G}} &= 0, && \\ z_t z_{t+1} &= q^{-1}\, z_{t+1} z_t, && 0 \le t < \mathscr{G}, \\ z_t^p &= 0, && 0 \le t \le \mathscr{G}. \end{aligned}$$

It has $\dim \mathscr{B}(\mathfrak{L}_q(1,\mathscr{G})) = p^{\mathscr{G}+3}$; indeed a PBW-basis is

$$B = \{x_1^{m_1} y_1^{m_2} z_{\mathscr{G}}^{n_{\mathscr{G}}} \dots z_1^{n_1} z_0^{n_0} : 0 \le m_i, n_j < p\}.$$

## 3. Finite dimensional Nichols algebras of diagonal type.

Rank 2 [Heckenberger-Wang]: classified (5 tables for $p = 2, 3, 5, 7$, $> 7$).

Rank 3 [Wang]: classified (3 tables for $p = 2, 3$, $> 3$).

Rank 4 [Wang]: classified.

Rank 5, 6, 7 [Yuan-Qian-Wang]: classified.

Rank $> 7$: open?

| | Dynkin diagrams, $p = 2$ | fixed parameters |
|---|---|---|
| 1 | $\overset{q}{\circ} \qquad \overset{r}{\circ}$ | $q, r \in \Bbbk^*$ |
| 2 | $\overset{q}{\circ}\overset{q^{-1}}{\text{———}}\overset{q}{\circ}$ | $q \in \Bbbk^* \setminus \{1\}$ |
| 3 | $\overset{q}{\circ}\overset{q^{-1}}{\text{———}}\overset{1}{\circ} \quad \overset{1}{\circ}\overset{q}{\text{———}}\overset{1}{\circ}$ | $q \in \Bbbk^* \setminus \{1\}$ |
| 4 | $\overset{q}{\circ}\overset{q^{-2}}{\text{———}}\overset{q^2}{\circ}$ | $q \in \Bbbk^* \setminus \{1\}$ |
| 5 | $\overset{q}{\circ}\overset{q^{-2}}{\text{———}}\overset{1}{\circ} \quad \overset{q^{-1}}{\circ}\overset{q^2}{\text{———}}\overset{1}{\circ}$ | $q \in \Bbbk^* \setminus \{1\}$ |
| 6 | $\overset{\zeta}{\circ}\overset{q^{-1}}{\text{———}}\overset{q}{\circ} \quad \overset{\zeta}{\circ}\overset{\zeta^{-1}q}{\text{———}}\overset{\zeta q^{-1}}{\circ}$ | $\zeta \in G'_3,\ q \in \Bbbk^* \setminus \{1, \zeta, \zeta^2\}$ |
| 7 | $\overset{\zeta}{\circ}\overset{\zeta}{\text{———}}\overset{1}{\circ} \quad \overset{\zeta^{-1}}{\circ}\overset{\zeta^{-1}}{\text{———}}\overset{1}{\circ}$ | $\zeta \in G'_3$ |
| 10 | $\overset{\zeta^2}{\circ}\overset{\zeta}{\text{———}}\overset{1}{\circ} \quad \overset{\zeta^3}{\circ}\overset{\zeta^{-1}}{\text{———}}\overset{1}{\circ} \quad \overset{\zeta^3}{\circ}\overset{\zeta^{-2}}{\text{———}}\overset{\zeta}{\circ}$ | $\zeta \in G'_9$ |
| 11 | $\overset{q}{\circ}\overset{q^{-3}}{\text{———}}\overset{q^3}{\circ}$ | $q \in \Bbbk^* \setminus \{1\},\ q \notin G'_3$ |
| 14 | $\overset{\zeta}{\circ}\overset{\zeta^2}{\text{———}}\overset{1}{\circ} \quad \overset{\zeta^{-2}}{\circ}\overset{\zeta^{-2}}{\text{———}}\overset{1}{\circ}$ | $\zeta \in G'_5$ |
| 16 | $\overset{\zeta^5}{\circ}\overset{\zeta^{-3}}{\text{———}}\overset{\zeta}{\circ} \quad \overset{\zeta^5}{\circ}\overset{\zeta^{-2}}{\text{———}}\overset{1}{\circ} \quad \overset{\zeta^3}{\circ}\overset{\zeta^2}{\text{———}}\overset{1}{\circ} \quad \overset{\zeta^3}{\circ}\overset{\zeta^4}{\text{———}}\overset{\zeta^{-4}}{\circ}$ | $\zeta \in G'_{15}$ |
| 17 | $\overset{\zeta}{\circ}\overset{\zeta^{-3}}{\text{———}}\overset{1}{\circ} \quad \overset{\zeta^{-2}}{\circ}\overset{\zeta^3}{\text{———}}\overset{1}{\circ}$ | $\zeta \in G'_7$ |

| | Dynkin diagrams, $p = 3$. | fixed parameters |
|---|---|---|
| 1 | $\overset{q}{\circ}\ \overset{r}{\circ}$ | $q, r \in \Bbbk^*$ |
| 2 | $\overset{q}{\circ}\overset{q^{-1}}{\text{——}}\overset{q}{\circ}$ | $q \in \Bbbk^* \setminus \{1\}$ |
| 3 | $\overset{q}{\circ}\overset{q^{-1}}{\text{——}}\overset{-1}{\circ}\quad \overset{-1}{\circ}\overset{q}{\text{——}}\overset{-1}{\circ}$ | $q \in \Bbbk^* \setminus \{-1, 1\}$ |
| 4 | $\overset{q}{\circ}\overset{q^{-2}}{\text{——}}\overset{q^2}{\circ}$ | $q \in \Bbbk^* \setminus \{-1, 1\}$ |
| 5 | $\overset{q}{\circ}\overset{q^{-2}}{\text{——}}\overset{-1}{\circ}\quad \overset{-q^{-1}}{\circ}\overset{q^2}{\text{——}}\overset{-1}{\circ}$ | $q \in \Bbbk^* \setminus \{-1, 1\}$, $q \notin G'_4$ |
| $6'$ | $\overset{1}{\circ}\overset{q}{\text{——}}\overset{q^{-1}}{\circ}\quad \overset{1}{\circ}\overset{q^{-1}}{\text{——}}\overset{q}{\circ}$ | $q \in \Bbbk^* \setminus \{1, -1\}$ |
| $6'''$ | $\overset{1}{\circ}\overset{-1}{\text{——}}\overset{-1}{\circ}$ | |
| $9'$ | $\overset{\zeta}{\circ}\overset{\zeta}{\text{——}}\overset{-1}{\circ}\quad \overset{1}{\circ}\overset{-\zeta}{\text{——}}\overset{-1}{\circ}\quad \overset{1}{\circ}\overset{\zeta}{\text{——}}\overset{1}{\circ}$ | $\zeta \in G'_4$ |
| 11 | $\overset{q}{\circ}\overset{q^{-3}}{\text{——}}\overset{q^3}{\circ}$ | $q \in \Bbbk^* \setminus \{-1, 1\}$ |
| 12 | $\overset{\zeta}{\circ}\overset{-\zeta}{\text{——}}\overset{-1}{\circ}\quad \overset{\zeta^2}{\circ}\overset{-\zeta^{-1}}{\text{——}}\overset{-1}{\circ}\quad \overset{\zeta^2}{\circ}\overset{\zeta}{\text{——}}\overset{\zeta^{-1}}{\circ}$ | $\zeta \in G'_8$ |
| $13'$ | $\overset{-\zeta}{\circ}\overset{\zeta^{-1}}{\text{——}}\overset{-1}{\circ}\quad \overset{1}{\circ}\overset{\zeta}{\text{——}}\overset{-1}{\circ}\quad \overset{1}{\circ}\overset{\zeta^{-1}}{\text{——}}\overset{-\zeta^2}{\circ}\quad \overset{-\zeta^{-1}}{\circ}\overset{-\zeta}{\text{——}}\overset{-\zeta^2}{\circ}$ | $\zeta \in G'_8$ |
| 14 | $\overset{\zeta}{\circ}\overset{\zeta^2}{\text{——}}\overset{-1}{\circ}\quad \overset{-\zeta^{-2}}{\circ}\overset{\zeta^{-2}}{\text{——}}\overset{-1}{\circ}$ | $\zeta \in G'_5$ |
| 15 | $\overset{\zeta}{\circ}\overset{\zeta^{-3}}{\text{——}}\overset{-1}{\circ}\quad \overset{-\zeta^{-2}}{\circ}\overset{\zeta^3}{\text{——}}\overset{-1}{\circ}\quad \overset{-\zeta^{-2}}{\circ}\overset{-\zeta^3}{\text{——}}\overset{-1}{\circ}\quad \overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\text{——}}\overset{-1}{\circ}$ | $\zeta \in G'_{20}$ |
| $16'$ | $\overset{1}{\circ}\overset{-\zeta^{-1}}{\text{——}}\overset{-\zeta^2}{\circ}\quad \overset{1}{\circ}\overset{-\zeta}{\text{——}}\overset{-1}{\circ}\quad \overset{\zeta}{\circ}\overset{-\zeta^{-1}}{\text{——}}\overset{-1}{\circ}\quad \overset{\zeta}{\circ}\overset{-\zeta^3}{\text{——}}\overset{-\zeta^{-3}}{\circ}$ | $\zeta \in G'_5$ |
| 17 | $\overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\text{——}}\overset{-1}{\circ}\quad \overset{-\zeta^{-2}}{\circ}\overset{-\zeta^3}{\text{——}}\overset{-1}{\circ}$ | $\zeta \in G'_7$ |

**4. No finite group collapses in** $\operatorname{char} p > 0$**.**

Let $G$ be a finite group and $\rho \in \operatorname{Irr} G$ of dimension $d$.
Then $M = M(e, \rho) \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ is simple and has dimension $d$.

Notice that the braiding is the usual flip $\tau$.
Hence $\mathscr{B}(M(e, \rho)) \simeq S(M)/\langle M^p \rangle$ has dimension $p^d$ and

$$H := \mathscr{B}(M(e, \rho))\#\Bbbk G \simeq \mathscr{B}(M(e, \rho)) \rtimes \Bbbk G$$

is a pointed Hopf algebra with $G(H) \simeq G$ and $\dim H = p^d|G|$.

Similarly, if $g \in G$, $\mathcal{O} = \mathcal{O}_g^G$ *is a trivial rack* and $\varepsilon =$ trivial representation of $G^g$, then $M = M(\mathcal{O}, \varepsilon) \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ is simple and has dimension $|\mathcal{O}|$.

Again, the braiding is the usual flip $\tau$.
Hence $\mathscr{B}(M(\mathcal{O}, \varepsilon)) \simeq S(M)/\langle M^p \rangle$ has dimension $p^{|\mathcal{O}|}$ and

$$H := \mathscr{B}(M(\mathcal{O}, \varepsilon)) \# \Bbbk G$$

is a pointed Hopf algebra with $G(H) \simeq G$ and $\dim H = p^{|\mathcal{O}|}|G|$.

Finally, there are partial results on pointed Hopf algebras over abelian groups that have finite GK-dim .

One faces obstacles similar to those in positive characteristic. Actually, the results in the former setting inspired results in the latter.