Pointed Hopf algebras over simple groups

VI. Collapsing. Nichols algebras over alternating groups

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i. Nichols algebras of non-simple Yetter-Drinfeld modules (rank 2).

i.o Recall of the simple case.

Examples over solvable groups:

rack	cocycle	dim.	rack	cocycle	dim.
$\mathrm{Aff}(\mathbb{F}_3, 2)$	-1	12	$\mathscr{T}$	-1	72
$\mathrm{Aff}(\mathbb{F}_5, 2)$	-1	$1280 = 5.4^4$	$\mathscr{T}$	χ	$5{,}184$
$\mathrm{Aff}(\mathbb{F}_5, 3)$	-1	$1280 = 5.4^4$	$\mathcal{O}_2^4$	-1	576
$\mathrm{Aff}(\mathbb{F}_7, 3)$	-1	$326,592 = 7.6^6$	$\mathcal{O}_2^4$	χ_4	576
$\mathrm{Aff}(\mathbb{F}_7, 5)$	-1	$326,592 = 7.6^6$	$\mathcal{O}_4^4$	-1	576

Examples over non-solvable groups:
Rack: $\mathcal{O}_2^5$, cocycles -1 or χ_5, dimension $8,294,400$.

i.i Classification.

We need the following observation. Given two racks Y and Z, a rack operation on $X := Y \dot\cup Z$ such that (Y, Z) is a decomposition, is equivalent to a pair (ς, ϖ) of morphisms of racks $\varsigma : Y \to \operatorname{Aut} Z$, $\varpi : Z \to \operatorname{Aut} Y$ such that

$$
\begin{aligned}
y \triangleright \varpi_z(u) &= \varpi_{\varsigma_y(z)}(y \triangleright u), & \forall y, u \in Y,\ z \in Z,\\
z \triangleright \varsigma_y(w) &= \varsigma_{\varpi_z(y)}(z \triangleright w), & \forall y \in Y,\ z, w \in Z,
\end{aligned}
$$

In this case we denote $X := Y\, {}_{\varsigma}\!\coprod_{\varpi} Z$; ς is omitted if $\varsigma_y = \mathrm{id}_Z$ for all $y \in Y$, ϖ is omitted if $\varpi_z = \mathrm{id}_Y$ for all $z \in Z$.

Theorem A. [Heckenberger-Vendramin]

Let G be a finite non-abelian group and $V = V_1 \oplus V_2 \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$, where V_1 and V_2 are simple, the support of V generates G and $c^2_{|V_1 \otimes V_2} \neq \mathrm{id}$. Assume that $\dim \mathscr{B}(V) < \infty$. Then as a braided vector space, V is isomorphic to one of the Examples below.

Proof. Use the Weyl groupoid $\cdots$

In particular, assuming $\dim V_1 \leq \dim V_2$, the pair $(\dim V_1, \dim V_2)$ belongs to

$$\{(1,3),(1,4),(2,2),(2,3),(2,4)\}. \tag{1}$$

Decomposable racks and dimensions of the Nichols algebras.

Name	rack	dimension
D4	$\{1,2\}_{(34)}\coprod_{(12)}\{3,4\}$	64
D3-1a	$\mathcal{D}_3\coprod\{4\}$	$10,368 = 3^4 2^7$
D3-1b	$\mathcal{D}_3\coprod\{4\}$	$2,304 = 3^2 2^8$
D3-2	$\mathcal{D}_{3\ (45)}\coprod_{(132),(123)}\mathbb{I}_{4,5}$	$10,368,\ 2,304$
D4-2	$X = \mathcal{D}_{4\ (56)}\coprod_{\sigma_1,\sigma_2}\mathbb{I}_{5,6}$	$262,144 = 2^1 8$
T-1	$\mathcal{T}\coprod\{5\}$	$80,621,568 = 2^{12}\cdot 3^9$

i.ii Examples (rank 2).

Example D4. Let $X = \mathcal{D}_4 = \mathbb{I}_2 \,{}_\sigma\coprod_\sigma \mathbb{I}_2$, $\sigma \neq \mathrm{id}$. Concretely, $X = \{1,2\}_{(34)}\coprod_{(12)}\{3,4\}$. Then $\Bbbk X = V_1 \oplus V_2$, where V_1 is spanned by $(x_i)_{i\in\mathbb{I}_2}$, while V_2 is spanned by $(x_j)_{j\in\mathbb{I}_{3,4}}$. Let $p,q,r,t \in \Bbbk^\times$, $p \neq 1 \neq q$, and $\epsilon, \epsilon' \in \mathbb{G}_2$. Define a braiding on $\Bbbk X$ by

$$\begin{aligned}
&c_{|V_1\otimes V_1} \text{ is of diagonal type with matrix } \begin{pmatrix} q & \varepsilon q \\ \varepsilon q & q \end{pmatrix}, \\
&c_{|V_2\otimes V_2} \text{ is of diagonal type with matrix } \begin{pmatrix} p & \varepsilon' p \\ \varepsilon' p & p \end{pmatrix}, \\
&\left(c(x_i\otimes x_j)_{i\in\mathbb{I}_2, j\in\mathbb{I}_{3,4}}\right) = \begin{pmatrix} x_4\otimes x_1 & t^2x_3\otimes x_1 \\ \varepsilon' x_4\otimes x_2 & \varepsilon' t^2 x_3\otimes x_2 \end{pmatrix}, \\
&\left(c(x_j\otimes x_i)_{j\in\mathbb{I}_{3,4}, i\in\mathbb{I}_2}\right) = \begin{pmatrix} x_2\otimes x_3 & r^2x_1\otimes x_3 \\ \varepsilon x_2\otimes x_4 & \varepsilon r^2 x_1\otimes x_4 \end{pmatrix}.
\end{aligned} \tag{2}$$

Theorem. [Graña]

Let $(V, c) = (\Bbbk\mathcal{D}, c)$ where c is given by (2). Then $\dim \mathscr{B}(V) = 64$.

Proof. Up to a change of basis this is of diagonal type.

Example D3-1a.

Let $X = \mathcal{D}_3 \coprod \{4\}$. Then $\Bbbk X = V_1 \oplus V_2$, where V_1 is spanned by $(x_i)_{i\in\mathbb{I}_3}$, while V_2 is spanned by x_4.

Let $\epsilon \equiv -1$, cocycle on $\mathcal{D}_3$. Given $\omega \in \Bbbk^\times, \zeta \in \mathbb{G}_3, q_1, q_2 \in \Bbbk^\times$. define a braiding on $\Bbbk X$ by

$$\begin{aligned} c_{|V_1\otimes V_1} &= c^\epsilon, & c(x_4 \otimes x_4) &= -\omega x_4 \otimes x_4, \\ c(x_i \otimes x_4) &= q_1 \zeta^{i-1} x_4 \otimes x_i, & c(x_4 \otimes x_i) &= q_2 x_i \otimes x_4, \quad i \in \mathbb{I}_3. \end{aligned} \tag{3}$$

Thus $\Bbbk X = V_1 \oplus V_2$ is a decomposition of braided vector spaces where V_1 is $(\Bbbk\mathcal{O}_2^3, c^\epsilon)$, V_2 is a point with label $-\omega \in \mathbb{G}'_6$ and the braiding between them is prescribed in the second line of (3).

Theorem. If $\omega \in \mathbb{G}'_3$, $q_1 q_2 = -\omega^2$ and $(V, c) = (\Bbbk(\mathcal{D}_3 \coprod \{4\}), c)$ where c is given by (3), then

$$\dim \mathscr{B}(V) = 10,368 = 3^4 2^7.$$

Example D3-1b.
Let $X = \mathcal{D}_3 \coprod \{4\}$ as in the previous Example.
Let $V = V_1 \oplus V_2$, where $V_1 = \Bbbk\mathcal{D}_3$ is spanned by $(x_i)_{i\in\mathbb{I}_3}$, but now V_2 is $\Bbbk x_4 \otimes \Bbbk^2$; let $y_4 = x_4 \otimes (1,0)$, $y_5 = x_4 \otimes (0,1)$.

Let $\zeta \in \mathbb{G}_3$, $q_1, q_2 \in \Bbbk^\times$. Define a braiding on $\Bbbk X$ by

$$\begin{aligned} c_{|V_1\otimes V_1} &= c^\epsilon, & c_{|V_2\otimes V_2} &= -\tau, & \\ c(x_i\otimes y_4) &= \zeta^{i-1} y_5\otimes x_i, & c(x_i\otimes y_5) &= q_1\zeta^{2(i-1)} y_4\otimes x_i, & \\ c(y_4\otimes x_i) &= q_2 y_4\otimes x_i, & c(y_5\otimes x_i) &= q_2 x_i\otimes y_5, & i \in \mathbb{I}_3. \end{aligned} \tag{4}$$

Thus $V = V_1 \oplus V_2$ is a decomposition of braided vector spaces where V_1 is $(\Bbbk\mathcal{O}_2^3, c^\epsilon)$, $V_2 = (\Bbbk y_4 \oplus \Bbbk y_5, -\tau)$.

Theorem. Let (V, c) be the braided vector space with c given by (4). Assume that $q_1 q_2^2 = 1$. Then

$$\dim \mathscr{B}(V) = 2{,}304 = 3^2 2^8.$$

Example D3-2. Let $X = \mathcal{D}_3 \coprod_{(45)}^{(132),(123)} \mathbb{I}_{4,5}$; let $\sigma = (132)$. Let $V = \Bbbk X = V_1 \oplus V_2$, where $V_1 = \Bbbk\mathcal{D}_3$ is spanned by $(x_i)_{i\in\mathbb{I}_3}$ and V_2 is spanned by x_4, x_5.

Let $\zeta \in \mathbb{G}_3$, $a_1, q_1, q_2 \in \Bbbk^\times$. Define a braiding on V by

$$\begin{aligned} c_{|V_1\otimes V_1} &= c^\epsilon, & c(x_i\otimes x_j) &= a_1\zeta^{2-\delta_{ij}}x_j\otimes x_i, \quad i,j \in \mathbb{I}_{4,5}; \\ c(x_i\otimes x_4) &= \zeta^{i-1}x_5\otimes x_i, & c(x_i\otimes x_5) &= q_1\zeta^{2(i-1)}x_4\otimes x_i, \\ c(x_4\otimes x_i) &= q_2 x_{\sigma(i)}\otimes x_4, & c(x_5\otimes x_i) &= q_2 x_{\sigma^{-1}(i)}\otimes x_5, \qquad i \in \mathbb{I}_3. \end{aligned}$$

Theorem. Let (V, c) be the braided vector space as above.

- If $\zeta \in \mathbb{G}_3'$, $a_1 = -\zeta^2$ and $q_1q_2^2 = \zeta^2$, then $\dim \mathscr{B}(V,c) = 10,368$.

- If $\zeta = 1$, $a_1 = -1$ and $q_1q_2^2 = 1$, then $\dim \mathscr{B}(V,c) = 2,304$.

Example D4-2. Let $X = \mathcal{D}_4 \,{}_{(56)}\coprod_{\sigma_1,\sigma_2} \mathbb{I}_{5,6}$.

Here we number $\mathcal{D}_4$ as follows: $\mathcal{D}_4 = \{1,3\}\,{}_{\sigma}\coprod_{\sigma}\{2,4\}$, where $\sigma \neq \mathrm{id}$; i.e.,, we change the numeration in Example D4 by $2 \leftrightarrow 3$. Also, $\sigma_1 = (1234)$, $\sigma_2 = (1432)$.

Let $V = \Bbbk X = V_1 \oplus V_2$, with $V_1 = \Bbbk\mathcal{D}_4$ spanned by $(x_i)_{i\in\mathbb{I}_4}$ and V_2 by x_5, x_6. Let $q_1, q_2 \in \Bbbk^\times$, $\zeta_1, \zeta_2 \in \mathbb{G}_4$. Define $\mathfrak{q} \in Z^2(V, \Bbbk^\times)$ by

$$(\mathfrak{q}_{ij})_{i,j\in\mathbb{I}_4} = \begin{pmatrix} -1 & -\zeta_1^2 & -\zeta_1^2 & -\zeta_1^2 \\ -1 & -1 & -1 & -\zeta_1^2 \\ -\zeta_1^2 & -1 & -1 & -1 \\ -\zeta_1^2 & -\zeta_1^2 & -\zeta_1^2 & -1 \end{pmatrix},$$

$$(\mathfrak{q}_{ij})_{i,j\in\mathbb{I}_{5,6}} = \begin{pmatrix} -1 & -\zeta_2^3 \\ -\zeta_2^3 & -1 \end{pmatrix},$$

$$(\mathfrak{q}_{ij})_{i\in\mathbb{I}_{5,6},j\in\mathbb{I}_4} = \begin{pmatrix} 1 & q_2\zeta_1^3 & 1 & q_2\zeta_1 \\ \zeta_1^2 & q_2\zeta_1^3 & 1 & q_2\zeta_1^3 \end{pmatrix},$$

$$\mathfrak{q}_{ij} = \begin{cases} \zeta_2^{1-i}, & j=5, \\ q_1\zeta_2^{i-1} & j=6, \end{cases} \qquad i \in \mathbb{I}_4.$$

Theorem.
Let $(V, c^{\mathfrak{q}})$ be the braided vector space with $\mathfrak{q}$ given above. Assume that $\zeta_1\zeta_2 = q_1q_2$ and $\zeta_2 \in \mathbb{G}'_4$. Then

$$\dim \mathscr{B}(X, \mathfrak{q}) = 262,144 = 2^{18}.$$

Example T-1. Let $X = \mathcal{T} \coprod \{5\}$.
Let $V = \Bbbk X = V_1 \oplus V_2$, where $V_1 = \Bbbk\mathcal{T}$ is spanned by $(x_i)_{i \in \mathbb{I}_4}$, and V_2 is $\Bbbk x_5$. Let $a, q_1, q_2 \in \Bbbk^\times$. Define a braiding on V by

$$\begin{aligned} c_{|V_1 \otimes V_1} &= c^\epsilon, & c_{|V_2 \otimes V_2} &= a \,\mathrm{id}, \\ c(x_i \otimes x_5) &= q_1 x_5 \otimes x_i, & c(x_5 \otimes x_i) &= q_2 x_i \otimes x_5, \ i \in \mathbb{I}_4; \end{aligned} \tag{5}$$

$V = V_1 \oplus V_2$ is a decomposition of braided vector spaces where V_1 is $(\Bbbk\mathcal{T}, c^\epsilon)$ (ϵ is the cocycle -1), $V_2 = \Bbbk x_5$ has dimension 1.

Theorem.
Let (V, c) be the braided vector space with c given by (5). Assume that $-q_1 q_2 \in \mathbb{G}'_3$ and $a q_1 q_2 = 1$. Then

$$\dim \mathscr{B}(X, \mathfrak{q}) = 80,621,568.$$

ii. Collapsing.

ii.i Collapsing criteria. Notation: Let G be a finite group and $\mathcal{O}$ a conjugacy class of G.

• G *collapses* if any finite-dimensional pointed Hopf algebra H with $G(H) \simeq G$ is isomorphic to the group algebra $\Bbbk G$.

• $\mathcal{O}$ *falls* if $\dim \mathscr{B}(V) = \infty$ $\forall V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ with $\operatorname{supp} V = \mathcal{O}$.

Remark: G collapses if and only if every conjugacy class $\mathcal{O}$ of G falls.

• $\mathcal{O}$ *collapses* when $\dim \mathscr{B}(X, \sigma) = \infty$ for any 2-cocycle σ of any degree.

If the conjugacy class $\mathcal{O}$ collapses, then it falls; the converse is not true.

Criteria: C, D, F. We say that $\mathcal{O}$ is of type C, D, F when the corresponding property below holds:

(C) There are $H \le G$ and $r, s \in H \cap \mathcal{O}$ such that

$$\begin{array}{ccc} H = \langle \mathcal{O}_r^H, \mathcal{O}_s^H \rangle, & \mathcal{O}_r^H \neq \mathcal{O}_s^H, & rs \neq sr, \\ \min\{|\mathcal{O}_r^H|, |\mathcal{O}_s^H|\} > 2, & \text{or} & \max\{|\mathcal{O}_r^H|, |\mathcal{O}_s^H|\} > 4. \end{array}$$

(D) There are r, $s \in \mathcal{O}$ such that $\mathcal{O}_r^{\langle r,s \rangle} \neq \mathcal{O}_s^{\langle r,s \rangle} \& (rs)^2 \neq (sr)^2$.

(F) There are $r_a \in \mathcal{O}$, $a \in \mathbb{I}_4$, such that $\mathcal{O}_{r_a}^{\langle r_a : a \in \mathbb{I}_4 \rangle} \neq \mathcal{O}_{r_b}^{\langle r_a : a \in \mathbb{I}_4 \rangle}$ and $r_a r_b \neq r_b r_a$ for $a \neq b \in \mathbb{I}_4$.

Theorem. [A-Fantino-Graña-Vendramin, A-Carnovale-García]
Let $\mathscr{O}$ be a conjugacy class of a finite group G. If $\mathscr{O}$ is either of type C, D or F, then $\mathscr{O}$ collapses.

Sketch of the proof (type C). Let us say that a rack X is *of type C* when there are a decomposable subrack $Y = R \coprod S$ and elements $r \in R$, $s \in S$ such that

$$r \triangleright s \neq s, \qquad R = \mathscr{O}_r^{\mathsf{Inn}\, Y}, \qquad S = \mathscr{O}_s^{\mathsf{Inn}\, Y}, \tag{6}$$

$$\min\{|R|, |S|\} > 2 \qquad \text{or} \qquad \max\{|R|, |S|\} > 4. \tag{7}$$

If the conjugacy class $\mathscr{O}$ is of type C, then so is the underlying rack. Let G be a finite group and $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ such that X is isomorphic to a subrack of $\operatorname{supp} M$. We will check that $\mathscr{B}(M)$ has infinite dimension.

Let $Y = R \coprod S$ be as in the definition above. Let $K = \langle Y \rangle \leq G$. Then $M_Y := \oplus_{y \in Y} M_y \in {}^K_K\mathcal{YD}$, with

$$M_R := \oplus_{x \in R} M_x \qquad \text{and} \qquad M_S := \oplus_{z \in S} M_z$$

being Yetter-Drinfeld submodules of M_Y; then $R = \mathcal{O}_r^K$, $S = \mathcal{O}_s^K$. Let V, respectively W, be a simple Yetter-Drinfeld submodule of M_R, respectively M_S. Then

$\operatorname{supp} V = R$ (since $\operatorname{supp} V$ is stable under the conjugation of K), $\operatorname{supp} W = S$ and $\operatorname{supp}(V \oplus W) = Y$, that generates K.

Now $(\mathrm{id} - c_{W,V} c_{V,W})(V \otimes W) \neq 0$ because $rs \neq sr$.

We may assume that $\dim V \leq \dim W$. Now $\dim V \geq |R| > 2$ or $\dim W \geq |S| > 4$. Hence $(\dim V, \dim W)$ does not belong to the set (1). Thus $\dim \mathscr{B}(V \oplus W) = \infty$ by Theorem A and a fortiori $\dim \mathscr{B}(M) = \infty$.

Lemma. If a rack Z contains a subrack of type C, respectively projects onto a rack of type C, then Z is of type C.

$$\begin{tikzcd} Y \arrow[r, hook, "\iota"] & Z \arrow[d, "\pi"] \\ & X \end{tikzcd} \qquad X \text{ or } Y \text{ of type C} \Longrightarrow Z \text{ of type C.}$$

Proof. By definition, if Y is of type C, then so is Z.
Let $Y = R \coprod S \subset X$ be as in the definition of type C with $|R| \le |S|$, $|R| > 2$ or $|S| > 4$. Fix $\widetilde{r}, \widetilde{s} \in Z$ such that $\pi(\widetilde{r}) = r$, $\pi(\widetilde{s}) = s$. Define recursively

$$R_1 = \pi^{-1}(R),\ S_1 = \pi^{-1}(S),\ Y_1 = \pi^{-1}(Y),\ \ K_1 = \langle \varphi_y, y \in Y_1 \rangle \le \operatorname{Inn} Z,$$
$$R_2 = \mathcal{O}_{\widetilde{r}}^{K_1}, \qquad S_2 = \mathcal{O}_{\widetilde{s}}^{K_1}, \qquad Y_2 = R_2 \coprod S_2,\ K_2 = \langle \varphi_y, y \in Y_2 \rangle \le \operatorname{Inn} Z;$$
$$R_j = \mathcal{O}_{\widetilde{r}}^{K_{j-1}}, \quad S_j = \mathcal{O}_{\widetilde{s}}^{K_{j-1}}, \quad Y_j = R_j \coprod S_j, \ \ K_j = \langle \varphi_y, y \in Y_j \rangle \le \operatorname{Inn} Z.$$

Clearly, $R_1 \supseteq R_2 \supseteq \dots$ and $S_1 \supseteq S_2 \supseteq \dots$, hence $Y_i = R_i \coprod S_i$ is a rack decomposition. Now the sequence

$$Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_i \supseteq Y_{i+1} \supseteq \dots$$

stabilizes because Z is finite. Let $i \in \mathbb{N}$ such that $Y_i = Y_{i-1}$; then

$$\widetilde{R} := R_i = R_{i-1} = \mathcal{O}_{\widetilde{r}}^{K_{i-1}} \text{ and } \widetilde{S} := S_i = S_{i-1} = \mathcal{O}_{\widetilde{s}}^{K_{i-1}}.$$

Thus $\widetilde{Y} := \widetilde{R} \coprod \widetilde{S}$ is a subrack of Z that satisfies (6). We claim now that $\pi(Y_j) = Y$ for all $j \in \mathbb{N}$; hence $|R_j| \geq |R| > 2$ or $|S_j| \geq |S| > 4$, proving (7) for $\widetilde{Y}$.

Indeed, $\pi(R_1) = R$ because π is surjective.
Assume that $\pi(Y_j) = Y$; hence $\pi(R_j) = R$ and $\pi(S_j) = S$. Let $t \in R$. There exist $y_1, \dots, y_h \in Y$ such that $y_1 \triangleright (y_2 \triangleright \cdots \triangleright (y_h \triangleright r) \dots) = t$ by (7) for Y.
Pick $\widetilde{y}_1, \dots, \widetilde{y}_h \in Y_j$ such that $\pi(\widetilde{y}_\ell) = y_\ell$, $\ell \in \mathbb{I}_h$. Then

$$\begin{aligned} \widetilde{y}_1 \triangleright (\widetilde{y}_2 \triangleright \cdots \triangleright (\widetilde{y}_h \triangleright \widetilde{r}) \dots) &\in \mathcal{O}_{\widetilde{r}}^{K_j} = R_{j+1}, \qquad \text{hence} \\ \pi(\widetilde{y}_1 \triangleright (\widetilde{y}_2 \triangleright \cdots \triangleright (\widetilde{y}_h \triangleright \widetilde{r}) \dots) &= y_1 \triangleright (y_2 \triangleright \cdots \triangleright (y_h \triangleright r) \dots) \\ &= t \in \pi(R_{j+1}). \end{aligned}$$

ii.ii Collapsing alternating groups (and their conjugacy classes).

Let $m \in \mathbb{Z}_{\geq 2}$.
Recall that the type of $\sigma \in \mathbb{S}_m$ being $(1^{n_1}, 2^{n_2}, \dots, m^{n_m})$ means that the action of σ on $\mathbb{I}_m$ has n_1 fixed points, n_2 orbits with 2 elements and in general n_j orbits with j elements. Then $\sigma \in \mathbb{A}_m$ if and only if $\sum_{j \text{ even}} m_j$ is even.

Also, the conjugacy class $\mathcal{O}_\sigma^{\mathbb{S}_m}$ consists of all permutations with the same type as σ.

Let $\sigma \in \mathbb{A}_m \backslash \{e\}$ be of type $(1^{n_1}, 2^{n_2}, \dots, m^{n_m})$ and let $\mathcal{O} = \mathcal{O}_\sigma^{\mathbb{A}_m}$; thus $\mathcal{O}$ is a simple rack when $m \geq 5$.

It is known that either $\mathcal{O}_\sigma^{\mathbb{S}_m}$ splits as a disjoint union of two orbits in $\mathbb{A}_m$, or $\mathcal{O}_\sigma^{\mathbb{S}_m} = \mathcal{O}_\sigma^{\mathbb{A}_m}$, which happens either when $n_i > 0$ for some i even, or else $n_i > 1$ for some i odd.

Theorem. [A-Fantino-Graña-Vendramin] Let $\sigma \in \mathbb{A}_m \backslash \{e\}$, $m \geq 5$. Assume that the type of σ is different from

$$(3^2);\ (2^2, 3);\ (1^n, 3);\ (2^4);\ (1, 2^2);\ (1^2, 2^2);\ (1, p);\ (p);\ (8)$$

where p is prime. Then the rack $\mathcal{O}$ is of type D, hence it collapses.

The classes of types (3^2) and $(1^n, 3)$ collapse (are of type C).

Theorem. [Fantino] Let p be a prime number, $p \geq 5$, and $m \in \{p, p+1\}$. Let $\mathcal{O}$ be a conjugacy class of p-cycles in $\mathbb{A}_m$.

• If $m = p$, then $\mathcal{O}$ is of type D if and only if $p \geq 13$ and $p = \frac{r^k - 1}{r-1}$, with r a prime power and k is a natural number.

• If $m = p+1$, then $\mathcal{O}$ is of type D if and only if $p \geq 7$ and $p = \frac{r^k - 1}{r-1}$, with r a prime power and k is a natural number.

Theorem. [A-Fantino-Graña-Vendramin] $\mathbb{A}_m$, $m \geq 5$, collapses.

Example. $\mathbb{A}_5$ collapses.

- The types of conjugacy classes of $\mathbb{S}_5$ are
$$(1^5),\ (1^3, 2),\ (1^2, 3),\ (1, 2^2),\ (1, 4),\ (2, 3),\ (5).$$
- The types of conjugacy classes of $\mathbb{A}_5$ are
$$(1^5),\ (1^2, 3),\ (1, 2^2),\ (5).$$

Remark. If G is any finite group, then $\dim \mathscr{B}(M(\{e\}, \rho)) = \infty$ and $\dim \mathscr{B}(M(\mathcal{O}_g, \varepsilon)) = \infty$. Here e is the unit and ε is the counit.

Indeed, in both cases we elements $0 \neq m \in M(\mathcal{O}_g, \rho)$ such that $c(m \otimes m) = m \otimes m$, thus $\dim \mathscr{B}(\Bbbk m) = \infty$.

$(1^2,3)$: This class is of type C.

(5): Let σ be of type (5). We know that $\mathcal{O}_\sigma^{\mathbb{S}_5}$ splits as a disjoint union of two orbits in $\mathbb{A}_5$. Hence there is j: $\sigma^j \in \mathcal{O}_\sigma^{\mathbb{A}_5} \setminus \{\sigma\}$; it can be shown that $j = -1$. Now the centralizer C of σ in $\mathbb{A}_5$ is $\langle\sigma\rangle \simeq \mathbb{Z}/5$, hence $\widehat{C} \simeq \mathbb{Z}/5$.

If $\rho = \varepsilon$ is trivial, then we know that $\dim \mathscr{B}(M(\mathcal{O}_\sigma, \rho)) = \infty$.
If $\rho \neq \varepsilon$ is non-trivial, then $\omega := \rho(\sigma)$ has order 5. Let $U = \Bbbk\sigma \oplus \Bbbk\sigma^{-1}$; this is a braided vector space of Cartan type but not of finite type. Hence $\dim \mathscr{B}(U) = \infty$ and $\mathscr{B}(M(\mathcal{O}_\sigma, \rho)) = \infty$.

$(1,2^2)$: similar.

ii.iii Conjugacy classes of symmetric groups. Let $\sigma \in \mathbb{S}_m \backslash \mathbb{A}_m$ be of type $(1^{n_1}, 2^{n_2}, \dots, m^{n_m})$ and $\mathcal{O} = \mathcal{O}_\sigma^{\mathbb{S}_m}$; $\mathcal{O}$ is a simple rack.

Theorem. [A-Fantino-Graña-Vendramin] If the type of σ is different from

$$(2,3); \qquad (2^3); \qquad (1^n, 2), \tag{9}$$

then the rack $\mathcal{O}$ is of type D, hence it collapses.

Example. To see that $\mathbb{S}_5$ collapses, it remains to discard the class of type $(2,3)$.

Example. To see that $\mathbb{S}_6$ collapses, it remains to discard the class of type $(1^4, 2)$ and cocycle either -1 or χ_6; this last is the famous $FK(6)$.

Notice that for σ of type $(1^4, 2)$ and μ of type (2^3), the racks $\mathcal{O}_\sigma^{\mathbb{S}_6}$ and $\mathcal{O}_\mu^{\mathbb{S}_6}$ are isomorphic via the outer automorphism of $\mathbb{S}_6$.