Pointed Hopf algebras over simple groups

V. Nichols algebras over finite groups. Racks.

Nicolás Andruskiewitsch
CIEM–CONICET, Córdoba, Argentina

Department of Mathematics and Data Science,
Vrije Universiteit Brussel, Belgium

VUB-Leerstoel 2025-2026

Vrije Universiteit Brussels, October 16, 2025.

i. Yetter-Drinfeld modules and racks.

i.i Yetter-Drinfeld modules over groups.

Assume that $\Bbbk = \overline{\Bbbk}$.

If A is a finite-dimensional algebra, then $\operatorname{Irr} A :=$ $\{$isomorphism classes of simple objects in ${}_A\mathcal{M}\}$. Idem for abelian categories.

Let H be a finite-dimensional Hopf algebra. The Drinfeld double of H is a quasitriangular Hopf algebra such that

$$ {}_H^H\mathcal{YD} \simeq {}_H\mathcal{M} $$

as braided tensor categories; $D(H) \simeq H^* \otimes H$ as vector spaces.

Proposition. The category ${}_H^H\mathcal{YD}$ is semisimple $\iff D(H)$ is semisimple $\iff H$ and H^* are semisimple.

Let G be a finite group and $g \in G$. We set

- $\widehat{G} :=$ Hom$_{\text{groups}}(G, \Bbbk^{\times})$ (multiplicative characters).
- $\mathcal{O}_g = \{xgx^{-1} | x \in G\}$ is the conjugacy class of g, and
- $G^g = \{x \in |xg = gx\}$ the isotropy subgroup of g.

Lemma. $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ iff $M \in {}_{\Bbbk G}\mathcal{M}$, $M = \oplus_{g\in G} M_g$ is G-graded and

$$g \cdot M_h = M_{ghg^{-1}}.$$

Hence $\operatorname{supp} M := \{g \in G : M_g \neq 0\}$ is stable by conjugation.

- For a conjugacy class $\mathcal{O}$ of G, set $M_{\mathcal{O}} = \oplus_{g\in\mathcal{O}} M_g$. Then $M_{\mathcal{O}}$ is a Yetter-Drinfeld submodule of M and

$$M = \oplus_{\mathcal{O} \text{ conjugacy class}} M_{\mathcal{O}}.$$

- If M is indecomposable, then $\operatorname{supp} M =: \mathcal{O}$ is a single conjugacy class and $M = M_{\mathcal{O}}$.

- Let $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ and $g \in \operatorname{supp} M$. Then G^g acts on M_g.

Let U_g be a G^g-submodule of M_g. Then for any $h = xgx^{-1}$, $U_h := x \cdot U_g$ is a G^h-submodule of M_h.

(If $h = ygy^{-1}$, then $y^{-1}x \in G^g$, thus $x \cdot U_g = y \cdot U_g$ and similarly $x \cdot U_g$ is G^h-stable). Therefore,

$$U := \oplus_{h \in \mathcal{O}_g} U_h \qquad \text{is a subobject of } M \text{ in } {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}.$$

Thus, if M is indecomposable, respectively simple, in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$, then M_g is indecomposable, respectively simple, in ${}_{\Bbbk G^g}\mathcal{M}$.

Corollary. If G is abelian, every $M \in \operatorname{Irr} {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ has dimension 1. Therefore, if $\operatorname{char} \Bbbk \not| \, |G|$, then any $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ is of diagonal type.

Let $\rho : G^g \to \mathrm{End}(V)$ be a fin.-dim. representation and let

$$M(g,\rho) := \mathrm{Ind}_{G^g}^{G} V = \Bbbk G \otimes_{\Bbbk G^g} V.$$

Clearly, $\dim M(g,\rho) = [G : G^g] \dim \rho$. Set $y \triangleright z := yzy^{-1}$, if $y, z \in G$.

Then $M(g,\rho) \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ with action and grading given by

$$\begin{aligned} &h \rightharpoonup x \otimes v = hx \otimes v && \text{(the induced structure)}, \\ &x \otimes v \in M(g,\rho)_{x \triangleright g}, && h \in G, x \in G, v \in V. \end{aligned}$$

Theorem. If $\rho \in \mathrm{Irr}\, G^g$, then $M(g,\rho) \in \mathrm{Irr}\, {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ and any irreducible in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ arises in this way.

If $\mathrm{char}\, \Bbbk$ does not divide $|G|$, then ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ is semisimple.

The first part follows from the preceding discussion. Fix a subset $\mathcal{Q} \subset G$ that intersects each conjugacy class exactly once. It is easy to see that the objects $M(g,\rho)$ for $g \in \mathcal{Q}$ and $\rho \in \operatorname{Irr} G^g$ are mutually non isomorphic. If $\operatorname{char} \Bbbk$ does not divide $|G|$, then

$$\sum_{g\in\mathcal{Q}} \sum_{\rho\in\operatorname{Irr} G^g} (\dim M(g,\rho))^2 = \sum_{g\in\mathcal{Q}} \sum_{\rho\in\operatorname{Irr} G^g} ([G:G^g]\dim\rho)^2$$

$$= \sum_{g\in\mathcal{Q}} [G:G^g]^2 \sum_{\rho\in\operatorname{Irr} G^g} (\dim\rho)^2 = \sum_{g\in\mathcal{Q}} [G:G^g]^2 |G^g|$$

$$= \sum_{g\in\mathcal{Q}} |\mathcal{O}_g|^2 |G^g| = \sum_{g\in\mathcal{Q}} |\mathcal{O}_g||G| = |G| \sum_{g\in\mathcal{Q}} |\mathcal{O}_g|$$
$$= |G|^2 = \dim D(\Bbbk G).$$

Hence $D(\Bbbk G)$ is semisimple.

Let $g \in G$, $\rho : G^g \to \mathrm{End}(V)$, $\rho \in \mathrm{Irr}\, G^g$ and $M = M(g, \rho)$. Fix a set $(h_i)_{i \in \mathbb{I}_s}$ of representatives of G/G^g in G. Then $M = \oplus_{i \in \mathbb{I}_s} h_i V$. We compute the braiding of M. For $i, j \in \mathbb{I}_s$, $v, w \in V$,

$$c(h_i v \otimes h_j w) = (h_i \triangleright g) \rightharpoonup (h_j w) \otimes h_i v = h_k\, \rho(\gamma)(w) \otimes h_i v,$$

where $k \in \mathbb{I}_s$ and $\gamma \in G^g$ are determined by

$$(h_i \triangleright g) h_j = h_k \gamma.$$

It is easy to see that

$$(h_i \triangleright g) \triangleright (h_j \triangleright g) = h_k \triangleright g.$$

Searching for a class of braided vector spaces that efficiently encompasses these Yetter-Drifeld modules, we arrive at the following notion.

i.ii. Racks

A rack is a pair $(X, \triangleright)$, where $X \neq \emptyset$ is a set and $\triangleright : X \times X \to X$ is an operation on X such that

- $x \triangleright \_\_$ is a bijection for all $x \in X$,
- $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$ for all $x, y, z \in X$.

Morphisms of racks are maps that preserve the operation $\triangleright$.

Main examples: X a conjugacy class in G, $x \triangleright y := xyx^{-1}$. More generally, a union of conjugacy classes, i.e., a subset of G stable by the adjoint.

Except explicitly said, all racks are finite and arise as (unions of) conjugacy classes of a finite group.

- A rack X is abelian if $x \triangleright y = y$ for all $x, y \in X$.

Affine racks.

Let A be an abelian group, $T \in \operatorname{Aut} A$; define $\triangleright$ by

$$x \triangleright y = (\mathrm{id} - T)x + Ty, \qquad\qquad x, y \in A.$$

Then $(A, \triangleright)$ is an *affine* rack, denoted $\operatorname{Aff}(A, T)$; it is isomorphic to the subrack $A \times \mathrm{id}$ of $A \rtimes \langle T \rangle$.

If T is multiplication by $m \in \mathbb{Z}$, then $\operatorname{Aff}(A, m) := \operatorname{Aff}(A, T)$.

- The dihedral rack is $\mathcal{D}_n := \operatorname{Aff}(\mathbb{Z}/n, -1)$.

Twisted conjugacy classes.

Let N, C be finite groups with C acting on N by group automorphisms, and let $G = N \rtimes C$. If $(m,z),(n,y) \in G$, then

$$(m,z)(n,y)(m,z)^{-1} = (m(z \cdot n)(zyz^{-1} \cdot m^{-1}), zyz^{-1}).$$

When C is abelian, it follows that

$$\mathcal{O}_{(n,y)} = \bigcup_{z \in C} \mathcal{C}^y_{z \cdot n} \times \{y\},$$

where $\mathcal{C}^y_n$ is the orbit of $n \in N$ under the action of N on itself given by

$$m \rightharpoonup_y n := m\, n\, (y \cdot m^{-1}).$$

Note that $\bigcup_{z \in <y>} \mathcal{C}^y{}_{z \cdot n} = \mathcal{C}^y{}_n$. For, $n^{-1} \rightharpoonup_y n = y \cdot n$, and the claim follows. Note also that $m \rightharpoonup_y n$ is *not* the same as $m \triangleright n$.

i.iii. Braided vector spaces from racks.

A **2-cocycle** on a rack X is a function $\mathfrak{q} : X \times X \to \Bbbk^{\times}$ such that

$$\mathfrak{q}_{x \triangleright y, x \triangleright z}\, \mathfrak{q}_{x,z} = \mathfrak{q}_{x, y \triangleright z}\, \mathfrak{q}_{y,z}$$

for all $x, y, z \in X$.

For example, any constant function $X \times X \to \Bbbk^{\times}$ is a 2-cocycle.

Let $\mathfrak{q} : X \times X \to \Bbbk^{\times}$ be a function and $V = \Bbbk X$ with basis $(e_x)_{x \in X}$. Let $c^{\mathfrak{q}} \in GL(V \otimes V)$ be given by

$$c^{\mathfrak{q}}(e_x \otimes e_y) = q_{x,y}\, e_{x \triangleright y} \otimes e_x, \qquad x, y \in X.$$

Then, $\mathfrak{q}$ is a 2-cocycle if and only if $(V, c^{\mathfrak{q}})$ is a braided vector space.

Notice: if G is a finite group, $g \in G$ and $\chi \in \widehat{G^g}$, then $M(g, \chi)$ as a braided vector space is of the form $(\Bbbk X, c^{\mathfrak{q}})$.

More generally, a **a non-abelian 2-cocycle of degree $n \geq 2$** on a rack X is a function $\mathfrak{q} : X \times X \to \mathbf{GL}(n, \Bbbk)$ such that

$$\mathfrak{q}_{x \triangleright y, x \triangleright z}\, \mathfrak{q}_{x,z} = \mathfrak{q}_{x, y \triangleright z}\, \mathfrak{q}_{y,z}$$

for all $x, y, z \in X$.

Let $\mathfrak{q} : X \times X \to \mathbf{GL}(n, \Bbbk)$ be a function and $V = \Bbbk X \otimes \Bbbk^n$. Let $c^{\mathfrak{q}} \in GL(V \otimes V)$ be given by

$$c^{\mathfrak{q}}(e_x v \otimes e_y w) = e_{x \triangleright y}\, q_{x,y}(w) \otimes e_x v, \qquad x, y \in X,\ v, w \in \Bbbk^n.$$

Then, $\mathfrak{q}$ is a non-abelian 2-cocycle if and only if $(V, c^{\mathfrak{q}})$ is a braided vector space.

Notice: if G is a finite group, $g \in G$ and $\rho \in \operatorname{Irr} G^g$, then $M(g, \rho)$ as a braided vector space is of the form $(\Bbbk X \otimes \Bbbk^n, c^{\mathfrak{q}})$.

i.iv. Finite simple racks

Let X be a (finite) rack. A *decomposition* of X is a pair of subracks (Y, Z) such that $X = Y \dot\cup Z$; X is *decomposable* if it admits a decomposition, *indecomposable* otherwise.

A finite rack X is *simple* if it has at least 2 elements, and for any surjective morphism of racks $\pi : X \to Y$, either π is an isomorphism or Y has just one element.

Notice: if Z is a finite rack, then there exists a finite simple rack X and a surjective morphism of racks $\pi : Z \to X$.

Theorem. [Joyce; A-Graña with help of Guralnick] Let X be a finite simple rack with $|X|$ elements. Then either of the following holds:

(I) $|X| = p^t$ where p is a prime and $t \in \mathbb{N}$. There are two possibilities:

(a) $t = 1$ and $X \simeq \mathbb{I}_p$ is the permutation rack of the cycle $\varsigma = (1, 2, \dots, p)$, i.e., $x \triangleright y = \varsigma(y)$ for all $x, y \in X$. (this can not be realized as a conjugacy class in a group).

(b) X is the affine rack $(\mathbb{F}^t, T)$, where T is the companion matrix of a monic irreducible polynomial $f \in \mathbb{F}[\mathbb{X}]$ of degree t, different from $\mathbb{X}$ and $\mathbb{X} - 1$.

(II) $|X|$ is divisible by at least two primes. Then, there exist

a simple non-abelian group L, $\quad t \in \mathbb{N}$, $\quad$ and $\theta \in \operatorname{Aut} L$,

such that X is a twisted conjugacy class of type (G, T), where $G = L^t$ and $T \in \operatorname{Aut}(L^t)$ acts by

$$T(\ell_1, \ldots, \ell_t) = (\theta(\ell_t), \ell_1, \ldots, \ell_{t-1}), \qquad \ell_1, \ldots, \ell_t \in L.$$

Namely. $X = \mathcal{O}_{T,n}$ is the orbit of $n \in N = L^t$ under the action $\rightharpoonup_T$ of $N = L$ on itself by

$$m \rightharpoonup_T n := m\, n\, (T \cdot m^{-1}).$$

Furthermore, L and t are unique, and T only depends on its conjugacy class in $\operatorname{Out}(L^t) = \operatorname{Aut}(L^t)/\operatorname{Inn}(L^t)$. If $m, n \in X$ then

$$m \triangleright n = mT(nm^{-1}).$$

i.v. Questions.

From now on, we assume that $\operatorname{char} \Bbbk = 0$.

To classify the finite-dimensional pointed Hopf algebras over non-abelian groups, we need to address the following:

Question I: For any finite group G, any famlies $(g_i)_{i\in\mathbb{I}_s}$ in G and $(\rho_i)_{i\in\mathbb{I}_s}$ in $\operatorname{Irr} G^{g_i}$, decide if $\dim \mathscr{B}(\oplus_{i\in\mathbb{I}_s} M(g_i, \rho_i))$ is finite.

A more economical approach is to address:

Question II: For any finite rack X (realizable as a subrack of a finite group) and any family $\mathfrak{q}_Y$ of 2-cocycles of degree $n_Y \geq 1$ on the components Y of X decide if $\dim \mathscr{B}(\oplus_Y (\Bbbk Y \otimes \Bbbk^{n_Y}, c^{\mathfrak{q}_Y}))$ is finite.

i.vi. Answers.

Question I: As we saw, the problem was solved for finite abelian groups G and recently, as we will see, also for solvable groups.

Many efforts were made towards:

Question III: Solve Question I for finite non-abelian *simple* groups.

Indeed, Question III is intertwined with the following:

Question IV: Solve Question II for finite *simple* racks.

To solve these problems we currently have very rudimentary methods for computing Nichols algebras and some powerful indirect techniques for deciding that a Nichols algebra has infinite dimension.

The technique of abelian subracks. Let X be a rack and let $\mathfrak{q} : X \times X \to \Bbbk^{\times}$ be a 2-cocycle. Let $Y \le X$ be an abelian subrack and $\mathfrak{p} = (q_{ij})_{i,j\in Y}$. Then $(\Bbbk Y, c^{\mathfrak{p}})$ is of diagonal type so we know when $\dim \mathscr{B}(\Bbbk Y, c^{\mathfrak{p}}) < \infty$. For example:

- Taking $Y = \{x\}$, one has $\dim \mathscr{B}(\Bbbk X, c^{\mathfrak{q}}) < \infty$ implies $q_{xx} \in \mathbb{G}'_{\infty}$.

- Assume that $\mathfrak{q}$ is the constant cocycle $\omega \in \mathbb{G}'_N$, $N \ge 2$. Then $(\Bbbk Y, c^{\mathfrak{p}})$ is of Cartan type with matrix $\mathscr{A} = (a_{ij})$, where $a_{ij} = 2-N$ for all $i \ne j$.
Then $\dim \mathscr{B}(\Bbbk X, c^{\mathfrak{q}}) < \infty$ implies $N = 2$, or $N = 3$ and $|Y| = 2$.

Theorem. [A] Let G be a finite nilpotent group of odd order. Given a finite-dimensional $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$, one has: $\dim \mathscr{B}(M) < \infty$ if and only if $M \simeq M_0 \oplus M_1 \oplus \cdots \oplus M_t$ where:

- $\operatorname{supp} M_0 \subseteq Z(G)$ and M_0 is given by a family of YD-pairs $(g_i, \chi_i)_{i \in J}$ such that the connected components of the matrix $\mathfrak{q} = (q_{ij})_{i,j \in J}$ belong to the list in [Heckenberger].

- For $j \in \mathbb{I}_t$, $M_j \simeq M(\mathcal{O}_j, \chi_j)$ where $\mathcal{O}_j$ is not central and abelian as rack; $\chi_j \in \widehat{G^{x_j}}$ for a fixed $x_j \in \mathcal{O}_j$ that satisfies

$$\chi_j\left((g^{-1} \triangleright x_j)(g \triangleright x_j)\right) = 1 \qquad \text{for every } g \in G \backslash G^{x_j}.$$

and $q_j := \chi_j\left(x_j\right)$ has order $2 < N_j < \infty$. Also $\dim \mathscr{B}(\mathcal{O}_j, \chi_j) = N_j^{|\mathcal{O}_j|}$. Furthermore,

$$c_{|M_j \otimes M_i}\, c_{|M_i \otimes M_j} = \mathrm{id}_{M_i \otimes M_j}, \qquad i \neq j \in \mathbb{I}_{0,t}.$$

For an arbitrary solvable group G there is no such explicit description (yet) but:

Theorem. [A-Heckenberger-Vendramin] If $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ is simple and $\dim \mathscr{B}(M)$ is finite, then the underlying pair $(X, \mathfrak{q})$ belongs to the list below, or $|X| = 1$.

Theorem. [Heckenberger-Vendramin] If $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ is semisimple but not simple and $\dim \mathscr{B}(M)$ is finite, then M belongs to a list to be seen later (again related to Lie theory via Dynkin diagrams).

Corollary. [A-Heckenberger-Vendramin] if $|G|$ is odd and $M \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ has and $\dim \mathscr{B}(M)$ finite, then $\operatorname{supp} M$ is an abelian rack.

ii. Finite-dimensional Nichols algebras.

ii.i Indecomposable support.

The list:

Racks with prime order: Let Aff$(\mathbb{F}_p, d)$ with p prime, where

$$(\mathbb{F}_p, d) \in \{(\mathbb{F}_3, 2), (\mathbb{F}_5, 2), (\mathbb{F}_5, 3), (\mathbb{F}_7, 3), (\mathbb{F}_7, 5)\}.$$

Relevant cocycle: constant $\epsilon \equiv -1$.

Tetrahedron rack: $\mathscr{T} = \mathrm{Aff}(\mathbb{F}_4, \mathfrak{t})$, $\mathfrak{t} =$ multiplication by $w \in \mathbb{F}_4 \backslash \{0, 1\}$. $\mathscr{T} \simeq$ conjugacy classes of 3-cycles in $\mathbb{A}_4$.
Two relevant cocycles: $(\epsilon, q) = (1, -1)$ or $(-1, \omega)$, $\omega \in \mathbb{G}_3'$.

The conjugacy class $\mathscr{O}_2^4$ of transpositions in $\mathbb{S}_4$:
Two relevant cocycles: constant $\epsilon = -1$ or χ_4 .

The conjugacy class $\mathcal{O}_4^4$ of 4-cycles in $\mathbb{S}_4$:
Relevant cocycle: constant $\epsilon \equiv -1$.

The conjugacy class $\mathcal{O}_2^5$ of transpositions in $\mathbb{S}_5$:

- $\mathscr{B}(\mathcal{O}_2^5, \epsilon)$ and
- $\mathscr{B}(\mathcal{O}_2^5, \chi_5)$ (here χ_5 is analogous to χ_4).

Both Nichols algebras have dimension $2^{12}3^4 5^2 = 8,294,400$.

Notation: If $(V, c) = (\Bbbk X, c^{\mathfrak{q}})$ where X is a rack and $\mathfrak{q}$ is a 2-cocycle, then we set $\mathscr{B}(V, c) =: \mathscr{B}(X, \mathfrak{q})$.

The Nichols algebra $\mathscr{B}(\mathcal{O}_2^3, -1)$. Clearly, $\mathcal{O}_2^3 \simeq \mathrm{Aff}(\mathbb{F}_3, 2)$.

Set $x_0 = x_{(12)}$, $x_1 = x_{(23)}$ and $x_2 = x_{(13)}$. One has

$$\begin{aligned}\mathscr{B}(\mathcal{O}_2^3, -1) \simeq \Bbbk\langle x_0, x_1, x_2 | x_0^2, x_1^2, x_2^2,& \\ & x_0x_1 + x_1x_2 + x_2x_0, \\ & x_1x_0 + x_2x_1 + x_0x_2.\rangle\end{aligned}$$

The Poincaré polynomial is
$(1+t)^2(1+t+t^2) = 1 + 3t + 4t^2 + 3t^3 + t^4$. Hence

$\dim \mathscr{B}(\mathcal{O}_2^3, -1) = 12 = 3.2^2$, top degree $= 4 = 2^2$.

The Nichols algebra $\mathscr{B}(\mathrm{Aff}(\mathbb{F}_5,2),-1)$. One has

$$\begin{aligned}\mathscr{B}(\mathrm{Aff}(\mathbb{F}_5,2),-1) \simeq \Bbbk\langle x_0,x_1,x_2,x_3,x_4 |\\
&x_0^2,\quad x_1^2,\quad x_2^2,\quad x_3^2,\quad x_4^2,\\
&x_3x_2+x_2x_0+x_1x_3+x_0x_1,\\
&x_4x_0+x_2x_1+x_1x_4+x_0x_2,\\
&x_4x_1+x_3x_4+x_1x_0+x_0x_3,\\
&x_4x_2+x_3x_0+x_2x_3+x_0x_4,\\
&x_4x_3+x_3x_1+x_2x_4+x_1x_2,\\
&x_1x_0x_1x_0+x_0x_1x_0x_1\rangle\end{aligned}$$

One has $\dim \mathscr{B}(\mathrm{Aff}(\mathbb{F}_5,2),-1) = 1280 = 5.4^4$,
top degree $= 16 = 4^2$ and Poincaré polynomial

$$\begin{aligned}&t^{16}+5t^{15}+15t^{14}+35t^{13}+66t^{12}+105t^{11}+145t^{10}\\
&+175t^9+186t^8+175t^7+145t^6+105t^5+66t^4\\
&+35t^3+15t^2+5t+1.\end{aligned}$$

The Nichols algebra $\mathscr{B}(\mathsf{Aff}(\mathbb{F}_7, 3), -1)$.

This algebra has generators $x_0, x_1, x_2, x_3, x_4, x_5, x_6$, 21 relations in degree 2 and one in degree 6:

$$x_2x_0x_1x_2x_0x_1 + x_1x_2x_0x_1x_2x_0 + x_0x_1x_2x_0x_1x_2.$$

One has $\dim \mathscr{B}(\mathsf{Aff}(\mathbb{F}_7, 3), -1) = 326,592 = 7.6^6$, top degree $36 = 6^2$.

The Nichols algebras $\mathscr{B}(\mathsf{Aff}(\mathbb{F}_5, 3), -1)$ and $\mathscr{B}(\mathsf{Aff}(\mathbb{F}_7, 5), -1)$.

These are dual to $\mathscr{B}(\mathsf{Aff}(\mathbb{F}_5, 2), -1)$ and $\mathscr{B}(\mathsf{Aff}(\mathbb{F}_7, 3), -1)$ respectively. So they have the same Poincaré series, dimension and top degree.

The Nichols algebra $\mathscr{B}(\mathscr{T}, -1)$.

Recall $\mathscr{T} = \mathrm{Aff}(\mathbb{F}_4, \mathfrak{t})$, $\mathfrak{t} =$ multiplication by $w \in \mathbb{F}_4 \backslash \{0, 1\}$.
$\mathscr{T} \simeq$ conjugacy classes of 3-cycles in $\mathbb{A}_4$.

The algebra has dimension 72 and top degree 9. One has

$$\begin{aligned}\mathscr{B}(\mathscr{T}, -1) \simeq \Bbbk\langle x_0, x_1, x_2, x_3 |\\ & x_3x_2 + x_2x_1 + x_1x_3,\\ & x_3x_1 + x_1x_0 + x_0x_3,\\ & x_3x_0 + x_0x_2 + x_2x_3,\\ & x_2x_0 + x_0x_1 + x_1x_2,\\ & x_2x_1x_0x_2x_1x_0 + x_1x_0x_2x_1x_0x_2 + x_0x_2x_1x_0x_2x_1.\rangle\end{aligned}$$

The Poincaré polynomial

$$\begin{aligned}(1+t)^2(1+t+t^2)^2(1+t^3) = t^9 + 4t^8 + 8t^7 + 11t^6\\ + 12t^5 + 12t^4 + 11t^3 + 8t^2 + 4t + 1.\end{aligned}$$

The Nichols algebra $\mathscr{B}(\mathscr{T}, \chi)$.

Here χ is a suitable 2-cocycle that depends on $\omega \in \mathbb{G}_3'$ The algebra $\mathscr{B}(\mathscr{T}, \chi)$ has dimension 5184 and top degree 24. It can be presented by generators a, b, c, d with defining relations

$$
\begin{aligned}
& a^3, \quad b^3, \quad c^3, \quad d^3, \\
& -\omega^2 ab - \omega bc + ca, \quad -\omega^2 ac - \omega cd + da, \\
& \omega ad - \omega^2 ba + db, \quad \omega bd + \omega^2 cb + dc, \\
& a^2bcb^2 6 + abcb^2a + bcb^2a^2 + cb^2a^2b + b^2a^2bc \\
& + ba^2bcb + bcba^2c + cbabac + cb^2aca.
\end{aligned}
$$

The Hilbert series of $\mathscr{B}(\mathscr{T}, \chi)$ is $(6)_t^4 (2)_{t^2}^2$.

The Nichols algebra $\mathscr{B}(\mathcal{O}_2^4, -1)$. Set $x_0 = x_{(12)}$, $x_1 = x_{(13)}$, $x_2 = x_{(14)}$, $x_3 = x_{(23)}$, $x_4 = x_{(24)}$, $x_5 = x_{(34)}$. One has

$$
\begin{aligned}
\mathscr{B}(\mathcal{O}_2^4, -1) \simeq \Bbbk\langle x_0, x_1, x_2, x_3, x_4, x_5 | \\
& x_0^2, \quad x_1^2, \quad x_2^2, \quad x_3^2, \quad x_4^2, \quad x_5^2, \\
& x_0x_5 + x_5x_0, \quad x_1x_4 + x_4x_1, \quad x_2x_3 + x_3x_2, \\
& x_3x_0 + x_1x_3 + x_0x_1, \quad x_0x_3 + x_1x_0 + x_3x_1, \\
& x_4x_0 + x_2x_4 + x_0x_2, \quad x_0x_4 + x_2x_0 + x_4x_2, \\
& x_1x_2 + x_5x_1 + x_2x_5, \quad x_2x_1 + x_5x_2 + x_1x_5, \\
& x_3x_4 + x_5x_3 + x_4x_5, \quad x_4x_3 + x_5x_4 + x_3x_5\rangle.
\end{aligned}
$$

One has $\dim \mathscr{B}(\mathcal{O}_2^4, -1) = 576 = 3^2.2^6$, the top degree is 12 and the Poincaré polynomial is

$$
\begin{aligned}
&(1+t)^2(1+t+t^2)^2(1+t+t^2+t^3)^2 \\
&= t^{12} + 6t^{11} + 19t^{10} + 42t^9 + 71t^8 + 96t^7 + 106t^6 \\
&+ 96t^5 + 71t^4 + 42t^3 + 19t^2 + 6t + 1.
\end{aligned}
$$

The Nichols algebra $\mathscr{B}(\mathcal{O}_4^4, -1)$. One has

$$\begin{aligned}\mathscr{B}(\mathcal{O}_2^4, -1) \simeq \Bbbk\langle x_1, x_2, x_3, x_4, x_5, x_6 | \\ & x_1^2, \quad x_2^2, \quad x_3^2, \quad x_4^2, \quad x_5^2, \quad x_6^2 \\ & x_4x_3 + x_3x_4, \quad x_5x_2 + x_2x_5, \quad x_6x_1 + x_1x_6, \\ & x_3x_2 + x_2x_1 + x_1x_3, \quad x_4x_1 + x_2x_4 + x_1x_2, \\ & x_5x_1 + x_4x_5 + x_1x_4, \quad x_5x_3 + x_3x_1 + x_1x_5, \\ & x_6x_2 + x_2x_3 + x_3x_6, \quad x_6x_3 + x_5x_6 + x_3x_5, \\ & x_6x_4 + x_4x_2 + x_2x_6, \quad x_6x_5 + x_5x_4 + x_4x_6\rangle.\end{aligned}$$

The Poincaré polynomial, the dimension and the top degree are the same as those of $\mathscr{B}(\mathcal{O}_2^4, -1)$.

The Nichols algebra $\mathscr{B}(\mathcal{O}_2^5, -1)$.

This algebra is quadratic: it has 10 generators and 45 relations in degree 2. One has $\dim \mathscr{B}(\mathcal{O}_2^5, -1) = 8,294,400$, and top degree 40.

The Nichols algebras $\mathscr{B}(\mathcal{O}_2^4, \chi_4)$ and $\mathscr{B}(\mathcal{O}_2^5, \chi_5)$.

These are twist-equivalent to $\mathscr{B}(\mathcal{O}_2^4, -1)$ and $\mathscr{B}(\mathcal{O}_2^5, -1)$, respectively. So they have the same Poincaré series, dimension and top degree.