

Pointed Hopf algebras over simple groups

IV. The Weyl groupoid

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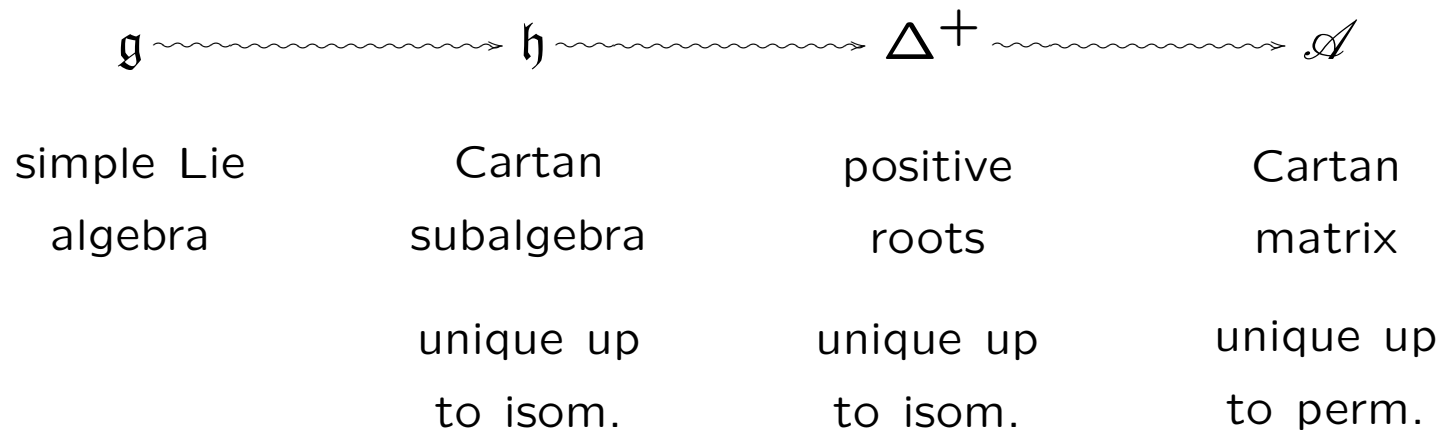
IV. The Weyl groupoid.

o. Overview.

Assume in this chapter that $\text{char } \mathbb{k} = 0$ and $\mathbb{k} = \bar{\mathbb{k}}$.

The famous Killing-Cartan classification of finite-dimensional simple Lie algebras requires two steps:

First, one attaches to a simple Lie algebra $\mathfrak{g}$ several invariants until getting a Cartan matrix; then classifies all possible ones.



Cartan matrices.

classical

$$A_\theta, \theta \geq 1 \quad \circ - \circ \cdots \circ - \circ - \circ$$

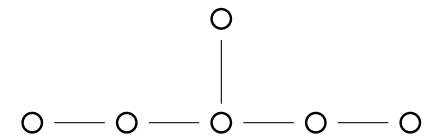
$$B_\theta, \theta \geq 2 \quad \circ - \circ \cdots \circ - \circ \Rightarrow \circ$$

$$C_\theta, \theta \geq 3 \quad \circ - \circ \cdots \circ - \circ \Leftarrow \circ$$

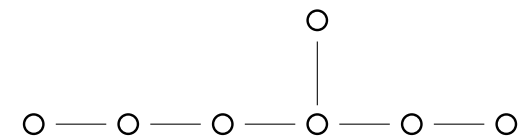
$$D_\theta, \theta \geq 4 \quad \begin{array}{ccccccc} & & & \circ & & & \\ & & & | & & & \\ \circ & - & \circ & \cdots & \circ & - & \circ \end{array}$$

exceptional

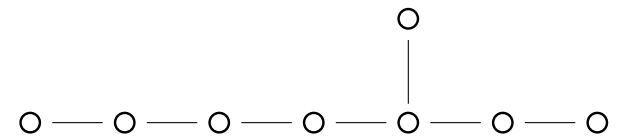
E_6



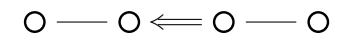
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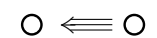
E_8



F_4



G_2



Second, given a Cartan matrix $\mathcal{A} = (a_{ij})_{i,j \in \mathbb{I}_\theta}$, one defines a simple Lie algebra $\mathfrak{g}(\mathcal{A})$, by generators and relations (that involve the entries of $\mathcal{A}$) the key ones being the Serre relations

$$\mathrm{ad}(x_i)^{1-a_{ij}}x_j = 0, \quad i \neq j;$$

(where ad is the adjoint representation) and checks that $\mathfrak{g}(\mathcal{A})$ is simple and finite-dimensional.

A key tool for the last verification is the Weyl group.

Recall that $\mathcal{A} = (a_{ij})_{i,j \in \mathbb{I}_\theta}$ is a **generalized** Cartan matrix if

$$\begin{aligned} a_{ii} &= 1, & i &\in \mathbb{I}_\theta, \\ a_{ij} &\in \mathbb{Z}_{\leq 0}, & i \neq j &\in \mathbb{I}_\theta, \\ a_{ij} = 0 &\implies & a_{ji} &= 0. \end{aligned}$$

The matrix $\mathcal{A}$ is encoded in a Dynkin diagram with vertices $\mathbb{I}_\theta$:

$$\dots \underset{i}{\overset{2}{\circ}} \text{---} \underset{j}{\overset{2}{\circ}} \dots$$

- no edge between the different vertices i and j if $a_{ij} = 0$;
- just an edge if $a_{ij}a_{ji} = 1$;
- if $1 < a_{ij}a_{ji} \leq 4$ and $|a_{ij}| \geq |a_{ji}|$, then there are $|a_{ij}|$ edges between i and j , with an arrow pointing towards i ;
- If $a_{ij}a_{ji} > 4$, then i and j are connected by a thick line decorated with the pair $(|a_{ij}|, |a_{ji}|)$.

When the Dynkin diagram is connected, $\mathcal{A}$ is called *indecomposable*. Also, $\mathcal{A}$ is *symmetrizable* if there are $d_i \in \mathbb{Z} \setminus 0$, $i \in \mathbb{I}_\theta$:

$$d_i a_{ij} = d_j a_{ji}, \quad \forall i, j \in \mathbb{I}_\theta.$$

Below, $u, v \in \mathbb{k}^\theta$ are column vectors; $u > 0$, respectively $u \geq 0$, means that all its entries are > 0 , respectively ≥ 0 .

Theorem. (Vinberg) [Kac, Thm. 4.3]

Let $\mathcal{A}$ be an indecomposable generalized Cartan matrix. Then exactly one of the following possibilities hold:

(Finite) $\det \mathcal{A} \neq 0$; there exists $u > 0$ such that $\mathcal{A}u > 0$; $\mathcal{A}v > 0$ implies $v > 0$ or $v = 0$;

(Affine) $\text{corank } \mathcal{A} = 1$; there exists $u > 0$ such that $\mathcal{A}u = 0$; $\mathcal{A}v \geq 0$ implies $\mathcal{A}v = 0$;

(Indefinite) there is $u > 0$ s.t. $\mathcal{A}u < 0$; $\mathcal{A}v \geq 0$, $v \geq 0 \implies v = 0$.

Given an indecomposable generalized Cartan matrix, the Kac-Moody algebra $\mathfrak{g}(\mathcal{A})$ is defined as in the second step of the classification, in particular via the Serre relations.

Theorem. [Kac, Prop. 4.9]

Let $\mathcal{A}$ be an indecomposable generalized Cartan matrix. Then the following conditions are equivalent:

- $\mathcal{A}$ is a generalized Cartan matrix of finite type;
- The Weyl group W is finite;
- Δ is finite;
- $\mathfrak{g}(\mathcal{A})$ is a simple finite-dimensional Lie algebra.

Quantum groups (quantized enveloping algebras of simple Lie algebras) were introduced defining them by generators and relations (that involve the entries of the corresponding Cartan matrix), in particular using a quantum version of the Serre relations.

In order to work the quantized enveloping algebras out, Lusztig (and other authors) introduced automorphisms generalizing the simple reflections that generate the Weyl group, getting a representation of the braid group.

A braided vector space (V, c) is of **diagonal type** if there exists a matrix $q = (q_{ij})_{i,j \in \mathbb{I}_\theta}$ and a basis $(v_i)_{i \in \mathbb{I}_\theta}$ of V such that

$$c(v_i \otimes v_j) = q_{ij} v_j \otimes v_i, \quad i, j \in \mathbb{I}_\theta.$$

The matrix q is encoded in a (sort of) Dynkin diagram

$$\dots \underset{i}{\underset{\circ}{q_{ii}}} \xrightarrow{\tilde{q}_{ij}} \underset{j}{\underset{\circ}{q_{jj}}} \dots \quad \text{where } \tilde{q}_{ij} = q_{ij}q_{ji}, \quad \text{no edge if } \tilde{q}_{ij} = 1.$$

We may (and will) assume that the Dynkin diagram is connected. Indeed let $\mathcal{X}$ be the set of connected components. For $J \in \mathcal{X}$, $V_J := \bigoplus_{j \in J} \mathbb{k} v_j$ is a braided subspace and we have a decomposition

$$(V, c) = \bigoplus_{J \in \mathcal{X}} V_J, \quad \text{with} \quad c_{J,K} c_{K,J} = \text{id}_{V_K \otimes V_J} \quad \forall K \neq J \in \mathcal{X}.$$

Hence

$$\mathcal{B}(V, c) = \bigotimes_{J \in \mathcal{X}} \mathcal{B}(V_J).$$

Definition. A braided vector space (V, c) is of **Cartan type** if it is of diagonal type and there exist $a_{ij} \in \mathbb{Z}$ such that

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}} v_j \otimes v_i, \quad i \neq j \in \mathbb{I}_\theta.$$

Assume that $q_{ii} \in \mathbb{G}'_\infty$ for all i . Choose $a_{ij} \in (-\text{ord } q_{ii}, 0]$ and set $a_{ii} = 2$. Then $\mathcal{A} = (a_{ij})_{i,j \in \mathbb{I}}$ is a generalized Cartan matrix.

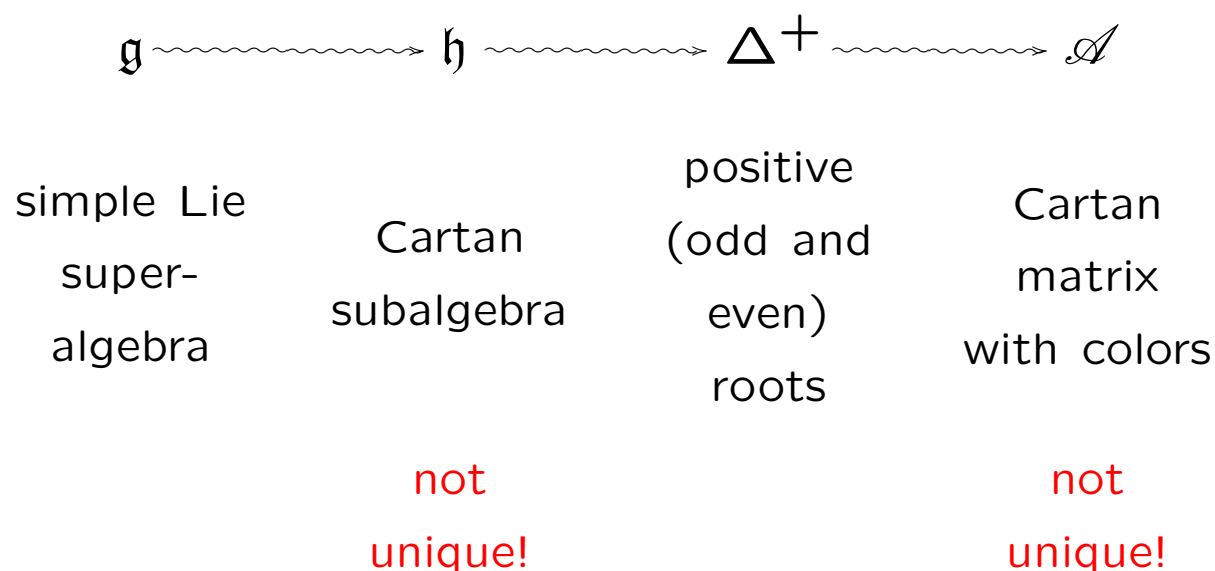
Theorem. $\dim \mathcal{B}(V) < \infty \iff \mathcal{A}$ is of finite type.

[A.-Schneider]: assuming restrictions on the orders of the q_{ij} 's and reducing to results of Lusztig, Rosso and De Concini–Procesi.

[Heckenberger]. Using quantum reflections.

What happens beyond Cartan type?

Finite-dimensional simple Lie superalgebras were classified by V. Kac. There are three classes; the *contragredient* ones are the closer to the simple Lie algebras. One may attach to a simple Lie superalgebra $\mathfrak{g}$ similar invariants but there are differences:



Types of contragredient simple Lie superalgebras, char 0:

classical

$\mathbf{A}(j-1|\theta-j)$

$\mathbf{BC}(j|\theta-j)$

$\mathbf{D}(j|\theta-j)$

exceptional

$\mathbf{D}(2, 1; \alpha)$

$\mathbf{F}(4)$

$\mathbf{G}(3)$

i. Coxeter groupoids.

i.i Quivers.

- A *quiver* is a collection $\mathcal{Q} = (\mathcal{Q}, \mathcal{X}, s, t)$, where $\mathcal{Q}$ and $\mathcal{X}$ are non-empty sets (of 'arrows' and 'points'), and $s, t : \mathcal{Q} \rightrightarrows \mathcal{X}$ are maps ('source' and 'target'): $a \in \mathcal{Q} \rightsquigarrow \overset{s(a)}{\bullet} \xrightarrow{a} \overset{t(a)}{\bullet}$
- The set $\mathcal{X}$ is the base of $\mathcal{Q}$.
- The category $\mathbf{Quiv}$ of quivers has morphisms $F : \mathcal{Q} \rightarrow \mathcal{Q}'$, $F = (F_a, F_p)$ where $F_a : \mathcal{Q} \rightarrow \mathcal{Q}'$, $F_p : \mathcal{X} \rightarrow \mathcal{X}'$ are maps compatible with sources and arrows.
- If $\mathcal{X} \neq \emptyset$ is a set, then we denote by $\mathbf{Quiv}_{\mathcal{X}}$ the category of quivers with base $\mathcal{X}$; the morphisms have $F_p = \text{id}_{\mathcal{X}}$.

- The product in $\text{Quiv}_{\mathcal{X}}$ is the pullback

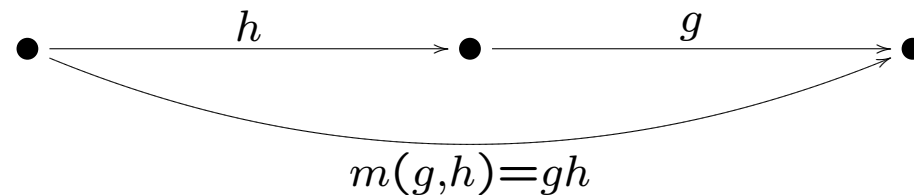
$$\mathcal{Q}_s \times_t \mathcal{Q}' = \{(g, h) \in \mathcal{Q} \times \mathcal{Q}' : s(g) = t(h)\}.$$

- The unit is $\mathcal{X} = (\mathcal{X}, \mathcal{X}, \text{id}_{\mathcal{X}}, \text{id}_{\mathcal{X}})$.
- The opposite quiver of $\mathcal{Q} = (\mathcal{Q}, \mathcal{X}, s, t)$ is $\mathcal{Q}^{\text{op}} = (\mathcal{Q}, \mathcal{X}, t, s)$.

i.ii Groupoids.

- A groupoid is a (small) category with all arrows invertible.
- That is, a groupoid is a collection $\mathcal{G} = (\mathcal{G}, \mathcal{X}, s, t, m, \text{id}, \iota)$, where
 - $\mathcal{G} = (\mathcal{G}, \mathcal{X}, s, t)$ is a quiver,
 - $m : \mathcal{G}_s \times_t \mathcal{G} \rightarrow \mathcal{G}$, $\text{id} : \mathcal{X} \rightarrow \mathcal{G}$ and $\iota : \mathcal{G} \rightarrow \mathcal{G}^{\text{op}}$ are morphisms in $\text{Quiv}_{\mathcal{X}}$, such that

m is associative with identity id and inverse ι ; it is denoted by composition, not by juxtaposition:



Occasionally we denote the groupoid by $\mathcal{G} \rightrightarrows \mathcal{X}$ and say that $\mathcal{G}$ is a groupoid *over* $\mathcal{X}$.

The definitions of morphisms of groupoids and subgroupoids are standard; a subgroupoid $\mathcal{H}$ over $\mathcal{Y}$ is *wide* when $\mathcal{Y} = \mathcal{X}$.

If $x, y \in \mathcal{X}$, then set $\mathcal{G}(x, y) = s^{-1}(x) \cap t^{-1}(y)$ (the Hom in the category $\mathcal{G}$), $x \sim y$ whenever $\mathcal{G}(x, y) \neq \emptyset$, and $\mathcal{G}^x = \mathcal{G}(x, x)$. Thus

- $\sim$ is an equivalence relation;
- $\mathcal{G}^x$ is a group;
- if $x \sim y$, then $\mathcal{G}(x, y)$ is a torsor over the group $\mathcal{G}^x$;
- $\mathcal{G}$ is determined by $\sim$ and the (isomorphism class of) the groups $\mathcal{G}^x$, one x for each class of $\sim$.

The groupoid $\mathcal{G}$ is connected if $x \sim y$ for all $x, y \in \mathcal{X}$;

- any groupoid is a disjoint union of connected ones, namely the subgroupoids supported by classes of $\sim$.

i.iii Coxeter groups.

○ A Coxeter matrix of size θ is a symmetric matrix $\mathbf{m} = (m_{ij})_{i,j \in \mathbb{I}_\theta}$ with entries in $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ such that

$$m_{ii} = 1 \quad \forall i \in \mathbb{I}_\theta, \quad m_{ij} \geq 2, \quad \forall i \neq j \in \mathbb{I}_\theta$$

○ The associated Coxeter system is the pair (W, S) , where W is the group presented by generators $S = \{s_1, \dots, s_\theta\}$ and relations

$$(s_i s_j)^{m_{ij}} = e, \quad i, j \in \mathbb{I}_\theta.$$

Here $(s_i s_j)^\infty$ means that there is no relation.

By abuse of notation, W is called a Coxeter group.

The classification of the finite Coxeter groups is well-known.

- Every Coxeter group is the restricted direct product of irreducible Coxeter groups [Bourbaki, Ch. IV, §1, n. 8].
- The list of all finite irreducible Coxeter groups (up to isomorphisms), [Bourbaki, Ch. VI, §4, Th. 1 and 2], is :
 - the Weyl groups of the simple finite-dimensional complex Lie algebras (B_n and C_n have isomorphic Weyl groups);
 - the dihedral groups $\mathbb{D}_n$, $n \neq 3, 4, 6$ (because $W(A_2) \simeq \mathbb{D}_3 \simeq \mathbb{S}_3$, $W(B_2) \simeq \mathbb{D}_4$, $W(G_2) \simeq \mathbb{D}_6$);
 - H_4 and H_5 .

i.iv Coxeter groupoids. Let $\mathbb{I}$ be a finite non-empty set.

- A *basic datum* is a pair $(\mathcal{X}, \rho)$ where $\mathcal{X} \neq \emptyset$ $\rho : \mathbb{I} \rightarrow \mathbb{S}_{\mathcal{X}}$ such that $\rho_i^2 = \text{id}$ for all $i \in \mathbb{I}$.

The notion of basic datum is equivalent to that of colored multi-graphs; it can be expressed via colored quivers: $(\mathcal{X}, \rho) \mapsto \mathcal{Q}_{\rho}$.

- A *Coxeter datum* is a triple $(\mathcal{X}, \rho, \mathbf{M})$, where $(\mathcal{X}, \rho)$ is a basic datum and $\mathbf{M} = (\mathbf{m}^x)_{x \in \mathcal{X}}$, $\mathbf{m}^x = (m_{ij}^x)_{i,j \in \mathbb{I}}$, is a family of Coxeter matrices such that

$$m_{ij}^x = m_{ij}^{\rho_i(x)}, \quad x \in \mathcal{X}, \quad i, j \in \mathbb{I}. \quad (1)$$

Definition. [Heckenberger-Yamane]

The *Coxeter groupoid* $\mathcal{W} = \mathcal{W}(\mathcal{X}, \rho, \mathbf{M})$ associated to $(\mathcal{X}, \rho, \mathbf{M})$ is the groupoid generated by $\mathcal{Q}_{\rho}$ with relations

$$(\sigma_i^x \sigma_j)^{m_{ij}^x} = \text{id}_x, \quad i, j \in \mathbb{I}, x \in \mathcal{X},$$

ii. The Weyl groupoid.

ii.i The case of a semisimple Yetter-Drinfeld module.

Let H be a Hopf algebra, $\theta \in \mathbb{Z}_{\geq 2}$ and $\mathcal{C}_\theta$ the class of all tuples

$$M = (M_1, M_2, \dots, M_\theta)$$

such that the M_i 's are simple in ${}^H_H\mathcal{YD}_{fd}$. Such a tuple gives rise to a semisimple $W \in {}^H_H\mathcal{YD}_{fd}$:

$$W = M_1 \oplus M_2 \oplus \dots \oplus M_\theta.$$

Let $(\alpha_1, \dots, \alpha_\theta)$ be the canonical basis of $\mathbb{Z}^\theta$. Define a $\mathbb{Z}^\theta$ -grading on W by $\deg M_j = \alpha_j$ for all $j \in \mathbb{I}_\theta$.

Then $T(W)$ and $\mathcal{B}(W)$ are $\mathbb{Z}^\theta$ -graded algebras in ${}^H_H\mathcal{YD}$.

Recall that $\mathcal{A}(W) := \mathcal{B}(W) \# H$, etc.; these are also $\mathbb{Z}^\theta$ -graded algebras by declaring $\deg H = 0$.

We fix $i \in \mathbb{I}_\theta$ and set

$$V = M_i, \quad U = \bigoplus_{j \in \mathbb{I}_\theta, j \neq i} M_j.$$

Thus $W \simeq U \oplus V$ and we have

$$\mathcal{A}(W) \simeq \mathcal{B}(Z_U) \# \mathcal{A}(V), \quad \mathcal{B}(W) \simeq \mathcal{B}(Z_U) \# \mathcal{B}(V)$$

where $Z_U = \text{ad}_c \mathcal{B}(V)(U)$. Then, consider for $j \in \mathbb{I}_\theta \setminus \{i\}$:

$$L_j := \text{ad } \mathcal{B}(V)(M_j), \quad \text{so that} \quad Z_U \simeq \bigoplus_{j \in \mathbb{I}_\theta \setminus \{i\}} L_j.$$

By definition L_j is $\mathbb{Z}^\theta$ -graded with $\text{supp } L_j \subseteq \alpha_j + \mathbb{Z}_{\geq 0} \alpha_i$. Also,

$$L_j = \bigoplus_{k \geq 0} L_j^k, \quad \text{where} \quad L_j^k = L_j \cap \mathcal{B}^k(W).$$

Assume that

$$\dim L_j < \infty \quad \forall j \in \mathbb{I}_\theta \setminus \{i\}. \quad (\mathcal{F}_i)$$

(this holds for instance if $\dim \mathcal{B}(W) < \infty$). Then there exist m_j s.t. $L_j = \bigoplus_{k \in \mathbb{I}_{0,m_j}} L_j^k$ and $L_j^{m_j} \neq 0$. Now define

$$-a_{ij}^M := \sup\{h \in \mathbb{N}_0 \mid \alpha_j + h\alpha_i \in \text{supp } L_j\}, \quad i \neq j, \quad a_{ii}^M = 2.$$

Let $s_{i,M} \in GL(\theta, \mathbb{Z})$ be given by

$$s_{i,M}(\alpha_j) = \alpha_j - a_{ij}^M \alpha_i, \quad j \in \mathbb{I}_\theta;$$

$s_{i,M}$ is a reflection, i.e., $s_{i,M}^2 = \text{id}$ and the rank of $\text{id} - s_{i,M}$ is 1.

Lemma. • L_j is a simple object in ${}_{A(V)}^{A(V)}\mathcal{YD}$.

• $L_j^{m_j}$ is a simple object in ${}_H^H\mathcal{YD}$ that generates L_j in $\mathcal{B}(V)\mathcal{M}$.

Define $\mathcal{R}_i(M) := (M'_1, \dots, M'_\theta)$, (the i -th reflection of M), by

$$M'_j = \begin{cases} L_j^{m_j} & \text{if } j \neq i, \\ M_i^* = V^* & \text{if } j = i. \end{cases}$$

Because of the Lemma, $\mathcal{R}_i(M) \in \mathcal{C}_\theta$.

Let $W' = \bigoplus_{j \in \mathbb{I}_\theta} M'_j$; it is $\mathbb{Z}^\theta$ -graded by $\deg M'_j = s_{i,M}(\alpha_j)$ $j \in \mathbb{I}_\theta$.

Lemma. The tuple $\mathcal{R}_i(M)$ satisfies Condition $(\mathcal{F}_i)$,
 $s_{i,\mathcal{R}_i(M)} = s_{i,M}$ and $\mathcal{R}_i^2(M) \simeq M$.

Let M and $M' \in \mathcal{C}_\theta$. We say that

$$M \sim M'$$

if there exists $i \in \mathbb{I}_\theta$ such that Condition (F_i) holds for M and $\mathcal{R}_i(M) \simeq M'$. By the Lemma, the relation $\sim$ is symmetric. The equivalence relation $\approx$ generated by $\sim$ is called *Weyl equivalence*.

Definition. Let $M \in \mathcal{C}_\theta$. The Weyl groupoid $\mathcal{W}(M)$ has base (class of points) the equivalence class $\mathfrak{W}(M)$ of M under $\approx$, while the arrows are the compositions of the arrows

$$\underset{\bullet}{M'} \xrightarrow{s_{i,M'}} \underset{\bullet}{\mathcal{R}_i(M')},$$

for $M' \in \mathfrak{W}(M)$ satisfying (F_i) . Here $s_{i,M'} \in \text{Aut}(\mathbb{Z}^\theta)$ is the reflection $s_{i,M'}(\alpha_j) = \alpha_j - a_{ij}^{M'} \alpha_i$ for all $j \in \mathbb{I}_\theta$.

Let $M = (M_1, M_2, \dots, M_\theta) \in \mathcal{C}_\theta$ and $W = M_1 \oplus M_2 \oplus \dots \oplus M_\theta$.

Lemma. If $M' = (M'_1, M'_2, \dots, M'_\theta) \approx M$ and $W' = M'_1 \oplus M'_2 \oplus \dots \oplus M'_\theta$, then

$$\dim \mathcal{B}(W) = \dim \mathcal{B}(W').$$

Definition. The set of real roots of M is

$$\Delta^{re}(M) = \{w(\alpha_j) \mid w \in \text{Hom}(M', M), M' \in \mathfrak{W}(M)\} \subset \mathbb{Z}^\theta.$$

Definition. $\mathcal{B}(M)$ is *decomposable* if there exists a family $(W_\ell)_{\ell \in L}$, where L is a totally ordered set and the $W_\ell \in {}^H_H\mathcal{YD}_{fd}$ are irreducible, such that

$$\mathcal{B}(W) \simeq \bigotimes_{\ell \in L} \mathcal{B}(W_\ell).$$

If this is the case, then $\dim \mathcal{B}(W) < \infty$ if and only if L is finite and $\dim \mathcal{B}(W_\ell) < \infty$ for all $\ell \in L$.

Theorem. [Heckenberger-Schneider] If all finite tensor powers of W are semisimple objects in ${}^H_H\mathcal{YD}$, then $\mathcal{B}(W)$ is decomposable.

Example. If H is semisimple, or if W is of diagonal type (= all the M_i 's have dimension 1), then $\mathcal{B}(W)$ is decomposable.

ii.ii Applications to diagonal type.

Assume that $\text{char } \mathbb{k} = 0$. Let $\mathfrak{q} = (q_{ij})_{i,j \in \mathbb{I}_\theta}$, with $q_{ij} \in \mathbb{k}^\times$ and $q_{ii} \in \mathbb{G}_\infty \setminus \{1\}$. Let $(V, c^{\mathfrak{q}})$ be a braided vector space with a basis $(v_i)_{i \in \mathbb{I}_\theta}$ of V such that

$$c(v_i \otimes v_j) = q_{ij} v_j \otimes v_i, \quad i, j \in \mathbb{I}_\theta.$$

We assume that the Dynkin diagram is connected.

In this case the Weyl groupoid gives enough information:

Theorem. [Heckenberger] $\dim \mathcal{B}(V) < \infty \iff \mathfrak{q} \in \text{List}$.

Another important result is the following.

Theorem. [Angiono] If $\dim \mathcal{B}(V) < \infty$, then the defining relations of $\mathcal{B}(V)$ are explicitly known (essentially powers of root vectors and generalized quantum Serre relations).

