Pointed Hopf algebras over simple groups
III. Computing Nichols algebras

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i. **First methods.**

Throughout this section, H is a Hopf algebra and (V, c) is a finite dimensional braided vector space realizable in ${}^H_H\mathcal{YD}$. Recall the Nichols algebra $\mathscr{B}(V) = \mathscr{B}(V, c) = T(V)/\widetilde{I}(V)$, where

$$I(V) = \sum_{I \in \mathfrak{S}} I = \widetilde{I}(V) = \sum_{J \in \widetilde{\mathfrak{S}}} J = \bigoplus_{n \geq 0} \ker \Omega^n$$

Here Ω^n is the (image of) the quantum symmetrizer and

$$\mathfrak{S} = \{I \subset \oplus_{n \geq 2} T^n(V) : I \text{ is a homogeneous ideal and coideal}\},$$
$$\widetilde{\mathfrak{S}} = \{I \in \mathfrak{S} : I \text{ is a Yetter-Drinfeld submodule of } T(V)\}.$$

Remark. A morphism of braided vector spaces $\phi : (V, c) \to (W, c)$ induces a morphism of braided Hopf algebras $\Phi : \mathscr{B}(V) \to \mathscr{B}(W)$, e.g., because the induced morphism of braided Hopf algebras $T(\phi) : T(V) \to T(W)$ intertwines the respective actions of the braid groups. Furthermore,

♦ if ϕ is injective, then $\Phi : \mathscr{B}(V) \to \mathscr{B}(W)$ is injective, because the image of Φ, which is the subalgebra of $\mathscr{B}(W)$ generated by V, is a pre- and post-Nichols algebra.

♦ if ϕ is surjective, then $\Phi : \mathscr{B}(V) \to \mathscr{B}(W)$ is surjective, because $\mathscr{B}(W)$ is generated by $W = \phi(V)$.

Thus, if $V \hookrightarrow W$ is a braided subspace (not necessarily a Yetter-Drinfeld submodule) such that $\dim \mathscr{B}(V) = \infty$, then $\dim \mathscr{B}(W) = \infty$.

i.i Brute force.

The first problem in the study of Nichols algebras is that we ignore a priori when the ideal $I(V)$ is finitely generated (and it could actually happen that it is not so; the second, that we ignore the degrees of a minimal set of homogeneous generators (that could be arbitrarily large).

To overcome this obstacle, we can appeal to the following basic observation: Let $I \in \widetilde{\mathfrak{S}}$; then $\mathcal{B} = T(V)/I$ is a pre-Nichols algebra of V, hence there is a surjective map $\mathcal{B} \twoheadrightarrow \mathscr{B}(V)$. Therefore

$$\text{GK-dim}\, \mathscr{B}(V) \leq \text{GK-dim}\, \mathcal{B},$$

in particular $\dim \mathcal{B} < \infty$ implies $\dim \mathscr{B}(V) < \infty$.

More systematically, we propose:

Definition. Let $d \in \mathbb{Z}_{\geq 2}$. The d-th approximation of $\mathscr{B}(V)$ is the pre-Nichols algebra $\mathscr{B}_d(V) := T(V)/I_d(V)$ where $I_d(V)$ is the ideal of $T(V)$ generated by

$$\bigoplus_{0 \leq j \leq d} I^j(V) = \bigoplus_{0 \leq j \leq d} \ker \Omega^j.$$

It is easy to see that $I_d(V) \in \widetilde{\mathfrak{S}}$, hence $\mathscr{B}_d(V)$ is a pre-Nichols algebra of V and $\operatorname{GK-dim} \mathscr{B}(V) \leq \operatorname{GK-dim} \mathscr{B}_d(V)$, in particular $\dim \mathscr{B}_d(V) < \infty$ implies $\dim \mathscr{B}(V) < \infty$.
For instance, the quadratic approximation is

$$\mathscr{B}_2(V) := T(V)/\ker\langle(\mathrm{id} + c)\rangle;$$

the cubic approximation is

$$\begin{aligned} \mathscr{B}_3(V) = T(V)/\langle \ker\langle(\mathrm{id} + c) \\ + \ker(\mathrm{id} + c_1 + c_2 + c_1c_2 + c_2c_1 + c_1c_2c_1)\rangle, \text{ etc.} \end{aligned}$$

II.i.ii Bilinear forms and derivations

Let (W, c) be a braided vector space provided with non degenerate bilinear form $(\,|\,) : V \otimes W \to \Bbbk$ satisfying

$$(c(v_1 \otimes v_2)|w_1 \otimes w_2) = (v_1 \otimes v_2|c(w_1 \otimes w_2))\,.$$

Here and below we extend $(\,|\,)$ to $(\,|\,) : T(V) \otimes T(W) \to \Bbbk$ by

$$\begin{aligned} (1|1) &= 1, \\ (T^n(V)|T^m(W)) &= 0, && \text{if } n \neq m, \\ (v_1 \otimes \ldots \otimes v_n|w_1 \otimes \ldots \otimes w_n) &= \prod_{i \in \mathbb{I}_n} (v_i|w_i), \end{aligned}$$

if $v_1, \ldots v_n \in V$, $w_1, \ldots w_n \in W$. Clearly, this is again non degenerate and $(\sigma \cdot x|y) = (x|\sigma \cdot y)$ for all $x \in T^n(V)$, $y \in T^n(W)$, $\sigma \in \mathbb{B}_n$, $n \geq 2$.

Set $(\!|\ |\ |\!) : T(V) \otimes T(W) \to \Bbbk$ by $(\!| x \,|\, y |\!) := (x|\Omega(y)) = (\Omega(x)|y)$, for $x \in T(V)$, $y \in T(W)$, i.e.,

$$\begin{aligned}
(\!| 1 \,|\, 1 |\!) &:= 1, && \\
(\!| v \,|\, w |\!) &:= (v|w), && v \in V, w \in W, \\
(\!| T^n(V) \,|\, T^m(W) |\!), &:= 0 && \text{if } n \neq m, \\
(\!| x \,|\, y |\!) &:= (x|\Omega^n(y)) = (\Omega^n(x)|y), && x \in T^n(V), y \in T^n(W), n \geq 2.
\end{aligned}$$

Clearly, the radicals of the form $(\!|\ |\ |\!)$ coincide with the defining ideals of the Nichols algebras:

$$\begin{aligned}
\mathrm{rad}_{\mathrm{left}}(\!|\ |\ |\!) &= \{x \in T(V) \mid (\!| x \,|\, y |\!) = 0 \ \forall y \in T(W)\} \\
&= \oplus_{n \geq 0} \ker(\Omega^n_{|T^n(V)} = I(V), \\
\mathrm{rad}_{\mathrm{right}}(\!|\ |\ |\!) &= \{y \in T(W) \mid (\!| x \,|\, y |\!) = 0 \ \forall x \in T(V)\} \\
&= \oplus_{n \geq 0} \ker(\Omega^n_{|T^n(W)} = I(W).
\end{aligned}$$

Thus $(\!|\ |\ |\!)$ induces a bilinear form $(\!|\ |\ |\!) : \mathscr{B}(V) \otimes \mathscr{B}(W) \to \Bbbk$, which is non-degenerate.

Application. This description of $\mathscr{B}(V)$ as $T(V)/\operatorname{rad}_{\mathrm{left}}(\!|\ |\ |\!)$ allows to interpret the algebra $\mathbf{f}$ in [Lusztig, Introduction to quantum groups] as a Nichols algebra.

Proposition. For $x, u \in \mathscr{B}(V)$ and $y, z \in \mathscr{B}(W)$, we have

$$(\!| x \mid y \cdot z |\!) = (\!| x^{(1)} \mid y |\!)\, (\!| x^{(2)} \mid z |\!),$$
$$(\!| x \cdot u \mid y |\!) = (\!| x \mid y^{(1)} |\!)\, (\!| u \mid z^{(2)} |\!),$$

Sketch of the proof. Below, $i + j = n$. Recall

$$\mathfrak{S}^n_{i,j} := \sum_{\sigma \in X^n_{i,j}} M_n(\sigma) \in \Bbbk\mathbb{B}_n$$

where $X^n_{i,j} \subset \mathbb{S}_n$ is the set of all (i,j)-shuffles. Let $\Omega_{i,j} := \varrho_n(\mathfrak{S}_{i,j})$. It can be shown that

$$\Omega^n = (\Omega^i \otimes \Omega^j)\Omega_{i,j}.$$

Recall that the (i,j)-graded component of the comultiplication Δ: $\Delta_{i,j} : C(i+j) \to C(i) \otimes C(j)$, $i, j \geq 0$, is given by $\Delta_{i,j} = \Omega_{i,j}$. We use a Sweedler-like notation:

$$\Omega_{i,j}(x) = \Omega_{i,j}(x)_{(i)} \otimes \Omega_{i,j}(x)_{(j)} \in T^i(V) \otimes T^j(V),$$

for $x \in T^n(V)$.

Hence, for $x \in T^n(V)$, $y \in T^i(W)$ and $z \in T^j(W)$, we have

$$\begin{aligned}(\!|x \,|\, y \cdot z|\!) &= (\Omega^n(x) \,|\, y \cdot z) = ((\Omega^i \otimes \Omega^j)\Omega_{i,j}(x) \,|\, y \cdot z) \\ &= \left(\Omega^i \Omega_{i,j}(x)_{(i)} \,|\, y\right) \left(\Omega^j \Omega_{i,j}(x)_{(j)} \,|\, z\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}(\!|x^{(1)} \,|\, y|\!)\, (\!|x^{(2)} \,|\, z|\!) &= \sum_{k+\ell=n} (\!|\Omega_{k,\ell}(x)_{(k)} \,|\, y|\!)\, (\!|\Omega_{k,\ell}(x)_{(\ell)} \,|\, z|\!) \\ &= (\!|\Omega_{i,j}(x)_{(i)} \,|\, y|\!)\, (\!|\Omega_{i,j}(x)_{(j)} \,|\, z|\!) \\ &= \left(\Omega^i \Omega_{i,j}(x)_{(i)} \,|\, y\right) \left(\Omega^j \Omega_{i,j}(x)_{(j)} \,|\, z\right).\end{aligned}$$

Skew derivations.

Here is a useful tool to verify that some $r \in \mathscr{B}^n(V)$ is not 0.

For $f \in V^*$ we set

$$\partial_f = (\mathrm{id} \otimes f)\Delta^{n-1,1} : \mathscr{B}^n(V) \to \mathscr{B}^{n-1}(V).$$

Fix a basis $(x_i)_{i\in\mathbb{I}}$ of V and let $(f_i)_{i\in\mathbb{I}}$ be its dual basis. Set $\partial_i = \partial_{f_i}$, $i \in \mathbb{I}$.

Suppose that there is a family $(g_i)_{i\in\mathbb{I}}$ in $G(H)$ such that $\delta(x_i) = g_i \otimes x_i$, for $i \in \mathbb{I}$. Then

$$\partial_i(xy) = x\partial_i(y) + \partial_i(x)\, g_i \cdot y, \qquad x, y \in \mathscr{B}(V), \qquad i \in \mathbb{I}.$$

Poincaré duality. Let now $\mathcal{R} = \oplus_{n\geq 0}\mathcal{R}^n$ be a connected graded, locally finite, Hopf algebra in ${}^H_H\mathcal{YD}$. Then $\dim \mathcal{R} < \infty$ if and only if there exists $M \in \mathbb{Z}_{\geq 1}$ s.t.:

$$\mathcal{R}^M \neq 0 \qquad \text{and} \qquad \mathcal{R}^{M+j} = 0 \ \forall j \in \mathbb{Z}_{>0}.$$

Lemma. $\dim \mathcal{R}^M = 1$ and $\dim \mathcal{R}^i = \dim \mathcal{R}^{M-i}$ for all $i \in \mathbb{I}_{0,M}$.

Sketch of the proof. (i) Let $\Lambda \in \mathcal{R}^M \backslash 0$. Then

$$x\Lambda = 0 = \varepsilon(x)\Lambda = \Lambda x, \qquad \forall x \in \mathcal{R}^i, i \in \mathbb{I}_M;$$

while if $x \in \mathcal{R}^0 = \Bbbk$, then $x\Lambda = \varepsilon(x)\Lambda = \Lambda x$. Hence Λ is an integral of R; but the space of integrals of a Hopf algebra in ${}^H_H\mathcal{YD}$ has dimension ≤ 1. Hence $\dim \mathcal{R}^M = 1$.

(ii) Now pick a non-zero element $\int \in (\mathcal{R}^*)^M$; this is an integral in R^*, hence the bilinear form $(\,|\,) : \mathcal{R} \times \mathcal{R} \to \Bbbk$ given by

$$(x|y) = \langle \textstyle\int, xy \rangle$$

is non-degenerate. Observe that

$$(x|y) = 0 \text{ if } x \in \mathcal{R}^d,\ y \in \mathcal{R}^e,\ d + e \neq M.$$

Thus, the restriction of the bilinear form $(\,|\,)$ to $\mathcal{R}^d \times \mathcal{R}^{M-d}$ is non-degenerate, implying the claim.

A rough algorithm:

Compute the pre-Nichols algebra $\mathcal{R} = \mathscr{B}_2(V)$; i.e., compute first $I_2(V) = \ker(\mathrm{id} + c)$ and then try to compute the homogeneous components $\mathcal{R}^n$ of $\mathcal{R}$. If lucky to find that $\mathcal{R}^{M+1} = 0$, then $\dim \mathcal{R} < \infty$. Thus, $\dim \mathcal{R}^M = 1$. Check with skew derivations if $\mathcal{R} = \mathscr{B}(V)$.

If not lucky, proceed with the cubic approximation $\mathcal{R} = \mathscr{B}_3(V)$ . . . and so on.

Example.

Let $n \in \mathbb{Z}_{\geq 3}$ and let V_n be the vector space with basis y_τ, $\tau \in \mathbb{S}_n$ a transposition $\tau = (i, j)$, $i \neq j$. Then $V \in {}^{\Bbbk\mathbb{S}_n}_{\Bbbk\mathbb{S}_n}\mathcal{YD}$ by

$$\delta(y_\tau) = \tau \otimes y_\tau, \qquad\qquad \sigma \rightharpoonup y_\tau = \mathrm{sgn}(\sigma) y_{\sigma\tau\sigma^{-1}}.$$

The ideal I generated by $\ker(\Omega^2)$ is generated by the elements

$$\begin{aligned} y_\tau^2 &\quad \forall\tau, &(1)\\ y_\tau y_{\tau'} + y_{\tau'} y_\tau &\quad \text{if } \tau\tau' = \tau'\tau, &(2)\\ y_\tau y_{\tau'} + y_{\tau'} y_{\tau''} + y_{\tau''} y_\tau &\quad \text{if } \tau\tau' = \tau''\tau. &(3)\end{aligned}$$

Let $\mathcal{R}(n) := T(V_n)/I$, a Hopf algebra in ${}^{\Bbbk\mathbb{S}_n}_{\Bbbk\mathbb{S}_n}\mathcal{YD}$.

Theorem. $\mathcal{R}(3) \simeq \mathscr{B}(V_3)$ has dimension 12.

Set $y_0 = (12)$, $y_1 = (23)$ and $y_2 = (23)$. By direct computations using the relations we have that

$$
\begin{aligned}
y_0y_1y_0 &= -y_1y_2y_0 = y_1y_0y_1 = -y_0y_2y_1,\\
y_0y_1y_2 &= -y_0y_2y_0 = y_2y_1y_0 = -y_2y_0y_2,\\
y_1y_0y_2 &= -y_2y_1y_2 = y_2y_0y_1 = -y_1y_2y_1,
\end{aligned}
$$

and the other monomials in degree 3 vanish since in all of them appears y_i^2 for some i. This in turn implies

$$
\begin{aligned}
y_0y_1y_0y_2 &= -y_1y_2y_0y_2 = y_1y_0y_1y_2 = -y_0y_2y_1y_2\\
&= y_0y_2y_0y_1 = -y_0y_1y_2y_1 = -y_2y_1y_0y_1\\
&= y_2y_0y_2y_1 = -y_2y_0y_1y_0 = -y_1y_0y_2y_0\\
&= y_2y_1y_2y_0 = y_1y_2y_1y_0,
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
y_0y_1y_0y_1 &= y_1y_2y_0y_1 = y_1y_0y_1y_0 = y_0y_2y_1y_0 \\
&= y_0y_1y_2y_0 = y_2y_0y_2y_0 = y_0y_2y_0y_2 \\
&= y_2y_1y_0y_2 = y_1y_0y_2y_1 = y_2y_1y_2y_1 \\
&= y_2y_0y_1y_2 = y_1y_2y_1y_2 = 0,
\end{aligned}
$$

and the other monomials in degree 4 vanish since in all of them appears y_i^2 for some i. Moreover, the monomials in (4) are annihilated by multiplying them with any of the y_i, and then

$$\mathcal{R}(3)^n = 0 \ \forall n \geq 5.$$

With this, we get the set of generators of R_3^2 consisting of

$$\{1,\ y_0,\ y_1,\ y_2,\ y_0y_1,\ y_1y_2,\ y_0y_2,\ y_1y_0,\ y_0y_1y_0,\ y_0y_1y_2,\ y_1y_0y_2,\ y_0y_1y_0y_2\}. \quad (5)$$

It can be proved that this set is indeed a basis by various methods (it is enough to check that $\mathcal{R}(3)^4 \neq 0$ and $\dim \mathcal{R}(3)^3 = 4$).

We check now that $\mathcal{R}(3) \simeq \mathscr{B}(V_3)$. Since $I \subseteq \ker \Omega$, there exists a surjective map $\pi : \mathcal{R}(3) \twoheadrightarrow \mathscr{B}(V_3)$. Let N be such that

$$\mathscr{B}^N(V_3) \neq 0, \qquad \mathscr{B}^i(V_3) = 0 \qquad \forall i > N.$$

By Poincaré duality, $\dim \mathscr{B}(V_3)^N = 1$, $\dim \mathscr{B}^i(V_3) = \dim \mathscr{B}^{N-i}(V_3)$. We have the possibilities:

$N = 4$, and then $\dim \mathscr{B}^3(V_3) = \dim V_3 = 3$, hence π is an isomorphism unless $\dim \mathscr{B}^2(V_3) < 4$.

$N = 3$, and then $\dim \mathscr{B}^2(V_3) = \dim \mathscr{B}^1(V_3) = 3$.

$N = 2$, and then $\dim \mathscr{B}^2(V_3) = \dim \mathscr{B}^0(V_3) = 1$.

We see that in any case π is an isomorphism unless $\dim T(2) < 4$, but $\dim \mathscr{B}^2(V_3)$ is the codimension of $\ker \Omega^2$, which is 4.

Theorem. $\mathcal{R}(4) \simeq \mathscr{B}(V_4)$ has dimension $576 = 24^2$.

Theorem. $\mathcal{R}(5) \simeq \mathscr{B}(V_5)$ has dimension 8.294.400.

Problem.

$\mathcal{R}(6) \simeq \mathscr{B}(V_6)$?

$\dim \mathcal{R}(6) < \infty$?

$\dim \mathscr{B}(V_6) < \infty$?

i.ii. Cocycle deformations and twisting.

Recall that if (A, μ) is an algebra and (C, Δ) is a coalgebra, the map $* : \mathrm{hom}(C, A) \times \mathrm{hom}(C, A) \to \mathrm{hom}(C, A)$ (called the convolution product), given by

$$T * S := \mu \circ (T \otimes S) \circ \Delta,$$

is an associative multiplication on $\mathrm{hom}(C, A)$ with unit $u\varepsilon$.

Definition. A linear map $\phi : H \otimes H \to \Bbbk$, which is is invertible with respect to the convolution, is a unitary 2-cocycle if

$$\phi(x_{(1)} \otimes y_{(1)})\, \phi(x_{(2)} y_{(2)} \otimes z) = \phi(y_{(1)} \otimes z_{(1)})\, \phi(x \otimes y_{(2)} z_{(2)}),$$
$$\phi(x \otimes 1) = \phi(1 \otimes x) = \varepsilon(x),$$

for all $x, y, z \in H$.

If ϕ is a unitary 2-cocycle ϕ, then the multiplication $\cdot_\phi$ given by

$$x \cdot_\phi y = \phi(x_{(1)} \otimes y_{(1)})\, x_{(2)} y_{(2)}\, \phi^{-1}(x_{(3)} \otimes y_{(3)}), \qquad x, y \in H,$$

is associative and unital with the same unit as H.
Let $H_\phi = (H, \cdot_\phi, \Delta)$, with the new multiplication and the given comultiplication.

Lemma. H_ϕ is a Hopf algebra.

Exercise. Let G be a group. A unitary 2-cocycle on $\Bbbk G$ is determined by a cocycle $\phi \in Z^2(G, \Bbbk^\times)$, i.e., a map $\phi : G \times G \to \Bbbk^\times$ such that

$$\begin{aligned} \phi(g,h)\phi(gh,t) &= \phi(h,t)\phi(g,ht), \\ \phi(g,e) = \phi(e,g) &= 1, \qquad\qquad g,h,t \in G. \end{aligned}$$

Theorem. [Majid-Oeckl, Theorem 2.7, Corollary 3.4]
Let $\phi : H \otimes H \to \Bbbk$ be an invertible unitary 2-cocycle.

(a) There is an equivalence $\mathcal{T}_\phi : {}^H_H\mathcal{YD} \to {}^{H_\phi}_{H_\phi}\mathcal{YD}$ of braided categories, $V \mapsto V_\phi$, which is the identity on the underlying vector spaces, morphisms and coactions, and transforms the action of H on V to $\cdot_\phi : H_\phi \otimes V_\phi \to V_\phi$, given for $h \in H_\phi$, $v \in V_\phi$ by

$$h \cdot_\phi v = \phi(h_{(1)}, v_{(-1)})(h_{(2)} \cdot v_{(0)})_{(0)}\, \phi^{-1}((h_{(2)} \cdot v_{(0)})_{(-1)}, h_{(3)}).$$

The monoidal structure on $\mathcal{T}_\phi$ is given by the natural transformation $b_{V,W} : (V \otimes W)_\phi \to V_\phi \otimes W_\phi$

$$b_{V,W}(v \otimes w) = \phi(v_{(-1)}, w_{(-1)})v_{(0)} \otimes w_{(0)}, \quad v \in V, w \in W.$$

(b) $\mathcal{T}_\phi$ preserves Nichols algebras: $\mathscr{B}(V)_\phi \simeq \mathscr{B}(V_\phi)$ as objects in ${}^{H_\phi}_{H_\phi}\mathcal{YD}$. In particular, the Hilbert-Poincaré series of $\mathscr{B}(V)$ and $\mathscr{B}(V_\phi)$ are the same.

Application. We say that two matrices $\mathfrak{q} = (q_{ij})_{i,j\in\mathbb{I}_\theta}$ and $\mathfrak{q}' = (q'_{ij})_{i,j\in\mathbb{I}_\theta}$ with invertible entries are *twist-equivalent* if

$$q_{ii} = q'_{ii}, \qquad i \in \mathbb{I}_\theta \qquad \text{and} \qquad q_{ij}q_{ji} = q'_{ij}q'_{ji}, \qquad i \neq j \in \mathbb{I}_\theta.$$

Let V and V' be the braided vector spaces of diagonal type associated to twist-equivalent matrices $\mathfrak{q}$ and $\mathfrak{q}'$, respectively.

Corollary. [AS3, Proposition 3.9]
The Hilbert-Poincaré series of $\mathscr{B}(V)$ and $\mathscr{B}(V')$ coincide.

Proof. One defines a suitable cocycle ϕ on the group $\mathbb{Z}$ and applies the Theorem.

i.iii. The splitting technique.

Let H be a Hopf algebra. Let $V, U \in {}^H_H\mathcal{YD}$ and

$$W = V \oplus U;$$

this is a decomposition of W as above (any decomposition can be realized over a suitable H provided that c_W is rigid). Set

$$\mathcal{A}(W) = \mathscr{B}(W)\#H, \quad \mathcal{A}(V) = \mathscr{B}(V)\#H, \quad \mathcal{A}(U) = \mathscr{B}(U)\#H.$$

The natural maps of Hopf algebras in ${}^H_H\mathcal{YD}$

$$\pi_{\mathscr{B}(V)} : \mathscr{B}(W) \to \mathscr{B}(V) \qquad \text{and} \qquad \iota_{\mathscr{B}(V)} : \mathscr{B}(V) \to \mathscr{B}(W)$$

induce–by tensoring with id_H–morphisms of Hopf algebras

$$\begin{aligned} & \pi_{\mathcal{A}(V)} : \mathcal{A}(W) \to \mathcal{A}(V), && \pi_{\mathcal{A}(V)} := \pi_{\mathscr{B}(V)}\#\,\mathrm{id}_H, \\ \text{and} \quad & \iota_{\mathcal{A}(V)} : \mathcal{A}(V) \to \mathcal{A}(W), && \iota_{\mathcal{A}(V)} := \iota_{\mathscr{B}(V)}\#\,\mathrm{id}_H, \end{aligned}$$

Now $\pi_{\mathcal{A}(V)}\iota_{\mathcal{A}(V)} = \mathrm{id}_{\mathcal{A}(V)}$, hence by Radford-Majid we have that

$$\mathcal{K} = \mathcal{A}(W)^{\mathrm{co}\,\pi_{\mathcal{A}(V)}}$$

is a Hopf algebra in ${}^{\mathcal{A}(V)}_{\mathcal{A}(V)}\mathcal{YD}$ with the adjoint action and the coaction

$$\delta = (\pi_{\mathcal{A}(V)} \otimes \mathrm{id})\Delta_{\mathcal{A}(W)},$$

so that $\mathcal{A}(W)$ is the bosonization of $\mathcal{K}$ by $\mathcal{A}(V)$:

$$\mathcal{A}(W) \simeq \mathcal{K}\#\mathcal{A}(V).$$

Proposition. [Rosso, Proposition 22] [HS-adv, Proposition 8.6]. $\mathcal{K} \simeq \mathscr{B}(Z_U)$, where

$$Z_U := \mathrm{ad}_c\,\mathscr{B}(V)(U) \in {}^{\mathcal{A}(V)}_{\mathcal{A}(V)}\mathcal{YD}.$$

In fact, $\mathscr{B}(W)$ is the braided bosonization $\mathcal{K}\#\mathscr{B}(V)$, i.e.,

$$\mathscr{B}(W) \simeq \mathscr{B}(Z_U)\#\mathscr{B}(V).$$

This result can be used in two directions, both assuming that $\mathscr{B}(V)$ is known:

- To compute $\mathscr{B}(W)$ by computing first $\mathscr{B}(Z_U)$,
- To compute $\mathscr{B}(Z_U)$ by computing first $\mathscr{B}(W)$.