Pointed Hopf algebras over simple groups
II. Nichols algebras

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0. Overview Let H be a Hopf algebra and $V \in {}^H_H\mathcal{YD}_{fd}$. In the previous chapter we introduced the Nichols algebra as the image of a canonical map $\Omega : T(V) \to T^c(V)$ from the tensor algebra to the quantum shuffle algebra. In the first section of this chapter, we give several characterizations of Nichols algebras:

- as the only graded connected Hopf algebra in ${}^H_H\mathcal{YD}$ that is simultaneously post and pre-Nichols;
- a description of Ω in terms of quantum symmetrizers;
- in the context of Hopf bimodules;
- as a radical of a suitable bilinear form, or using skew derivations.

Despite these numerous definitions, there are currently only a few techniques available to explicitly compute some distinctive features of a Nichols algebra, such as its (Gelfand-Kirillov) dimension, its defining relations, whether it is Noetherian, etc. These techniques apply only to some specific features of some specific classes of braided vector spaces:

- braided vector spaces of diagonal type;
- braided vector spaces associated to racks.
- braided vector spaces associated to blocks.

We introduce these classes in the fourth section, leaving a comprehensive presentation of their state of the art for later chapters.

A fruitful reduction is to deal with braided Hopf algebras, i.e., Hopf algebras in the category of braided vector spaces, as we explain in the second section. A few elementary examples are discussed in the third section.

i. Nichols algebras.

We fix a Hopf algebra H (with bijective antipode) and $V \in {}^H_H\mathcal{YD}_{fd}$.

i.i (Pre- and post-)Nichols algebras. We have considered:

• The tensor algebra $T(V)$, a graded connected Hopf algebra in ${}^H_H\mathcal{YD}$ with comultiplication $\Delta : T(V) \to T(V) \otimes T(V)$ given by

$$\Delta(v) = v \otimes 1 + 1 \otimes v, \ \forall v \in V,$$

where $T(V) \otimes T(V)$ is an algebra in ${}^H_H\mathcal{YD}$ with the product altered by the braiding c.

• The quantum shuffle algebra $T^c(V) = {}^{\#}T(V^*)$ of V whose homogeneous components are $T^{(n)}(V) = T^n(V)$, but the multiplication of $T^c(V)$ is transpose to the comultiplication of $T(V)$ and vice versa.

- The algebra $T(V)$ is generated by V,
- the quantum shuffle algebra $T^c(V)$ is strictly graded,
- there exists a map $\Omega: T(V) \to T^c(V)$ of graded Hopf algebras in ${}^H_H\mathcal{YD}$ which is the identity in $V = T^1(V)$.

A graded connected Hopf algebra $\mathcal{E} = \bigoplus_{n\in\mathbb{N}_0} \mathcal{E}^n$ in ${}^H_H\mathcal{YD}$ such that $\mathcal{E}^1 \simeq V$ is a

- *pre-Nichols* algebra of V if $\mathcal{E}(1) \simeq V$ generates the algebra $\mathcal{E}$;

- *post-Nichols* algebra of V if it is strictly graded, or equivalently $\mathrm{Prim}(\mathcal{E}) = \mathcal{E}^1 \simeq V$.

In other words, $\mathcal{B}$ is a pre-Nichols algebra of V if and only if $\mathcal{B} \simeq T(V)/I$, where I is an homogeneous Hopf ideal of $T(V)$ stable by the antipode and a Yetter-Drinfeld submodule of $T(V)$ such that $I \cap \Bbbk \oplus V = 0$;

in turn, $\mathcal{R}$ is a post-Nichols algebra of V if and only if there exists an injective morphism $\mathcal{R} \to T^c(V)$ of graded Hopf algebras in ${}^H_H\mathcal{YD}$ which is the identity on V.

Proposition. There exists a unique up to isomorphisms graded connected Hopf algebra $\mathcal{E} = \bigoplus_{n\in\mathbb{N}_0} \mathcal{E}^n$ with $\mathcal{E}(1) \simeq V$ such that

- $\mathcal{E}(1) \simeq V$ generates the algebra $\mathcal{E}$ and
- $\operatorname{Prim}(\mathcal{E}) = \mathcal{E}^1 \simeq V$.

In other words, there exists a unique pre-Nichols algebra which is also post-Nichols.

Proof. We consider

$$\mathfrak{S} = \{I \subset \oplus_{n\geq 2} T^n(V) : I \text{ is a homogeneous ideal and coideal}\},$$
$$\widetilde{\mathfrak{S}} = \{I \in \mathfrak{S} : I \text{ is a Yetter-Drinfeld submodule of } T(V)\},$$
$$I(V) = \sum_{I\in\mathfrak{S}} I, \qquad \widetilde{I}(V) = \sum_{J\in\widetilde{\mathfrak{S}}} J.$$

That is, $I(V)$, respectively $\widetilde{I}(V)$, is the largest ideal in $\mathfrak{S}$, resp. $\widetilde{\mathfrak{S}}$. Let $\mathcal{E} := T(V)/\widetilde{I}(V)$ and $\pi : T(V) \to \mathcal{E}$ the natural projection. We claim that $V = \mathcal{P}(\mathcal{E})$.

Since Δ is a homogeneous map, we have that $\mathcal{P}(\mathcal{E}) = \oplus_{n\geq 1}\mathcal{P}^n(\mathcal{E})$, where $\mathcal{P}^n(\mathcal{E}) = \mathcal{P}(\mathcal{E}) \cap T^n(V)$.

Let $\mathcal{X} = \pi^{-1}\left(\oplus_{n\geq 2}\mathcal{P}^n(\mathcal{E})\right)$; this is a graded Yetter-Drinfeld submodule of $T(V)$, and

$$\Delta(x) \in x\otimes 1 + 1\otimes x + T(V)\otimes \widetilde{I}(V) + \widetilde{I}(V)\otimes T(V)$$

for all $x \in \mathcal{X}$. Hence the ideal generated by $\widetilde{I}(V)$ and $\mathcal{X}$ is in $\widetilde{\mathfrak{S}}$, $\mathcal{X} \subset \widetilde{I}(V)$ by the maximality of $\widetilde{I}(V)$, so $\pi(\mathcal{X}) = 0$ and $\mathcal{P}(\mathcal{E}) = \mathcal{P}^1(\mathcal{E}) = V$ as claimed.

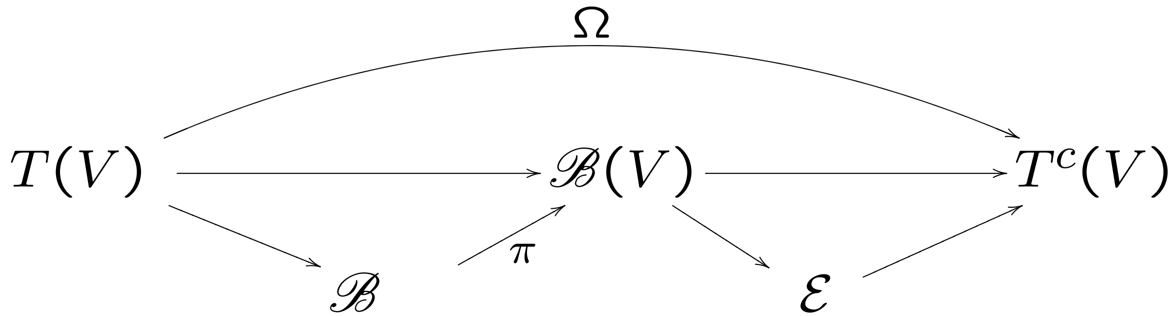
Let $R = \oplus_{n\geq 0} R(n)$ be a pre-Nichols algebra of V which is also post-Nichols. By the first assumption, there exists $I \in \widetilde{\mathfrak{S}}$ such that $R \simeq T(V)/I$; since $I \subseteq \widetilde{I}(V)$, we have a morphism $\varphi : R \to \mathcal{E}$ of Hopf algebras in ${}^H_H\mathcal{YD}$. By the second assumption and the Lemma below, φ is injective. Hence $R \simeq \mathcal{E}$ as braided Hopf algebras in ${}^H_H\mathcal{YD}$. □

Lemma. [Montgomery, Lemma 5.3.3] A morphism of pointed coalgebras which is injective in the first term of the coalgebra filtration is injective.

The Nichols algebra of V will be denoted by $\mathscr{B}(V)$.

Remark. $I(V) = \widetilde{I}(V)$, i.e., $\mathscr{B}(V) \simeq T(V)/I(V)$.

Since $I(V) \supseteq \widetilde{I}(V)$, we have a surjective map $\mathscr{B}(V) \to T(V)/I(V)$ which is injective in the first term of the coalgebra filtration by the Proposition. Then the Lemma applies.



Corollary. $\mathscr{B}(V) \simeq \operatorname{Im}\Omega$.

Remark. (1) If $\mathcal{B}$ is a pre-Nichols algebra of V, then there exists a surjective map of graded Hopf algebras $\mathcal{B} \to \mathscr{B}(V)$, which is an isomorphism of Yetter-Drinfeld modules in degree 1.
$\mathfrak{Pre}(V)$: poset of pre-Nichols, $\leq$ is $\twoheadrightarrow$; min. $T(V)$, max. $\mathscr{B}(V)$.

(2) If $\mathcal{R}$ is a post-Nichols algebra of V, then $\mathscr{B}(V)$ is isomorphic to the subalgebra $\Bbbk\langle V\rangle$ of $\mathcal{R}$ generated by V.
$\mathfrak{Post}(V)$: poset of post-Nichols, $\leq$ is $\subseteq$; min. $\mathscr{B}(V)$, max. $T^c(V)$.

i.ii Quantum symmetrizers. Notation: If $\ell < n \in \mathbb{N}_0$, then we set $\mathbb{I}_{\ell,n} = \{\ell, \ell+1, \dots, n\}$, $\mathbb{I}_n = \mathbb{I}_{1,n}$.

We now describe the map $\Omega = \sum_{n\in\mathbb{N}_0} \Omega^n : T(V) \to T^c(V)$, where $\Omega^n : T^n(V) \to T^{c,n}(V)$. Clearly, $\Omega^0 = \mathrm{id}_{\Bbbk}$ and $\Omega^1 = \mathrm{id}_V$.

The projection $\pi : \mathbb{B}_n \to \mathbb{S}_n$ sending $\sigma_i \mapsto s_i := (i, i+1)$, $i \in \mathbb{I}_{n-1}$, admits a set-theoretical section $M_n : \mathbb{S}_n \to \mathbb{B}_n$ determined by

$$M(s_i) = \sigma_i,\ i \in \mathbb{I}_{n-1},\ M(\tau\omega) = M(\tau)M(\omega),\ \text{if } \ell(\tau\omega) = \ell(\tau) + \ell(\omega),$$

where ℓ is the length of an element of $\mathbb{S}_n$ with respect to the set of generators $s_1, \dots, s_{n-1}$. The map M is called the Matsumoto section. In other words, if $\omega = s_{i_1} \dots s_{i_M}$ is a reduced expression of $\omega \in \mathbb{S}_n$, then $M(\omega) = \sigma_{i_1} \dots \sigma_{i_M}$.

We consider the following elements of the group algebra $\Bbbk\mathbb{B}_n$:

$$\mathfrak{S}_n := \sum_{\sigma \in \mathbb{S}_n} M(\sigma), \qquad \mathfrak{S}^n_{i,j} := \sum_{\sigma \in X^n_{i,j}} M(\sigma),$$

where $X^n_{i,j} \subset \mathbb{S}_n$ is the set of all (i,j)-shuffles. The element $\mathfrak{S}_n$ is called the quantum symmetrizer.

Let $\varrho_n : \Bbbk\mathbb{B}_n \to \operatorname{End} T^n(V)$ be the representation of the group algebra of $\mathbb{B}_n$ discussed in the first chapter.

Proposition. $\Omega^n = \varrho_n(\mathfrak{S}_n)$.

Since $\mathscr{B}(V) = \bigoplus_{n \geq 0} T^n(V)/\ker \Omega^n$, we have

$$\mathscr{B}^2(V) = T^2(V)/\ker(\mathrm{id} + c),$$
$$\mathscr{B}^3(V) = T^3(V)/\ker(\mathrm{id} + c_1 + c_2 + c_1c_2 + c_2c_1 + c_1c_2c_1), \text{ etc.}$$

where $c_1 = c \otimes \mathrm{id}$, $c_2 = \mathrm{id} \otimes c$.

If $C = \bigoplus_{n\geq 0} C(n)$ is a graded coalgebra with comultiplication Δ, we denote by $\Delta_{i,j} : C(i+j) \to C(i) \otimes C(j)$, $i, j \geq 0$, the (i, j)-graded component of the map Δ.

Proposition. For the coalgebra $\mathscr{B}(V)$,

$$\Delta_{i,j} = \varrho_n(\mathfrak{S}_{i,j}).$$

i.iv Hopf bimodules and Yetter-Drinfeld modules

Recall that a bimodule over an algebra A is a left and right A-module such that the actions $A \otimes M \to M$ and $M \otimes A \to M$ commute:

$$(a \cdot m) \cdot b = a \cdot (m \cdot b), \qquad \forall a, b \in A,\ m \in M.$$

Also, a bicomodule over a coalgebra C is a left and right comodule such that the coactions $\lambda : M \to C \otimes M$ and $\rho : M \to M \otimes C$ commute;

$$\begin{array}{ccc} M & \xrightarrow{\quad \rho \quad} & M \otimes C \\ {\scriptstyle \lambda} \downarrow & \circlearrowleft & \downarrow {\scriptstyle \lambda \otimes \mathrm{id}} \\ C \otimes M & \xrightarrow[\mathrm{id} \otimes \rho]{} & C \otimes M \otimes C \end{array}$$

The categories ${}_A\mathcal{M}_A$ of A-bimodules and ${}^C\mathcal{M}^C$ of C-bicomodules are tensor categories:

$$ {}_A\mathcal{M}_A \times {}_A\mathcal{M}_A \to {}_A\mathcal{M}_A, \quad (M,N) \mapsto M \otimes_A N, \quad M,N \in {}_A\mathcal{M}_A; $$

$$ {}^C\mathcal{M}^C \times {}^C\mathcal{M}^C \to {}^C\mathcal{M}^C, \quad (M,N) \mapsto M \Box_C N, \quad M,N \in {}^C\mathcal{M}^C; $$

Definition. A Hopf bimodule over a Hopf algebra H is an H-bimodule M which is also an H-bicomodule, such that the left and right coactions are morphisms of bimodules.

Lemma. The category ${}^H_H\mathcal{M}^H_H$ Hopf bimodule over a Hopf algebra H is a braided tensor one:

Lemma. There is an equivalence of braided tensor categories ${}^H_H\mathcal{M}^H_H \to {}^H_H\mathcal{YD}$ given by $M \mapsto M^{\mathrm{co}\,H} = \{m \in M : \rho(m) = m \otimes 1\}$.

Given $M \in {}^H_H\mathcal{M}^H_H$, its tensor algebra is

$$\begin{aligned} T_H(M) &= H \oplus M \oplus (M \otimes_H M) \oplus \dots \\ &= \oplus_{n\in\mathbb{N}_0} T^n_H(M), \end{aligned}$$

where $T^0_H(M) = H$, $T^1_H(M) = M$, $T^{n+1}_H(M) = T^n_H(M) \otimes_H M$.

Then $T_H(M)$ is a Hopf algebra with comultiplication extending the comultiplication of H and given by

$$\Delta(m) = m_{(-1)} \otimes m_{(0)} \otimes m_{(1)}.$$

Definition. [Nichols 1976]
The **type one bialgebra** of $M \in {}^H_H\mathcal{M}^H_H$ is the graded Hopf algebra $\mathfrak{B}(M)$ obtained as the quotient of $T_H(M)$ by the maximal graded Hopf ideal that intersects trivially $H \oplus M$.

Lemma.[Nichols 1976]
Let $V = M^{\mathrm{co}\,H}$. Then $\mathscr{B}(V) \simeq \mathfrak{B}(M)^{\mathrm{co}\,H}$.

i.ii Bilinear forms and derivations Next meeting.

ii Braided Hopf algebras

ii.1 The category of braided vector spaces

Given Hopf algebras H and K, $V \in {}^H_H\mathcal{YD}_{fd}$ and $W \in {}^K_K\mathcal{YD}_{fd}$, the associated braided vector spaces (V, c) and (W, c) might be isomorphic as such even if H, K, the actions and the coactions are quite different.

To deal with this appropppriately, Takeuchi prposed the following notion:

Definition. A braided bialgebra is collection $(A, \mu, u, \Delta, \varepsilon, c)$ s. t.
∘ (A, μ, u) is an algebra over $\Bbbk$,
∘ (A, Δ, ε) is a coalgebra over $\Bbbk$,
∘ (A, c) is a braided vector space,
∘ $A, \mu, u, \Delta, \varepsilon$ *commute* with c,
∘ $u : \Bbbk \to A$ and $\varepsilon : A \to \Bbbk$ are morphisms of algebras,
∘ $\Delta \circ \mu = (\mu \otimes \mu)(\mathrm{id} \otimes c \otimes \mathrm{id})(\Delta \otimes \Delta)$.

• A braided Hopf algebra is a braided bialgebra having an antipode (an inverse of the identity for the convolution product).

ii.ii Quasitriangular and co-quasitriangular bialgebras

Recall that a quasitriangular Hopf algebra (H, R) consists of a Hopf algebra H and an invertible element $R \in H \otimes H$, called the *R-matrix*,such that the following conditions are fulfilled:

$$\begin{aligned} (\Delta \otimes \mathrm{id})(R) &= R_{13}R_{23}, & (\epsilon \otimes \mathrm{id})(R) &= 1, \\ (\mathrm{id} \otimes \Delta)(R) &= R_{13}R_{12}, & (\mathrm{id} \otimes \epsilon)(R) &= 1, \\ \Delta^{\mathrm{cop}}(h) &= R\Delta(h)R^{-1}, & \forall h &\in H. \end{aligned}$$

Then the category of left H-modules is braided, with braiding arising from R.

Let (H, ϱ) be a *coquasitriangular* bialgebra, i.e., H is a bialgebra and $\varrho : H \otimes H \to \Bbbk$ is a convolution-invertible linear map satisfying:

$$\varrho(xy \otimes z) = \varrho(x \otimes z_{(1)})\varrho(y \otimes z_{(2)}), \tag{1}$$

$$\varrho(x \otimes yz) = \varrho(x_{(1)} \otimes z)\varrho(x_{(2)} \otimes y), \tag{2}$$

$$y_{(1)}x_{(1)}\varrho(x_{(2)} \otimes y_{(2)}) = \varrho(x_{(1)} \otimes y_{(1)})x_{(2)}y_{(2)}. \tag{3}$$

for any $x, y, z \in H$. The category ${}^H\mathcal{M}$ of left comodules over (H, ϱ) is braided; explicitly, if $V, W \in {}^H\mathcal{M}$, $v \in V$ and $w \in W$, then

$$c_{V,W}(v \otimes w) = \varrho(w_{(-1)} \otimes v_{(-1)})w_{(0)} \otimes v_{(0)}.$$

In fact, there is a braided tensor functor ${}^H\mathcal{M} \to {}^H_H\mathcal{YD}$, which thus preserves algebras and Hopf algebras: if $V \in {}^H\mathcal{M}$, then $V \in {}^H_H\mathcal{YD}$ with the given coaction and the action

$$h \cdot v = \varrho(h, v_{(-1)})v_{(0)}, \qquad h \in H,\, v \in V \tag{4}$$

This is an action by (1), the associativity holds by (2) and the Yetter-Drinfeld compatibility by (3).

ii.iii Realizations

Definition. A *realization* of a braided vector spaces (V, c) over a Hopf algebras H (or a bialgebra B) is the data of an action and a coaction of H on V such that $V \in {}^H_H\mathcal{YD}$ and the categorical braiding coincides with c.

Theorem. Let (V, c) be a braided vector space.
(1) [Fadeev-Reshetikhin-Takhtajan] There is a universal coquasitriangular bialgebra B such that $V \in {}^B\mathcal{M}$ and the categorical braiding coincides with c.

(2) [Hayashi-Schauenburg] If (V, c) is *(rigid)*, then there is a universal coquasitriangular Hopf algebra H such that $V \in {}^H\mathcal{M}$ and the categorical braiding coincides with c.

iii. Examples.

iii.i Symmetric braided vector spaces.

Let (V, c) be a braided vector space.

Definition. We say that c is a symmetry if $c^2 = \mathrm{id}$.

Lemma. Assume $\operatorname{char} \Bbbk = 0$. If c is a symmetry, then $\mathscr{B}(V) \simeq T(V)/\langle \ker \Omega^2 \rangle$; i.e., $\mathscr{B}(V)$ is quadratic.

Proof. The representation $\varrho_n : \mathbb{B}_n \to \operatorname{End} T^n(V)$ factorizes through $\mathbb{S}_n$ because c is a symmetry. Thus $\Omega^n = \sum_{\sigma \in \mathbb{S}_n} \varrho_n(\sigma)$ and

$$\ker \Omega^n = \ker \varrho_n(\mathfrak{I}^n) \qquad \text{where} \qquad \mathfrak{I}^n = \frac{1}{n!} \sum_{\sigma \in \mathbb{S}_n} \sigma,$$

which makes sense because $\operatorname{char} \Bbbk = 0$. Clearly $\mathfrak{I}^n$ is an idempotent and the usual proof of linear algebra applies.

Examples. (char $\Bbbk = 0$). Let $\tau \in GL(V^{\otimes 2})$ be the usual flip.

- If $c = \tau$, then $\mathscr{B}(V) \simeq S(V)$, the symmetric algebra of V.
- If $c = -\tau$, then $\mathscr{B}(V \simeq \Lambda(V)$, the exterior algebra V.
- If $V = V_0 \oplus V_1$ is a super vector space and $c =$ super τ, then $\mathscr{B}(V) \simeq S(V_0)\underline{\otimes}\Lambda(V_1)$, the supersymmetric algebra of V.

Observe that when char $\Bbbk = p > 0$ and $c = \tau$, one has $\mathscr{B}(V) \simeq S(V)/\langle\{v^p : v \in V\}\rangle$.

Definition. We say that c is of *Hecke-type* with label $q \in \Bbbk^{\times}$, if $(c-q)(c+1)=0$.

Lemma. Assume $\operatorname{char} \Bbbk = 0$. If c is of Hecke-type with label q, which is either 1 or not a root of 1, then $\mathscr{B}(V)$ is quadratic, i.e.,

$$\mathscr{B}(V) \simeq T(V)/\langle \ker \Omega^2 \rangle.$$

Moreover, $\mathscr{B}(V)$ is a Koszul algebra and its Koszul dual is the Nichols algebra $\mathscr{B}(V^*)$ corresponding to the braiding $q^{-1}c^t$.

Sketch of the proof. In this case, the representation of the braid group $\rho_n : \mathbb{B}_n \to \operatorname{Aut}(V^{\otimes n})$ factorizes through the Hecke algebra $\mathcal{H}_q(n)$, which is semisimple by assumption on q, for all $n \geq 0$. Again $\ker \Omega^n = \ker$ of a suitable idempotent and the proof goes as in the symmetric case.

iii.ii Decompositions Let (V, c) be a braided vector space.

A decomposition of V is a family of subspaces $(V_i)_{i\in\mathbb{I}_\theta}$ where

$$V = V_1 \oplus \cdots \oplus V_\theta, \qquad V_i \neq 0,$$
$$c(V_i \otimes V_j) = V_j \otimes V_i, \qquad i, j \in \mathbb{I}_\theta,\ \theta \geq 2.$$

We shall discuss techniques to investigate $\mathscr{B}(V)$ when V is decomposable. Here is the simplest case. Let
$c_{ij} := c_{|V_i \otimes V_j} : V_i \otimes V_j \to V_j \otimes V_i$.

Lemma. If $c_{ji}c_{ij} = \mathrm{id}_{V_i \otimes V_j}$ for $i \neq j \in \mathbb{I}_\theta$, then

$$\mathscr{B}(V) \simeq \mathscr{B}(V_1) \underline{\otimes} \mathscr{B}(V_2) \underline{\otimes} \dots \mathscr{B}(V_\theta)$$

where $\underline{\otimes}$ means the multiplication altered by c.

iii.iii Quantum lines.

Notation: $\mathbb{G}_n$ is the group of n-th roots of 1, $\mathbb{G}'_n$ is the subset of $\mathbb{G}_n$ consisting of primitive roots, $\mathbb{G}_\infty = \cup_{n\in\mathbb{N}}\mathbb{G}_n$, $\mathbb{G}'_\infty = \mathbb{G}_\infty \backslash \{1\}$. Assume that char $\Bbbk = 0$. Let $\Bbbk[q]$ be the polynomial ring in the indeterminate q. Given $n \in \mathbb{N}_0$ and $i \in \mathbb{I}_{0,n}$, we consider the elements of $\Bbbk[q]$ given by

$$(i)_q = \sum_{0\leq j\leq i-1} q^j, \quad (n)_q! = \prod_{1\leq i\leq n} (i)_q, \quad \binom{n}{i}_q = \frac{(n)_q!}{(i)_q!(n-i)_q!}.$$

Notice that

$$\text{if } q \in \mathbb{G}'_n: \quad \binom{n}{i}_q = 0,\ 0 < i < n, \text{ btt } \binom{d}{j}_q \neq 0,\ 0 \leq j \leq d < n;$$

$$\text{if } q \notin \mathbb{G}'_\infty: \quad \binom{n}{i}_q \neq 0\ \forall n, i.$$

Lemma. (Quantum binomial formula). Let A be an associative $\mathbb{k}[q]$-algebra, a and $b \in A$ such that $ba = qab$. Then

$$(a+b)^n = \sum_{1 \le i \le n} \binom{n}{i}_q a^i b^{n-i}, \qquad \text{if } n \ge 1. \tag{5}$$

By specialization, (5) holds for $q \in \mathbb{k}$. In particular, if a and b are elements of an associative algebra over $\mathbb{k}$, and q is a primitive n-th root of 1, such that $ba = qab$ then

$$(a+b)^n = a^n + b^n. \tag{6}$$

Application. Let $V = \Bbbk v$ be a braided vector space of dimension 1. Thus the braiding is just the scalar multiplication by some $q \in \Bbbk^{\times}$. Let $\Bbbk[X]$ be the polynomial ring.

Proposition. (1) If $q \in \mathbb{G}'_n$ for $n \in \mathbb{N}_{>1}$, then $\mathscr{B}(V) \simeq \Bbbk[X]/(X^n)$.

(2) Otherwise, $\mathscr{B}(V) \simeq \Bbbk[X]$.

Proof. By the quantum binomial formula for $A = T(V)\underline{\otimes}T(V)$, $a = v\otimes 1$ and $b = 1\otimes v$, since $(1\otimes v)(v\otimes 1) = q\, v\otimes v = q(v\otimes 1)(1\otimes v)$.

iii.iv Quantum linear spaces.

Let $(q_{ij})_{i,j\in\mathbb{I}_\theta}$ be a matrix with entries in $\Bbbk^\times$ such that

$$q_{ij}q_{ji} = 1, \quad i, j \in \mathbb{I}_\theta,\ i \neq j.$$

Let N_i be the order of q_{ii}, when $q_{ii} \in \mathbb{G}'_\infty$.

Let (V, c) be the braided vector space with basis $(v_i)_{i\in\mathbb{I}_\theta}$ and braiding

$$c(v_i \otimes v_j) = q_{ij}\, v_j \otimes v_i, \qquad i, j \in \mathbb{I}_\theta.$$

Proposition. $\mathscr{B}(V) \simeq \Bbbk\langle v_1, \dots, v_\theta\rangle$ modulo the relations $v_i^{N_i} = 0$, for $q_{ii} \in \mathbb{G}'_\infty$, $\qquad v_iv_j = q_{ij}v_jv_i$, $i < j \in \mathbb{I}_\theta$.

dim R is infinite unless $q_{ii} \in \mathbb{G}'_\infty$, in which case dim $R = \prod_{1\le i\le\theta} N_i$.

Proof. This follows from the case dim $V = 1$ and the result on decompositions.

Let H be a Hopf algebra.

If V is a braided vector space of dimension 1 with braiding determined by q, then any realization of V in ${}^H_H\mathcal{YD}$ is determined by

$$\chi \in \mathrm{Hom}_{\mathsf{alg}}(H, \Bbbk), \qquad \& \qquad g \in G(H) \cap Z(H) : \chi(g) = q.$$

A pair $(\chi, g) \in \mathrm{Hom}_{\mathsf{alg}}(H, \Bbbk) \times G(H) \cap Z(H)$ is called a YD-pair.

We say that (V, c) is of **diagonal type** if there exists a basis $(v_i)_{i\in\mathbb{I}_\ell}$ and a matrix $\mathfrak{q} = (q_{ij})_{i,j\in\mathbb{I}_\theta}$ such that

$$c(v_i \otimes v_j) = q_{ij}\, v_j \otimes v_i, \qquad i, j \in \mathbb{I}_\theta.$$

A *principal realization* of (V, c) in ${}^H_H\mathcal{YD}$ is a family of YD-pairs $(\chi_1, g_1), \dots, (\chi_\theta, g_\theta)$ such that

$$\chi_i(g_j) = q_{ji}, \qquad 1 \le i, j \le \theta.$$

Indeed, V is realized over H by declaring $v_i \in V_{g_i}^{\chi_i}$.

Notice however that if $\theta \geq 2$, then there are realizations that are not principal.

iv. Clases of braided vector spaces.

iv.i Diagonal type

These were introduced just before. By several reasons, this is the most important class. Note that if Γ is a finite abelian group and $\operatorname{char} \Bbbk = 0$, then any $V \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ is of diagonal type. This class will be discussed thoroughly.

iv.ii Racks

A rack is a set $X \neq \emptyset$ with a self-distributive operation $\triangleright$ such that $\varphi_x := x \triangleright \_\_$ is bijective for all $x \in X$.

If X is a rack and $\mathfrak{q} : X \times X \to \Bbbk^\times$ is a (suitable) 2-cocycle, then $\Bbbk X, c^{\mathfrak{q}})$ is a braided vector space where

$$c^{\mathfrak{q}} \in GL(\Bbbk X \otimes \Bbbk X), \qquad c^{\mathfrak{q}}(x \otimes y) = q_{xy}(x \triangleright y) \otimes x, \ \forall x, y \in X.$$

This kind of braiding will be discussed in depth as needed to understand pointed Hopf algebras over non-abelian groups.

iv.iii Blocks Let $\ell \in \mathbb{N}_{\geq 2}$ and $\epsilon \in \Bbbk^{\times}$. The **block** $\mathcal{V}(\epsilon, \ell)$ is a braided vector space with a basis $(x_i)_{i \in \mathbb{I}_\ell}$ such that

$$c(x_i \otimes x_j) = \begin{cases} \epsilon x_1 \otimes x_i, & j = 1 \\ (\epsilon x_j + x_{j-1}) \otimes x_i, & j \geq 2, \end{cases} \qquad i \in \mathbb{I}_\ell.$$

Theorem. [A.–Angiono–Heckenberger] GK-dim $\mathscr{B}(\mathcal{V}(\epsilon, \ell)) < \infty$ if and only if $\ell = 2$ and $\epsilon \in \{\pm 1\}$.

- $\mathscr{B}(\mathcal{V}(1,2)) = \Bbbk\langle x_1, x_2 | x_2 x_1 - x_1 x_2 + \frac{1}{2} x_1^2 \rangle$ **Jordan plane**.
- $\mathscr{B}(\mathcal{V}(-1,2)) = \Bbbk\langle x_1, x_2 | x_1^2, x_2 x_{12} - x_{12} x_2 - x_1 x_{12} \rangle$ **super Jordan plane**. Here $x_{12} = x_2 x_1 + x_1 x_2$. Note: GK-dim $\mathcal{B}(\mathcal{V}(\pm 1, 2)) = 2$.

The class of decomposable braided vector spaces $V = V_1 \oplus \cdots \oplus V_\theta$ where the V_i's are either blocks or points (i.e. dim 1) is important for the classification of finite GK-dim (in char $= 0$) and of finite dim (in char > 0).

iv.iii Miscellaneous The rigid braided vector spaces of dimension 2, (not of diagonal type) were classified by Hietarinta.

Theorem. (A.-Jury Giraldi) If $\mathscr{B}(V)$ has quadratic relations, then (V, c) is classified, and the explicit presentation of $\mathscr{B}(V)$, a PBW-basis, the dimension and the GK-dimension are given.

Most of the Nichols algebras appearing in this way have known algebra structure but there are some strange examples.