Pointed Hopf algebras over simple groups
I. The lifting method

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o. Overview A basic result for Hopf algebras reads as follows:

Theorem 1. *Let H be a Hopf algebra whose coradical H_0 is a Hopf subalgebra. Then H is a deformation (lifting) of the bosonization of H_0 by a post-Nichols algebra R in ${}^{H_0}_{H_0}\mathcal{YD}$:*

$$\operatorname{gr} H \simeq R\# H_0. \tag{1}$$

Thus, to understand Hopf algebras (whose coradical is a Hopf subalgebra) satisfying a given property $\mathfrak{P}$, one should

(a) Verify that $\mathfrak{P}$ propagates softly through (1),

(b) understand the cosemisimple Hopf algebras K, as well as the post-Nichols algebras $R \in {}^{K}_{K}\mathcal{YD}$ satisfying $\mathfrak{P}$,

(c) recover information on H from information on $\operatorname{gr} H \simeq R\# K$.

In these notes, the main focus is on Hopf algebras satisfying the property $\mathfrak{P} =$ having finite dimension, under the hypotheses:

Hypothesis A. *The base field $\Bbbk$ is algebraically closed and has characteristic 0.*

Hypothesis B. *H is pointed, i.e. $H_0 \simeq \Bbbk G$.*

Under Hypothesis A, it was conjectured that any post-Nichols algebra $R \in {}^K_K\mathcal{YD}$ is indeed a Nichols algebra (which is definitely false in positive characteristic), implying a drastic simplification in our analysis.

Hypothesis B is justified because our knowledge of cosemisimple Hopf algebras is at an early stage.

Other properties of Hopf algebras that can be studied within this approach are:
having finite Gelfand-Kirillov dimension,
having finite dimension in characteristic > 0,
being Noetherian,
having finitely generated cohomology, . . .

In this chapter we explain the above terminology, prove Theorem 1, and discuss the propagation of several properties through (1). Notice that we do not assume Hypothesis A, i.e., the field $\Bbbk$ is arbitrary.

i. Preliminaries We assume that the reader is familiar with the basic definitions of coalgebras and Hopf algebras.

i.i Coalgebras The comultiplication of a coalgebra C is denoted by Δ and the counit by ε; the kernel of ε is denoted by C^+.

The convolution product in $A = C^*$ is the transpose of Δ; thus A is an associative algebra (but the dual of an algebra is not a coalgebra unless it has finite dimension).

A subspace D of C is

- a left coideal if $\Delta(D) \subseteq C \otimes D$,
- a right coideal if $\Delta(D) \subseteq D \otimes C$,
- a coideal if $\Delta(D) \subseteq D \otimes C + C \otimes D$,
- a subcoalgebra if $\Delta(D) \subseteq D \otimes D$.

Let V be a vector space. Given $U \subseteq V$ and $W \subseteq V^*$, we denote

$$U^{\perp} := \{f \in V^* : f(u) = 0 \ \forall u \in U\},$$
$$W^{\perp} := \{u \in V : f(u) = 0 \ \forall f \in W\}.$$

Then a subspace D of a coalgebra C is

- a left coideal if and only if $D^{\perp}$ is a left ideal of C^*,
- a right coideal if and only if $D^{\perp}$ is a right ideal of C^*,
- an ideal if it is the kernel of a coalgebra map,
- a subcoalgebra if and only if it is a left and right coideal, if and only if $D^{\perp}$ is a two-sided ideal of C^*.

Lemma 1. (Cartier). Every coalgebra is the union of its finite-dimensional subcoalgebras.

A coalgebra is *simple* if it is different from 0 and has no proper subcoalgebras.

Example. A coalgebra C of dimension 1 is spanned by $g \in C$ such that $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$ (called a group-like element).

By Lemma 1, a simple coalgebra is finite-dimensional, hence the dual of a simple algebra.

Lemma 1 also implies that any coalgebra contains a simple subcoalgebra.

Definition. The *coradical* C_0 of a coalgebra C is the sum of all its simple subcoalgebras.

One has $$C_0 = \bigoplus_{S \text{ subcoalgebra de } C} S.$$

- If D is a subcoalgebra of C, then $D_0 = C_0 \cap D$.

A coalgebra is *cosemisimple* if and only if is a sum of simple subcoalgebras, i.e. if it coincides with its coradical. It can be shown that being cosemisimple is equivalent to the category of left (or right) comodules being semisimple.

Example. Given a set $X \neq \emptyset$, the vector space $\Bbbk X$ with basis $(e_x)_{x\in X}$ is a cosemisimple coalgebra by prescribing that the e_x's are group-likes.

A coalgebra C is *pointed* when its simple subcoalgebras have dimension 1, i.e., $C_0 \simeq \Bbbk X$ where $X = \{x \in C : x \text{ is group-like}\}$. A coalgebra C is *connected* if $\dim C_0 = 1$.

i.ii Filtrations. We start with some standard definitions. Let V be a vector space. A family $\mathcal{F} = (\mathcal{F}^n V)_{n\in\mathbb{Z}}$ of subspaces of V is an ascending, respectively descending filtration, of V if

$\mathcal{F}^n V \subseteq \mathcal{F}^{n+1} V$,	respectively	$\mathcal{F}^n V \supseteq \mathcal{F}^{n+1} V$	$\forall n \in \mathbb{Z}$;
$\mathcal{F}$	is *separated* if	$\cap \mathcal{F}^n V = 0$,	
	and *exhaustive* if	$\cup \mathcal{F}^n V = V$.	

We shall consider

- Ascending filtrations with $\mathcal{F}^{-1} V = 0$, hence $\mathcal{F}^{-n} V = 0$ $\forall n < 0$ and $\mathcal{F}$ is separated. For such $\mathcal{F}$, we just provide $\mathcal{F} = (\mathcal{F}^n V)_{n\geq 0}$.

- Descending filtrations with $\mathcal{F}^{-1} V = V$, hence $\mathcal{F}^{-n} V = V$ $\forall n < 0$ and $\mathcal{F}$ is exhaustive. For such $\mathcal{F}$, we just provide $\mathcal{F} = (\mathcal{F}^n V)_{n\geq 0}$.

If $\mathcal{F} = (\mathcal{F}^n V)_{n\in\mathbb{Z}}$ is an ascending filtration of V,
then $\mathcal{F}^\perp = \left((\mathcal{F}^n V)^\perp\right)_{n\in\mathbb{Z}}$ is a descending filtration of V^*.

If $\mathcal{F} = (\mathcal{F}^n V)_{n\in\mathbb{Z}}$ is a descending filtration of V,
then $\mathcal{F}^\perp = \left((\mathcal{F}^n V)^\perp\right)_{n\in\mathbb{Z}}$ is an ascending filtration of V^*.

Clearly, $\mathcal{F}^{-1}V = 0$ implies $\left(\mathcal{F}^{-1}V\right)^\perp = V^*$,
while $\mathcal{F}^{-1}V = V$, implies $\left(\mathcal{F}^{-1}V\right)^\perp = 0$.

Let now A be an algebra. By convention, an algebra filtration of A is a descending filtration $\mathcal{F} = (\mathcal{F}^n A)_{n\geq 0}$ of A such that

$$\mathcal{F}^p A \cdot \mathcal{F}^q A \subseteq \mathcal{F}^{p+q} A, \qquad \forall p, q \in \mathbb{N}_0.$$

Ascending algebra filtrations are defined similarly.

Let now C be a coalgebra. By convention, a coalgebra filtration of C is an ascending filtration $\mathcal{F} = (\mathcal{F}^n C)_{n\geq 0}$ of C such that

$$\Delta(\mathcal{F}^n C) \subseteq \sum_{p,q\in\mathbb{N}_0 \colon p+q=n} \mathcal{F}^p C \otimes \mathcal{F}^q C \qquad \forall n \in \mathbb{N}_0.$$

Descending coalgebra filtrations are defined similarly.

If $\mathcal{F}$ is a coalgebra filtration of the coalgebra C, then $\mathcal{F}^{\perp}$ is an algebra filtration of the algebra $A = C^*$.

If $\mathcal{F}$ is an algebra filtration of the algebra A and $\dim A < \infty$, then $\mathcal{F}^{\perp}$ is a coalgebra filtration of the coalgebra $C = A^*$.

The typical example of an algebra filtration is $(I^n)_{n\geq 0}$ where I is a (2-sided) ideal of A. We next discuss the coalgebra version of it. We start with the notion of wedge.

Let C be a coalgebra. For D, E, F subspaces of C, we set

$$
\begin{aligned}
D \wedge E &= \{c \in C : \Delta(c) \in D \otimes C + C \otimes E\} \quad = \Delta^{-1}(C \otimes E + D \otimes C), \\
&= \ker\left(C \xrightarrow{\Delta} C \otimes C \to C/D \otimes C/E\right) \\
&= (C^{\perp} \cdot E^{\perp})^{\perp} \qquad\qquad \text{(product in } C^*).
\end{aligned}
$$

Some properties:

- $D \wedge (E \wedge F) = (D \wedge E) \wedge F$.

- $D \wedge C^{+} = D = C^{+} \wedge D$.

- If D is a left coideal and F is a right coideal (in particular, if D and F are subcoalgebras), then $D \wedge E$ is a subcoalgebra and $D \wedge E \supset D \cup E$.

- If S, D, and E are subcoalgebras of C, where S is simple and $S \subseteq D \wedge E$, then $S \subseteq D$ or $S \subseteq E$.

- If $\mathcal{F} = (\mathcal{F}^n C)_{n \geq 0}$ is an (ascending) coalgebra filtration of C, then $\mathcal{F}^{n+1} C \subseteq \mathcal{F}^n C \wedge \mathcal{F}^0 C$.

We shall need the following result.

Lemma 2. Let $\mathcal{F} = (\mathcal{F}^n V)_{n \in \mathbb{N}_0}$ be an exhaustive filtration of the coalgebra C. Then $C_0 \subseteq \mathcal{F}^0 V$.

Proof. We have to prove: if S is a simple subcoalgebra, then $S \subseteq \mathcal{F}^0 V$. Since the filtration $\mathcal{F}$ is exhaustive, there exists $n \in \mathbb{N}_0$ such that $S \cap \mathcal{F}^n V \neq 0$; by simplicity, $S \subseteq \mathcal{F}^n V \subseteq \mathcal{F}^{n-1} C \wedge \mathcal{F}^0 C$. Hence $S \subseteq \mathcal{F}^{n-1} C$ or $S \subseteq \mathcal{F}^0 C$. By induction, $S \subseteq \mathcal{F}^0 C$.

Next, for a subcoalgebra D of C, we set

$$\wedge^0 D = 0, \quad \wedge^1 D = D, \quad \wedge^{n+1} D = (\wedge^n D) \wedge D.$$

This defines an ascending separated coalgebra filtration $\wedge^\bullet D$. Given a subcoalgebra E of D, $\wedge^n E \subseteq \wedge^n D$ for all $n \in \mathbb{N}_0$.

When $D = C_0$, we set $C_n = \wedge^{n+1} C_0$ and call this the *coradical filtration* of C.

Lemma 3. $\wedge^\bullet D$ is an exhaustive filtration if and only if $C_0 \subseteq D$.

Proof. If $\dim C < \infty$, then the coradical filtration is exhaustive since $\wedge^n (C_0)^\perp = J^{n+1}$ where J is the Jacobson radical of $A = C^*$. By Lemma 1, we conclude that the coradical filtration is exhaustive for arbitrary C.
Thus the filtration $\wedge^\bullet D$ is exhaustive for $C_0 \subseteq D$.
The converse follows from Lemma 2.

i.iii Gradings.

Let V be a vector space and let X be a set. A family $\mathcal{G} = (V(x))_{x \in X}$ of subspaces of V is an X-grading of V if

$$V = \oplus_{x \in X} V(x).$$

We say that $(V, \mathcal{G})$, or simply V, is an X-graded vector space. When $X = \mathbb{N}_0$, we simply say 'graded vector space'. A graded vector space is *connected* if $\dim V(0) = 1$.

Let V be an X-graded vector space. Its graded dual is

$$V^{\#} = \oplus_{x \in X} V(x)^*.$$

We say that V is *locally finite* if $\dim V(x) < \infty$ for all $x \in X$. When this is the case, $V^{\#}$ is locally finite too and $V^{\#\#} \simeq V$.

Thus we have a contravariant functor $V \mapsto V^{\#}$ from the category of locally finite X-graded vector spaces (with morphisms being linear maps preserving the grading) onto itself; clearly it sends injective maps to surjective maps and vice versa.

Let $\mathcal{F} = (\mathcal{F}^n V)_{n \in \mathbb{N}_0}$ be an ascending filtration of V. Set

$$\operatorname{gr}^n V := \mathcal{F}^n V / \mathcal{F}^{n-1} V, \qquad \operatorname{gr} V := \oplus_{n \in \mathbb{N}_0} \operatorname{gr}^n V.$$

We say that $\operatorname{gr} V$ is the graded vector space associated to $(V, \mathcal{F})$. Notice that, for an exhaustive filtration, $\dim V = \dim \operatorname{gr} V$.

Conversely, a graded vector space V has a canonical ascending filtration

$$\mathcal{F}^n V = \oplus_{m \leq n} V(m).$$

The graded vector space associated to this filtration is isomorphic to V.

Let now M be a monoid (with unit).

Let A be an algebra and let $\mathcal{G} = (A(m))_{m \in M}$ be an M-grading of A. We say that $(A, \mathcal{G})$, or simply A, is an M-graded algebra if

$$A(p) \cdot A(q) \subseteq A(p \cdot q), \qquad \forall p, q \in M.$$

Let C be a coalgebra and let $\mathcal{G} = (C(m))_{m \in M}$ be an M-grading of C. We say that $(C, \mathcal{G})$, or simply C, is an M-graded coalgebra if

$$\Delta(C(m)) \subseteq \sum_{p,q \in \mathbb{N}_0 \colon p \cdot q = m} C(p) \otimes C(q) \qquad \forall m \in M.$$

If A is an $\mathbb{N}_0$-graded algebra, then the unit $1 \in A(0)$;
if C is an $\mathbb{N}_0$-graded coalgebra, then $\varepsilon(A(n)) = 0$ for $n \neq 0$.

If C is an M-graded coalgebra, then the graded dual $C^{\#}$ is an M-graded algebra.

If A is a locally finite M-graded algebra, then the graded dual $A^{\#}$ is an M-graded coalgebra.

i.iv Coradically graded coalgebras.

Definition. We say that a graded coalgebra C is *coradically graded* if the coradical filtration coincides with the canonical filtration associated to the grading:

$$C_n = \oplus_{m \leq n} C(m).$$

Thus $C(0) = C_0$.

If in addition $\dim C(0) = 1$, then we say that C is *strictly graded*.

Lemma 4. Let C be a coalgebra. Then the graded coalgebra $\operatorname{gr} C$ associated to the coradical filtration is coradically graded.

Proof. See [Radford, 4.4.15].

Let $C = \oplus_{n \in \mathbb{N}_0} C(n)$ be a graded coalgebra with $\dim C(0) = 1$. Let us denote by 1 the group-like element in $C(0)$ and set

$$\mathcal{P}(C) = \{c \in C : \Delta(c) = c \otimes 1 + 1 \otimes c\},$$

the space of primitive elements of C. Notice that $\mathcal{P}(C) \subseteq C^+$.

Lemma 5. The following are equivalent:

(a) C is strictly graded.

(b) $\mathcal{P}(C) = C(1)$.

If in addition C is locally finite, then these are equivalent to

(c) The algebra $A = C^{\#}$ is generated by $A(1) = C(1)^*$.

Proof. Assume (a). Then $C(1) \subseteq C_1$, thus if $c \in C(1)$, then

$$\Delta(c) = c_1 \otimes 1 + 1 \otimes c_2,$$

where $c_1, c_2 \in C(1)$ since C is a graded coalgebra. Then $c = (\mathrm{id} \otimes \varepsilon)\Delta(c) = c_1$, and similarly $c = c_2$.

Assume (b). We have to prove that $C_n = \oplus_{m \leq n} C(m)$ for all $n \in \mathbb{N}_0$. If $n = 0$, then $C_0 \subset C(0)$ by Lemma 3 and the other inclusion is clear. The case $n = 1$ follows from

$$C_1 = C_0 + \mathcal{P}(C).$$

Indeed, $\supseteq$ is clear, so let $x \in C_1 = C_0 \wedge C_0$. We may assume that $x \in C(d)$ for some $d > 0$. Then

$$\Delta(x) = x_1 \otimes 1 + 1 \otimes x_2,$$

where $x_1, x_2 \in C(d)$; now $x = (\mathrm{id} \otimes \varepsilon)\Delta(x) = x_1 = x_2$, and $\subseteq$ follows. The rest of the proof: see [Sweedler, Section 11.2].

i.v Filtrations of Hopf algebras. Let B be a bialgebra and let H be a Hopf algebra.

A bialgebra filtration of B is an ascending filtration $\mathcal{F} = (\mathcal{F}^n B)_{n \geq 0}$ of B which is an algebra and a coalgebra filtration at the same time.

Graded bialgebras and graded Hopf algebras are defined similarly.

- If $\mathcal{F}$ is a bialgebra filtration, then $\mathrm{gr}_{\mathcal{F}} B$ is a graded bialgebra.

- If $\mathcal{F}$ is a bialgebra filtration of H, and $\mathcal{F}^n H$ is stable by the antipode for all n (called a Hopf algebra filtration), then $\mathrm{gr}_{\mathcal{F}} H$ is a graded Hopf algebra.

• If D is a sub-bialgebra of H, then $\wedge^\bullet D$ is a bialgebra filtration of H. If D is a Hopf subalgebra, then $\wedge^n D$ is stable by the antipode for all $n \in \mathbb{N}_0$ and $\mathrm{gr}_{\mathcal{F}} H$ is a graded Hopf algebra.

Example. Let $H_{[0]} := \Bbbk\langle H_0 \rangle$ be the subalgebra of H generated by the coradical H_0. Since the coradical is stable by the antipode, $\wedge^\bullet H_{[0]}$ is a Hopf algebra filtration called the *standard filtration*.

ii. Hopf algebras whose coradical is a Hopf subalgebra

ii.i Yetter-Drinfeld modules. In the 1970's, Yang and Baxter discovered the so called Quantum Yang-Baxter equation which is equivalent to the braid equation.

Definition. A braided vector space is a pair (V, c) where V is a vector space and $c : V \otimes V \to V \otimes V$ is a linear automorphism that satisfies the braid equation:

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c).$$

(Taking $R = \tau c$, R satisfies the QYBE iff c satisfies the braid equation).

Why are they called **braided** vector spaces?

Recall that the braid group in n strands (Artin 1928) is

$$\mathbb{B}_n = \langle \sigma_1, \dots, \sigma_{n-1} : \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| > 1, \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, |i-j| = 1\rangle.$$

There is a group epimorphism $\mathbb{B}_n \twoheadrightarrow \mathbb{S}_n$, $\sigma_i \mapsto s_i = (i, i+1)$.

If (V, c) is a braided vector space, then $\mathbb{B}_n$ acts on $V^{\otimes n}$ by

$$\sigma_i \longmapsto \mathrm{id}_{V^{\otimes(i-1)}} \otimes \mathbf{c} \otimes \mathrm{id}_{V^{\otimes(n-i-1)}}$$

Most applications of the qYBE arise from these representations.

In 1986, Drinfeld found a method to construct braided vector spaces. Let H be a Hopf algebra (with bijective antipode).

A Yetter-Drinfeld-module over H is a vector space V provided with

- a structure of H-module $\cdot : H \otimes V \to V$,

- a structure of H-comodule $\delta : V \to H \otimes V$, $\delta(v) = v_{(-1)} \otimes v_{(0)}$;

such that $\delta(h \cdot v) = h_{(1)} v_{(-1)} \mathcal{S}(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)}$, $\forall h \in H, v \in V$.

Morphisms of Yetter-Drinfeld modules are linear maps preserving the action and the coaction.

The category of Yetter-Drinfeld-modules over H, denoted by ${}^H_H\mathcal{YD}$, is a braided tensor category:

- If $V, W \in {}^H_H\mathcal{YD}$, then $V \otimes W := V \otimes_{\Bbbk} W$ with the tensor product module structure and the tensor product comodule structure.

- The braiding is given by

$$c_{V,W}(v \otimes w) = v_{(-1)} \cdot w \otimes v_{(0)}, \qquad v \in V,\ w \in W.$$

Thus any $V \in {}^H_H\mathcal{YD}$ is a braided vector space with braiding $c = c_{V,V}$.

Remark: The subcategory ${}^{H}_{H}\mathcal{YD}_{fd}$ of finite-dimensional objects in ${}^{H}_{H}\mathcal{YD}$ is *rigid*: thus any $V \in {}^{H}_{H}\mathcal{YD}_{fd}$ has a left dual ${}^{*}V$ and a right dual V^{*}.

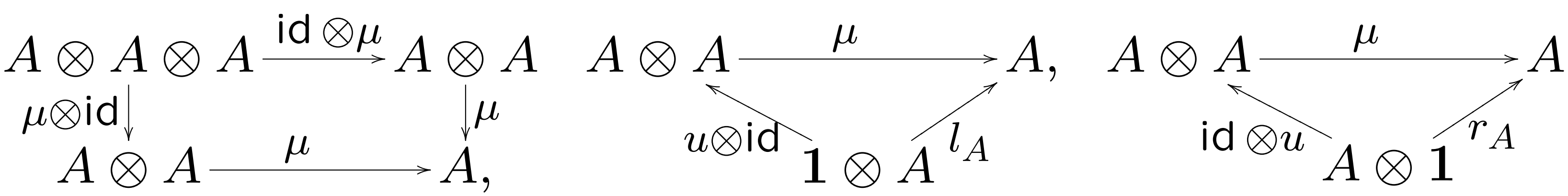
Accordingly, if $V = \oplus_{n\in\mathbb{N}_0} V(n)$ is a locally finite graded Yetter-Drinfeld module, then we set

$$V^{\#} = \oplus_{n\in\mathbb{N}_0} V(n), \qquad {}^{\#}V = \oplus_{n\in\mathbb{N}_0} {}^{*}V(n).$$

ii.iii Hopf algebras in ${}^H_H\mathcal{YD}$

Let $\mathscr{C}$ be a tensor category. Then we may define

- algebras, i.e., triples (A, μ, u) where $A \in \mathscr{C}$, $\mu : A \otimes A \to A$ and $u : \mathbf{1} \to A$ are morphisms in $\mathscr{C}$ that are associative and unital:



- coalgebras, i.e., triples (C, Δ, ε) where $C \in \mathscr{C}$, $\Delta : C \to C \otimes C$ and $\varepsilon : A \to \mathbf{1}$ are morphisms in $\mathscr{C}$ that are coassociative and counital;

If the tensor category $\mathscr{C}$ is braided, then we may also define:

- tensor products of associative algebras: if A and B are algebras in $\mathscr{C}$, then $A \otimes B$ is an algebra in $\mathscr{C}$:

$$\begin{array}{ccc} A \otimes B \otimes A \otimes B & \xrightarrow{\text{id} \otimes \mathbf{c} \otimes \text{id}} & A \otimes A \otimes B \otimes B \\ & \searrow^{\mu_{A \otimes B}} \quad\quad {}^{\mu_A \otimes \mu_B}\swarrow & \\ & A \otimes B & \end{array}$$

- Bialgebras, i.e., collections $(A, \mu, u, \Delta, \varepsilon)$ such that
(A, μ, u) is an algebra in $\mathscr{C}$,
(A, Δ, ε) is a coalgebra in $\mathscr{C}$,
Δ and ε are morphisms of algebras.

- Hopf algebras, i.e., bialgebras having an antipode.

Example. Let H be a Hopf algebra. A Hopf algebra in ${}^H_H\mathcal{YD}$ is

- an H-module R, with action $\cdot : H \otimes R \to R$, which is also an H-comodule with coaction $\delta : R \to H \otimes R$, $\delta(r) = r_{(-1)} \otimes r_{(0)}$; such that

$$\delta(h \cdot r) = h_{(1)} r_{(-1)} \mathcal{S}(h_{(3)}) \otimes h_{(2)} \cdot r_{(0)}, \qquad \forall h \in H, r \in R;$$

- which is also an (associative unital) algebra such that

$$\begin{aligned} h \cdot (rs) &= (h_{(1)} \cdot r)(h_{(2)} \cdot s), \\ \delta(rs) &= r_{(-1)} s_{(-1)} \otimes r_{(0)} s_{(0)}, \qquad \forall h \in H, r, s \in R; \\ h \cdot 1 &= \varepsilon(h) 1, \quad \delta(1) = 1 \otimes 1; \end{aligned}$$

- and a coalgebra with comultiplication Δ, $\Delta(r) = r^{(1)} \otimes r^2$, such that

$$\Delta(h \cdot r) = (h_{(1)} \cdot r^{(1)})(h_{(2)} \cdot r^{(2)}), \quad \forall h \in H, r \in R;$$
$$r_{(-1)} \otimes (r_{(0)})^{(1)} \otimes (r_{(0)})^{(2)}$$
$$= (r^{(1)})_{(-1)} (r^{(2)})_{(-1)} \otimes (r^{(1)})_{(0)} \otimes (r^{(2)})_{(0)};$$
$$\varepsilon_R(h \cdot r) = \varepsilon_H(h)\varepsilon_R(r); \quad \varepsilon_R(r) = r_{(-1)}\varepsilon_R(r_{(0)}).$$

- Furthermore, Δ is an algebra map, i.e.,

$$\Delta(rs) = r^{(1)}(r^{(2)})_{(-1)} \cdot s^{(1)} \otimes (r^{(2)})_{(0)} s^{(2)}, \qquad \forall r, s \in R;$$

- there exists an antipode $\mathcal{S} : R \to R$ (inverse of id_R wrt the convolution product).

ii.iii (Pre- and Post-) Nichols algebras. *The tensor algebra.* Let $V \in {}^H_H\mathcal{YD}$.

$$\rightsquigarrow V \otimes V \in {}^H_H\mathcal{YD} \rightsquigarrow T^n(V) = V \otimes T^{n-1}(V) \in {}^H_H\mathcal{YD}$$

$$\rightsquigarrow T(V) = \oplus_{n \in \mathbb{N}_0} T^n(V) \in {}^H_H\mathcal{YD}.$$

It is immediate that $T(V)$ is a (graded) algebra in ${}^H_H\mathcal{YD}$. Hence

$$T(V) \otimes T(V) \text{ is an algebra in } {}^H_H\mathcal{YD}$$

with the algebra structure twisted by the braiding c.

By the universal property of the tensor algebra, $\exists$ unique $\Delta : T(V) \to T(V) \otimes T(V)$ such that $\Delta(v) = v \otimes 1 + 1 \otimes v, \ \forall v \in V$.

Lemma 6. With respect to Δ, $T(V)$ is a Hopf algebra in ${}^H_H\mathcal{YD}$.

Definition. A graded Hopf algebra $\mathcal{B} = \oplus_{n\in\mathbb{N}_0}\mathcal{B}(n)$ in ${}^H_H\mathcal{YD}$ is a *pre-Nichols* algebra of V if

- it is connected, i.e., $\mathcal{B}(0) = \Bbbk$;
- $\mathcal{B}(1) \simeq V$ in ${}^H_H\mathcal{YD}$;
- $\mathcal{B}(1) \simeq V$ generates the algebra $\mathcal{B}$.

In other words, $\mathcal{B}$ is a pre-Nichols algebra of V if and only if there exists an homogeneous ideal I of $T(V)$ such that $\mathcal{B} \simeq T(V)/I$ and

- $I \cap \Bbbk \oplus V = 0$;
- I is a Yetter-Drinfeld submodule of $T(V)$;
- $\Delta(I) \subseteq I \otimes T(V) + T(V) \otimes I$ and $\mathcal{S}(I) \subseteq I$.

Let $V \in {}^H_H\mathcal{YD}_{fd}$, so that $T(V)$ is a locally finite graded Hopf algebra in ${}^H_H\mathcal{YD}$.

Definition. The quantum shuffle algebra of V is

$$T^c(V) := {}^{\#}T(V^*)$$

That is, the homogeneous components of $T^c(V)$ are the $T^n(V)$, but the multiplication of $T^c(V)$ is transpose to the comultiplication of $T(V)$ and vice versa.
Since the algebra $T(V)$ is generated by V, we have:

Lemma. (1) $T^c(V)$ is strictly graded,
(2) there exists a map $\Omega : T(V) \to T^c(V)$ of Hopf algebras in ${}^H_H\mathcal{YD}$ which is the identity in $V = T^1(V)$.

Definition. The Nichols algebra of V is the image of Ω.

Definition. A graded Hopf algebra $\mathcal{R} = \oplus_{n\in\mathbb{N}_0}\mathcal{R}(n)$ in ${}^H_H\mathcal{YD}$ is a post-Nichols algebra of V if

- it is connected, i.e., $\mathcal{R}(0) = \Bbbk$;
- $\mathcal{R}(1) \simeq V$ in ${}^H_H\mathcal{YD}$;
- $\mathcal{R}$ is strictly graded, or equivalently $\mathcal{P}(\mathcal{R}) = \mathcal{R}(1)$.

That is, $\mathcal{R}$ is a post-Nichols algebra of V if and only if there exists an injective morphism $\mathcal{R} \to T^c(V)$ of graded Hopf algebras in ${}^H_H\mathcal{YD}$ which is the identity on V.

ii.iv Bosonization (Radford biproduct) Hopf algebras in ${}^K_K\mathcal{YD}$ appear in nature by the following results of Radford and Majid:

- Let $H \underset{\iota}{\overset{\pi}{\rightleftarrows}} K$ be Hopf algebra maps such that $\pi\iota = \mathrm{id}_K$. Then

$$R = \{x \in H : (\mathrm{id} \otimes \pi)\Delta(x) = x \otimes 1\}$$

is a Hopf algebra in ${}^K_K\mathcal{YD}$.

- Conversely, let R be a Hopf algebra in ${}^K_K\mathcal{YD}$. Then $R\#K := R \otimes K$ with the semidirect multiplication and comultiplication is a Hopf algebra (called the **bosonization** of R by K) provided with Hopf algebra maps $\pi := \epsilon_R \otimes \mathrm{id} : R\#K \to K$, $\iota := u_R \otimes \mathrm{id} : K \to R\#K$.

- These constructions are reciprocal.

ii.v Proof of Theorem 1. We are now ready to sketch it. Recall

Let H be a Hopf algebra whose coradical H_0 is a Hopf subalgebra. Then H is a deformation (lifting) of the bosonization of H_0 by a post-Nichols algebra R in ${}^{H_0}_{H_0}\mathcal{YD}$:

$$\operatorname{gr} H \simeq R\# H_0.$$

Indeed, the coradical filtration is a Hopf algebra filtration, hence $\operatorname{gr} H$ is a graded Hopf algebra. The inclusion $\iota : H_0 \hookrightarrow \operatorname{gr} H$ and the projection $\pi : \operatorname{gr} H \to H_0$, which annihilates the components of positive degree, are Hopf algebra maps that satisfy $\pi\iota = \mathrm{id}_{H_0}$. Hence $\operatorname{gr} H \simeq R\# H_0$. Since π is homogeneous, R inherits the grading from $\operatorname{gr} H$ and turns out to be strictly graded.

Finally, by general reasons H is a deformation of $\operatorname{gr} H$.

iii. Applications. The lifting method.

Let H be a pointed Hopf algebra (or H_0 is a Hopf subalgebra). Recall that H is a deformation of $\operatorname{gr} H \simeq R\#\Bbbk G(H)$. Method to study when H has a property $\mathfrak{P}$:

Step 0: does property $\mathfrak{P}$ propagate well? Restrict to $\Bbbk G(H)$, or H_0, with property $\mathfrak{P}$.

Step 1: Classify all $V \in {}^{\Bbbk G(H)}_{\Bbbk G(H)}\mathcal{YD}$ (or $V \in {}^{\Bbbk H_0}_{\Bbbk H_0}\mathcal{YD}$) such that $\mathscr{B}(V)$ has property $\mathfrak{P}$.

Step 2: Classify all $\mathcal{R}$ post-Nichols algebra in ${}^{\Bbbk G(H)}_{\Bbbk G(H)}\mathcal{YD}$ (or in ${}^{\Bbbk H_0}_{\Bbbk H_0}\mathcal{YD}$) such that $\mathcal{R}$ has property $\mathfrak{P}$.

Step 3: Compute all liftings (deformations) of $\mathcal{R}\#\Bbbk G(H)$ (or $\mathcal{R}\#H_0$).

Finite dimension.

$\dim H < \infty \iff \dim \operatorname{gr} H < \infty \iff (\dim R < \infty \,\&\, |G(H)| < \infty)$.

Characteristic 0: **Conjecture.** $\mathcal{R} = \mathscr{B}(V)$.

Characteristic > 0: Conjecture does not hold.

Finite Gelfand-Kirillov dimension in characteristic 0:

$$\text{GK-dim}\, H < \infty \impliedby \text{GK-dim gr}\, H < \infty$$
$$\implies (\text{GK-dim}\, R < \infty \,\&\, G(H) \text{ nilpotent-by-finite}).$$

Noetherian: H Noetherian $\Longleftarrow$ $\operatorname{gr} H$ Noetherian $\overset{\Longrightarrow}{\Longleftarrow}$ R Noetherian & $G(H)$ polycyclic-by-finite???).

Finitely generated cohomology. $\dim H < \infty$. $\operatorname{Ext}^\bullet_H(\Bbbk,\Bbbk)$ fin. gen. $\overset{?}{\Longleftrightarrow}$ $\operatorname{Ext}^\bullet_{\operatorname{gr} H}(\Bbbk,\Bbbk)$ fin. gen. $\overset{?}{\Longleftrightarrow}$ $\operatorname{Ext}^\bullet_R(\Bbbk,\Bbbk)$ fin. gen.