On the classification of pointed Hopf algebras

Nicolás Andruskiewitsch

CIEM-CONICET, Córdoba, Argentina

Department of Mathematics and Data Science,

Vrije Universiteit Brussel, Belgium

Vrije Universiteit Brussel, September 29, 2025.

i. Classification / characterization

A classical theme in Mathematics is the classification (or at least characterization) of certain kind of mathematical objects.

By *classification* of objects in a class $\mathfrak{C}$ one usually means focusing on a significant equivalent relation $\sim$ on $\mathfrak{C}$ and finding invariants with respect to $\sim$, so that collecting them together one gets a bijective correspondance

$$\mathfrak{C}/\sim \longleftrightarrow \mathfrak{D}.$$

Example: Finite abelian simple groups, up to a group isomorphism, are in bijective correspondance with prime numbers.

The expectations are

• the elements in $\mathfrak{D}$ are easier to handle (or at least we think we understand them better);

• we may solve problems on $\mathfrak{C}$ by case-by-case considerations;

• or at least the process of classification provides some robust invariants that help to abord some difficult questions;

• there is the possibility of discovering new examples of importance.

The closer the equivalence relation $\sim$, the more difficult the classification. Sometimes one starts by a more relaxed relation as a first approximation.

Example: Connected Lie groups, up to local isomorphisms, are in bijective correspondance with finite dimensional Lie algebras.

Example: Algebraic varieties over an algebraically closed field $\Bbbk$, up to birational transformations, are in bijective correspondance with field extensions of $\Bbbk$ of finite trascendence degree.

Then one goes on classifying the invariants in the first approximation, sometimes restricting to a subclass of $\mathfrak{C}$.

Example: Simple finite dimensional Lie algebras, up to isomorphisms, are classified by connected Dynkin diagrams.

Along the way, we encounter an important invariant: the Cartan subalgebra of a simple Lie algebra, instrumental in many questions about simple Lie algebras and groups.

Sometimes we end having a full classification. . . :
Example: Simple complex Lie groups, up to isomorphisms, are classified by pairs formed by a connected Dynkin diagram and a subgroup of P/Q (weight lattice modulo the root lattice).

. . . but sometimes this is out of reach; then we have to content ourselves with a *characterization* (or structure description).

Example: Finite dimensional Lie algebras, up to isomorphisms, are determined as semidirect products of a solvable Lie algebra and a semisimple one.

But solvable Lie algebras can not be classified in a profitable way.

Example: The classification of the finite groups (up to isomorphisms) is a wild problem. Indeed the classification of the 2-step (nilpotent) p-groups with non-cyclic center is wild (Sergejchuk).

As a substitute, one may consider:

- any finite group is an iterated extension of simple groups (Jordan-Hölder theorem);

- The classification of the finite *simple* groups (up to isomorphisms) is known.

Theorem. Any finite simple group is isomorphic either to

- the alternating group $\mathbb{A}_n$, $n \geq 5$, or
- one of the 26 sporadic groups, or
- a finite simple group of Lie type.

By several reasons, this is a remarkable achievement:

- it is a meta-theorem, not proved by a single mathematician but by a large community of them (starting with Galois);

- its developing provoked a very large and substantial number of developments in other areas of mathematics and computer science;

- it has a very wide rank of applications.

ii. Hopf algebras

These objects have origins in three different sources:

• A. Borel (1953) developped an abstract formalism towards axiomatizing techniques introduced by H. Hopf to compute the cohomology of Lie groups.

• P. Cartier (1955) introduced the notion of hyperalgebra towards axiomatizing the algebra of distributions studied by J. Dieudonné seeking to a positive characteristic version of the dictionary Lie groups–Lie algebras.

• G. I. Kac (1958) started a remarkable program extending properties of the C^*-algebra of functions on a finite group to the noncommutative setting.

In the late 1950s and early 1960s, the theory developed independently along the three paths mentioned above. Some notewhorty achievements:

- In the topological side, Milnor and Moore obtained a classification of cocommutative connected Hopf (super)algebras.

- Cartier (1962) obtained classifications of commutative, respectively cocommutative, Hopf algebras (in char 0).

- Kostant (early 60's, cf. Sweedler's book) found independently the classification of cocommutative Hopf algebras (in char 0). He gave in the paper *Groups over* $\mathbb{Z}$ (whose goal was to give a uniform definition of finite groups of Lie type) the definition of Hopf algebra used today.

- G. I. Kac got a series of remarkable results on semisimple Hopf algebras neither commutative nor cocommutative.

Let $\Bbbk$ be a field.

A **Hopf algebra** is a collection (H, m, Δ) where

- (H, m) is an associative algebra with unit $u : \Bbbk \to H$,
- (H, Δ) is a coassociative coalgebra with counit $\varepsilon : H \to \Bbbk$,
- Δ and ε are algebra maps,
- there exists an algebra map $\mathcal{S} : H \to H^{\mathrm{op}}$ satisfying suitable axioms.

First invariants of a Hopf algebra H:

- The group of group-like elements:

$$G(H) = \{x \in H\backslash 0 : \Delta(x) = x \otimes x\}.$$

- The Lie algebra of primitive elements:

$$\mathcal{P}(H) = \{x \in H : \Delta(x) = x \otimes 1 + 1 \otimes x\}.$$

- The coradical

$$H_0 = \sum_{C \subseteq H:\ \text{simple subcoalgebra}} C.$$

Examples of Hopf algebras:

- Group algebras (cocommutative).

- Enveloping algebras (cocommutative).

- Algebras of polynomial functions on affine algebraic groups (commutative).

Theorem. (Cartier-Kostant-Gabriel). $\Bbbk = \overline{\Bbbk}$, char $\Bbbk = 0$.
H cocommutative Hopf algebra $\implies H \simeq U(\mathfrak{g}) \rtimes \Bbbk G$,
where $\mathfrak{g} = \mathrm{Prim}(H)$ and $G = G(H)$ acts on $\mathfrak{g}$ by conjugation.

Theorem. (Cartier). $\mathbb{k} = \overline{\mathbb{k}}$, char $\mathbb{k} = 0$.
H finitely generated commutative Hopf algebra $\implies H \simeq \mathbb{k}[\mathbb{G}]$ where $\mathbb{G} = \mathrm{Hom}_{\mathrm{alg}}(H, \mathbb{k})$.

The study of affine algebraic groups through their (Hopf) algebras of functions was undertaken in the 60's and 70's by Geothendieck and his school (Demazure, Gabriel; SGA3) and G. Hochshild.

The study of Hopf algebras neither commutative nor cocommutative was started by M. Sweedler (student of Kostant) in the late '60 in his book *Hopf algebras.*

A number of basic fundamental results were obtained in the 70's and 80's by several mathematicians, e.g.

Sweedler, Heynemann, Larson, Radford, Taft, Nichols, Wilson, Zöller, Takeuchi, Schneider, etc.

In 1981, N. Reshetikhin, P. Kulish and V. Sklyanin discovered quantum SL_2.

Shortly after, Drinfeld and Jimbo discovered independently quantum versions of $U(\mathfrak{g})$ for $\mathfrak{g}$ simple.

In his report presented (by P. Cartier!) at the ICM 1986, Drinfeld wrote:

I believe that most of the examples of noncommutative noncocommutative Hopf algebras invented independently of integrable quantum system theory are counterexamples rather than "natural" examples (however there are remarkable exceptions ...)

iii. (Finite-dimensional) (pointed) Hopf algebras. Assume that $\Bbbk$ is algebraically closed and that $\operatorname{char} \Bbbk = 0$.
A Hopf algebra H is *pointed* if

- any simple subcoalgebra has dimension one, or equivalently,

- any simple comodule has dimension one, or equivalently,

- the coradical (the largest cosemisimple subcoalgebra) is isomorphic to the group algebra $\Bbbk G(H)$.

This talk is an introduction to the course on f.d. pointed Hopf algebras. Let us first try to answer the questions:

- Why $\Bbbk = \overline{\Bbbk}$?
- why pointed?
- Why $\operatorname{char} \Bbbk = 0$?
- why finite-dimensional?

Generally speaking, even if the proposed method applies in a wider context, these conditions facilitate the understanding of the main steps of the approach without technical complications.

The case when "$\Bbbk$ is not algebraically closed" follows from the "$\Bbbk = \overline{\Bbbk}$ case" (via Galois cohomology).

When "$\operatorname{char} \Bbbk > 0$" there are specific features, e.g. the restricted universal enveloping algebras or non-semisimplicity of finite group algebras, requiring specific techniques.

The hypothesis *pointed* allows to use the vast knowledge of group theory; less is known about general cosemisimple Hopf algebras.

There is work in progress about classification of pointed Hopf algebras either Noetherian, or of finite GK-dim. Again, lack of semisimplicity makes things harder . . . and more interesting.

The coradical filtration.

Let H be a Hopf algebra. Recall the coradical

$$H_0 = \sum_{C \subseteq H\text{: simple subcoalgebra}} C.$$

Define $\quad H_1 = H_0 \wedge H_0 := \{x \in H : \Delta(x) \in H_0 \otimes H + H \otimes H_0\};$

recursively $\quad H_{n+1} = H_n \wedge H_0 := \{x \in H : \Delta(x) \in H_n \otimes H + H \otimes H_0\}.$

$\Longrightarrow \quad 0 = H_{-1} \subseteq H_0 \subseteq H_1 \subseteq \dots H_n \subseteq H_{n+1} \dots$ exhaustive coalgebra filtration.

$\Longrightarrow \quad \operatorname{gr} H = \oplus_{n \geq 0} H_n / H_{n-1}$ graded coalgebra.

Now assume that H_0 is a Hopf subalgebra (e.g., H pointed).

$$\Longrightarrow 0 = H_{-1} \subseteq H_0 \subseteq H_1 \subseteq \dots H_n \subseteq H_{n+1} \dots$$

is a Hopf algebra filtration;

$$\Longrightarrow \operatorname{gr} H = \oplus_{n\geq 0} \operatorname{gr}^n H, \text{ where } \operatorname{gr}^n H := H_n/H_{n-1},$$

is a graded Hopf algebra.

Facts:

- There is an isomorphism of Hopf algebras $\operatorname{gr} H \simeq R\# H_0$, where R is a Hopf algebra in the braided tensor category ${}^{H_0}_{H_0}\mathcal{YD}$.

- $R = \oplus_{n\geq 0} R^n$, where $R^n = R \cap \operatorname{gr}^n H$, is a coradically graded algebra called the *diagram* of H.

- $V = R^1 \in {}^{H_0}_{H_0}\mathcal{YD}$ is called the *infinitesimal braiding* of H.

What is ${}^{H_0}_{H_0}\mathcal{YD}$? This is the braided tensor category of Yetter-Drinfeld modules over H_0. Namely, $V \in {}^{H_0}_{H_0}\mathcal{YD} \iff$

- V is a left H_0-module.
- V is a left H_0-comodule.
- Compatibility: $\delta(h \cdot v) = h_{(1)}v_{(-1)}\mathcal{S}(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)}$.

Example: If $H_0 = \Bbbk G$, G a group, then $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD} \iff$

- $\cdot : G \times V \to V$,
- $V = \oplus_{g \in G} V_g$, $\gamma \cdot V_g = V_{\gamma g \gamma^{-1}}$.

Fact: The subalgebra of R generated by $V = R^1$ is isomorphic to the *Nichols algebra of* V.

What is the Nichols algebra of V? This is a graded Hopf algebra in ${}^{H_0}_{H_0}\mathcal{YD}$: $\mathscr{B}(V) = \oplus_{n\geq 0} \mathscr{B}^n(V)$ such that

- $\mathscr{B}^0(V) = \Bbbk$,
- $\mathscr{B}^1(V) \simeq V$.
- V generates $\mathscr{B}(V)$,
- $\mathcal{P}(\mathscr{B}(V)) = V$.

Summarizing, given a Hopf algebra H with coradical being a Hopf subalgebra (for instance, H pointed), we have several invariants:

- H_0 (for instance $G(H)$),
- $\operatorname{gr} H$,
- the diagram R,
- the infinitesimal braiding V,
- its Nichols algebra $\mathscr{B}(V) \hookrightarrow R$. Notice:

$\dim H < \infty \iff \dim \operatorname{gr} H < \infty \iff (\dim R < \infty \ \& \dim H_0 < \infty)$.

In the pointed case, $\dim H_0 < \infty \equiv |G(H)| < \infty$.

iv. The lifting method. [A-Schneider]).

The idea of the method is to recover all possible Hopf algebras from the previous invariants. Here are the steps of the method:

Step I. Classify all $V \in {}^H_H\mathcal{YD}$ s. t. $\dim \mathscr{B}(V) < \infty$; for them find a presentation by generators and relations.

Step II. Is $\mathscr{B}(V) = R$?

Step III. Given V, classify all H s. t. $\operatorname{gr} H \simeq \mathscr{B}(V)\# H_0$.

We focus on Step I.

Let G be a finite group. Then ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ is a semisimple category (since $\operatorname{char}\Bbbk = 0$). We have to compute / determine the dimension of $\mathscr{B}(V)$ for any $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$, fin.dim.

We distinguish two cases:

∘ $V = V_1 \oplus \cdots \oplus V_\theta$, $\theta \geq 2$.

One deals with this case using the Weyl groupoid ([Heckenberger], [A-Heckenberger-Schneider]).
Outcome: All V like this with $\dim \mathscr{B}(V) < \infty$ are known ([Heckenberger], [Heckenberger-Vendramin]).

∘ V is simple. Difficult! Except when $\dim V = 1$ . . .

It is natural to consider classes of groups.

G **abelian:** Under this assumption all steps were solved ([A-Schneider], [Heckenberger], [Angiono], [Angiono-García Iglesias]).

$|G|$ **odd, hence** G **solvable:** Essentially solved by reduction to the abelian case [Heckenberger-Meir-Vendramin], [A-Heckenberger-Vendramin].

G **solvable:** the determination of the finite dimensional Nichols algebras essentially follows from ([Heckenberger-Vendramin], [Heckenberger-Meir-Vendramin], [A-Heckenberger-Vendramin]).

G **simple:** In contrast with the abelian case, the Nichols algebras of $V \in {}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ simple are very difficult to compute and essentially no (genuine) example is known.

However, several criteria were found that, once certain conditions are met, ensure that the Nichols algebra of a simple V has infinite dimension.

Definition. A finite group G *collapses* when the only finite dimensional pointed Hopf algebra H with $G(H) \simeq G$ is the group algebra $\Bbbk G$.

Conjecture. A finite simple group collapses.

We address this conjecture by analyzing group by group and, for each group, conjugacy class by conjugacy class, applying the aforementioned techniques.

State of the art.

The following (simple) groups collapse:

- (A-Fantino-Graña-Vendramin) the alternating groups $\mathbb{A}_n$, $n \geq 5$;

- (A-Fantino-Graña-Vendramin), (Fantino-Vendramin) the sporadic groups, except for Fi_{22}, B (the baby Monster), M (the Monster);

- (Carnovale-Costantini) Suzuki and Ree groups;

- (A-Carnovale-García) The groups $PSL_n(q)$ with $n \geq 4$, $PSL_3(q)$ with $q > 2$, and $PSp_{2n}(q)$, $n \geq 3$.

For the remaining finite simple groups, there is a (short) list of conjugacy classes still open; some are work in progress.

Finally, there is a related, somehow parallel, viewpoint: to look at Nichols associated to simple racks. Again, we may apply the mentioned techniques to decide when a Nichols associated to a simple rack is infinite dimensional.

Indeed, the classification of finite simple racks is known and contains (properly) the list of conjugacy classes of finite simple groups.

This viewpoint might be useful for the classification of finite-dimensional Nichols algebras over arbitrary finite groups, a goal that is still very far from us–today.